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Hamiltonian connected balanced multipartite tournaments and hamiltonian connected partitions*

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Abstract

In this paper we give sufficient conditions for the existence of a partition of an r-balanced c-partite tournament into r strongly hamiltonian connected tournaments of order c (an hc-partition). We also prove that every r-balanced c-partite tournament with $c \geq 5$ and $r \geq 5$ is strongly hamiltonian connected if it has an hc-partition and minimum degree at least $\frac{c(r+12)}{4} + \frac{3r}{4}$. As a consequence of these theorems, we give sufficient conditions for balanced multipartite tournaments and regular balanced multipartite tournaments to be strongly hamiltonian connected.

1 Introduction, notation and preliminary results

Let $c \geq 3$ be an integer. A *c-partite or multipartite tournament* is a digraph obtained from a complete c-partite graph by orienting each edge. Recently multipartite tour-

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naments have received considerable attention from various authors [2, 3, 4, 5, 8, 9]. Let G be a c-partite tournament of order n with partite sets $V_1, V_2, \dots V_c$. We say that a c-partite tournament is *r-balanced* if each partite set has exactly r vertices, and we denote it by $G_{r,c}$. We follow all the definitions and notation of [1]. Let A and B be two non intersecting subdigraphs (or subsets of vertices) of a digraph D. Then we denote by (A, B) the set of arcs from a vertex of A to a vertex of B, and if $a \in (A, B)$, we say that a is an AB-arc.

For an oriented graph D, the *global irregularity of D* is defined as

$$i_g(D) = \max_{x,y \in V(D)} \left(\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} \right).$$

If $i_g(D) = 0$, then D is regular. For $x \in V(G)$ and $i \in [c]$, the *out-neighborhood* of x in V_i is $N_i^+(x) = V_i \cap N^+(x)$; the *in-neighborhood* of x in V_i is $N_i^-(x) = V_i \cap N^-(x)$; $d_i^+(x) = |N_i^+(x)|$; $d_i^-(x) = |N_i^-(x)|$; and $\delta(G) = \min_{x \in V(G)} \{d^-(x), d^+(x)\}$ is the *minimum degree* of G. The *local partite irregularity* of G is defined as

$$\mu(G) = \max_{x \in V(G)} \max_{i \in [c]} |d_i^+(x) - d_i^-(x)|.$$

Given a tournament T, a vertex $x \in V(T)$ is q-wicked for T if $\min\{d_T^+(x), d_T^-(x)\} \leq q$. Let $G_{r,c}$ be an r-balanced c-partite tournament. Notice that a maximal tournament in $G_{r,c}$ is a tournament of order c. For each integer $q \geq 0$ and each vertex $x \in V(G_{r,c})$, let $T_q^+(x)$ (respectively, $T_q^-(x)$) be the number of maximal tournaments of $G_{r,c}$ for which x is q-wicked because it has out-degree (respectively, in-degree) at most q in T. A *partition of* $G_{r,c}$ *into maximal tournaments* is a spanning subdigraph of $G_{r,c}$ formed by r pairwise vertex-disjoint tournaments of order c. Notice that if T is not strong, then T must have a $\lceil \frac{c-2}{4} \rceil$-wicked vertex. In [4] we gave sufficient conditions for a balanced multipartite tournament to have a *strong partition* (*st*-partition for short); that is, partitions for which every maximal tournament of the partition is strong, and in our results we used the following bounds of the number of maximal tournaments of $G_{r,c}$ for which x is q-wicked for $q = \lceil \frac{c-2}{4} \rceil$.

Theorem 1.1 (Theorem 2.3, [3]) *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 2$ and $c \geq 5$ such that for some integer $q \geq 0$, $\delta(G_{r,c}) \geq q\,(r + \mu(G_{r,c}))\,\frac{c-1}{c-2}$. Then, for every $x \in V(G_{r,c})$,*

$$T_q^+(x) \leq \sum_{k=0}^{q} \binom{c-1}{k} \left(\frac{d^+(x)}{c-1}\right)^k \left(\frac{d^-(x)}{c-1}\right)^{c-1-k}.$$

Theorem 1.2 (Theorem 2.4, [3]) *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 2$ and $c \geq 5$. If for some $q \geq 0$, $\delta(G_{r,c}) \geq q\,(r + \mu(G_{r,c}))\,\frac{c-1}{c-2}$ and $i_g(G_{r,c}) = r(c-1)\beta$ with $0 \leq \beta < \frac{c-2q-2}{c}$, then for every $x \in V(G_{r,c})$ we have that*

$$T_q^+(x) \leq \binom{c-1}{q+1} \left(\frac{r}{2}\right)^{c-1} \frac{(1+\beta)^{c-2-2q}\,(q+1)}{c(1-\beta) - 2q - 2}.$$

A digraph D is *strongly hamiltonian connected* (*hamiltonian connected* for short) if, for any two vertices x and y, there is a hamiltonian path from x to y and from y to x. In this article we study the partitions of $G_{r,c}$ into hamiltonian connected maximal tournaments. A partition is called *hamiltonian connected* (*hc-partition* for short) if every maximal tournament of the partition is hamiltonian connected. A vertex x is *hc-wicked* for a maximal tournament T if $\min\{d^+_T(x), d^-_T(x)\} \leq \lceil \frac{c}{4} \rceil + 1$. It can be proved, using the following theorem, that if a tournament is not hamiltonian connected, then it has an *hc*-wicked vertex.

Theorem 1.3 (Corollary 5.7 [7]) *A 4-connected tournament is hamiltonian connected.*

Using Theorems 1.1, 1.2 and 1.3, we give sufficient conditions for $G_{r,c}$ to have an *hc*-partition. We also prove that every r-balanced c-partite tournament, $G_{r,c}$, with $c \geq 5$ and $r \geq 5$ is hamiltonian connected if it has an *hc*-partition and minimum degree at least $\dfrac{c(r+12)}{4} + \dfrac{3r}{4}$. As a consequence of these theorems we give sufficient conditions for balanced multipartite tournaments and regular balanced multipartite tournaments to be hamiltonian connected.

2 Hamiltonian connected partitions

In this section we give sufficient conditions on the global irregularity and the local partite irregularity for a balanced multipartite tournament to have a hamiltonian connected partition. Let T be a tournament of order $c \geq 10$. First we prove Lemma 2.1 to affirm that if T has no *hc*-wicked vertices, then T is hamiltonian connected, and this fact will be used thoughout, without mentioning it.

Lemma 2.1 *Let T be a tournament of order $c \geq 10$. If $\delta(T) \geq \lceil \frac{c}{4} \rceil + 2$, then T is hamiltonian connected.*

Proof. Let T be a tournament of order $c \geq 10$ with $\delta(T) \geq \lceil \frac{c}{4} \rceil + 2$. By Theorem 1.3, it suffices to prove that T is 4-connected. For a contradiction, assume that T is not 4-connected. Then there is a set $S \subseteq V(T)$, with $|S| \leq 3$, such that $T - S$ is not strongly connected. Therefore there is a pair $x, y \in V(T) \setminus S$ such that there is no xy-path in $T - S$. Let $A_x \subseteq V(T) \setminus S$ (respectively, $A_y \subseteq V(T) \setminus S$) be the set of all the $z \in V(T) \setminus S$ such that there is an xz-path (respectively, zy-path) in $T - S$. Clearly $A_x \cap A_y = \emptyset$, and there is no arc from A_x to $V(T) \setminus (S \cup A_x)$ (respectively, from $V(T) \setminus (S \cup A_y)$ to A_y). Therefore $\sum\limits_{w \in A_x} d^+(w) \leq \binom{|A_x|}{2} + |S||A_x|$ and $\sum\limits_{w \in A_y} d^-(w) \leq \binom{|A_y|}{2} + |S||A_y|$. We may assume that $|A_x| \leq |A_y|$. Hence there is $w_0 \in A_x$ such that $d^+(w_0) \leq \frac{|A_x|-1}{2} + |S|$, and since $|A_x| \leq |A_y|$ and $|A_x| \leq \frac{c-|S|}{2}$, we have $d^+(w_0) \leq \frac{|A_x|-1+2|S|}{2} \leq \frac{c-|S|-2+4|S|}{4} = \frac{c+3|S|-2}{4}$. Since $|S| \leq 3$, we have

$d^+(w_0) \leq \frac{c+7}{4}$, which contradicts the fact that $\delta(T) \geq \lceil \frac{c}{4} \rceil + 2$. By Theorem 1.3, the lemma follows. □

Lemma 2.3 assures us that if for every vertex x in an r-balanced c-partite tournament, the number of tournaments for which x is hc-wicked because of its out-neighborhood is less than $\dfrac{r^{c-1}}{2rc}$, then the r-balanced c-partite tournament has an hc-partition. For its proof we need the following lemma.

Lemma 2.2 *(Lemma 1 [4]) The number of partitions of any $G_{r,c}$ into maximal tournaments is $(r!)^{c-1}$.*

Lemma 2.3 *Let $G_{r,c}$ be an r-balanced c-partite tournament. If $G_{r,c}$ does not have an hc-partition, then there exists a vertex x_0 such that*

$$\max\{T^+_{\lceil \frac{c}{4} \rceil+1}(x_0), T^-_{\lceil \frac{c}{4} \rceil+1}(x_0)\} \geq \frac{r^{c-1}}{2rc}.$$

Proof. If $G_{r,c}$ does not have an hc-partition, then every partition into maximal tournaments has a tournament T with an hc-wicked vertex; that is, $\delta(T) \leq \lceil \frac{c}{4} \rceil + 1$. By Lemma 2.2, there are $(r!)^{c-1}$ partitions. Since $G_{r,c}$ does not have an hc-partition, the number of partitions with an hc-wicked vertex is the number of all partitions. For a given $x \in V(G_{r,c})$, $T^+_{\lceil \frac{c}{4} \rceil+1}(x) + T^-_{\lceil \frac{c}{4} \rceil+1}(x)$ is the number of maximal tournaments for which x is an hc-wicked vertex. Notice that each maximal tournament of $G_{r,c}$ for which x is hc-wicked is contained in $((r-1)!)^{c-1}$ partitions.

Therefore,

$$((r-1)!)^{c-1} \sum_{x \in V(G_{r,c})} \left(T^+_{\lceil \frac{c}{4} \rceil+1}(x) + T^-_{\lceil \frac{c}{4} \rceil+1}(x) \right) \geq (r!)^{c-1}$$

and

$$\sum_{x \in V(G_{r,c})} \left(T^+_{\lceil \frac{c}{4} \rceil+1}(x) + T^-_{\lceil \frac{c}{4} \rceil+1}(x) \right) \geq r^{c-1}.$$

Since the order of $G_{r,c}$ is rc, by an averaging argument, there exists a vertex $x_0 \in G_{r,c}$ such that

$$T^+_{\lceil \frac{c}{4} \rceil+1}(x_0) + T^-_{\lceil \frac{c}{4} \rceil+1}(x_0) \geq \frac{r^{c-1}}{rc}.$$

From here the result follows. □

We now give sufficient conditions for an r-balanced c-partite tournament to have an hc-partition.

Theorem 2.1 *Let $G_{r,c}$ be a regular r-balanced c-partite tournament, with $c \geq 10$, $r \geq 2$ and $\mu(G_{r,c}) \leq \frac{r(c-2)}{2\lceil \frac{c}{4} \rceil+2} - r$. Then, $G_{r,c}$ has an hc-partition if*

$$2^{c-2} > rc \min\left\{ \binom{c-1}{\lceil \frac{c}{4} \rceil + 2} \frac{\lceil \frac{c}{4} \rceil + 2}{c - 2\lceil \frac{c}{4} \rceil - 4}, \sum_{k=0}^{\lceil \frac{c}{4} \rceil+1} \binom{c-1}{k} \right\}.$$

Proof. If $G_{r,c}$ does not have an hc-partition, then, by Lemma 2.3, there exists $x_0 \in V(G_{r,c})$ such that

$$T^+_{\lceil \frac{c}{4} \rceil+1}(x_0) \geq \frac{r^{c-1}}{2rc}. \tag{1}$$

Since $\mu(G_{r,c}) \leq \frac{r(c-2)}{2\lceil \frac{c}{4} \rceil+2} - r$, and $G_{r,c}$ is regular,

$$\begin{aligned} \left(\left\lceil \frac{c}{4} \right\rceil + 1\right)(r + \mu(G_{r,c}))\frac{c-1}{c-2} &\leq r\left(\left\lceil \frac{c}{4} \right\rceil + 1\right)\left(\frac{c-2}{2\lceil \frac{c}{4} \rceil + 2}\right)\frac{c-1}{c-2} \\ &= \frac{r(c-1)}{2} = \delta(G_{r,c}). \end{aligned}$$

Thus, by Theorem 1.1 and Inequality (1), for every $x \in V(G_{r,c})$

$$\frac{r^{c-1}}{2rc} \leq T^+_{\lceil \frac{c}{4} \rceil+1}(x) \leq \sum_{k=0}^{\lceil \frac{c}{4} \rceil+1} \binom{c-1}{k}\left(\frac{d^+(x)}{c-1}\right)^k\left(\frac{d^-(x)}{c-1}\right)^{c-1-k}.$$

Since $G_{r,c}$ is regular, for every $x \in V(G_{r,c})$, $d^+(x) = d^-(x) = \frac{r(c-1)}{2}$ and

$$\frac{r^{c-1}}{2rc} \leq \sum_{k=0}^{\lceil \frac{c}{4} \rceil+1} \binom{c-1}{k}\left(\frac{d^+(x)}{c-1}\right)^k\left(\frac{d^-(x)}{c-1}\right)^{c-1-k} = \sum_{k=0}^{\lceil \frac{c}{4} \rceil+1} \binom{c-1}{k}\left(\frac{r}{2}\right)^{c-1}. \tag{2}$$

Moreover, since $i_g(G_{r,c}) = r(c-1)\beta = 0$; $\delta(G_{r,c}) = \frac{r(c-1)}{2}$ and $\mu(G_{r,c}) \leq \frac{r(c-2)}{2\lceil \frac{c}{4} \rceil+2} - r$, it follows that $\delta(G_{r,c}) \geq q\,(r + \mu(G_{r,c}))\,\frac{c-1}{c-2}$ with $q = \lceil \frac{c}{4} \rceil + 1$. By Theorem 1.2 and Inequality (1), for every $x \in V(G_{r,c})$,

$$\begin{aligned} \frac{r^{c-1}}{2rc} \leq T^+_{\lceil \frac{c}{4} \rceil+1}(x) &\leq \binom{c-1}{\lceil \frac{c}{4} \rceil+2}\left(\frac{r}{2}\right)^{c-1}\frac{(1+\beta)^{c-4-2\lceil \frac{c}{4} \rceil}\left(\lceil \frac{c}{4} \rceil+2\right)}{c(1-\beta)-2\lceil \frac{c}{4} \rceil-4} \\ &= \binom{c-1}{\lceil \frac{c}{4} \rceil+2}\left(\frac{r}{2}\right)^{c-1}\frac{\lceil \frac{c}{4} \rceil+2}{c-2\lceil \frac{c}{4} \rceil-4}. \end{aligned} \tag{3}$$

Multiplying (2) and (3) by $2^{c-1}\dfrac{2rc}{r^{c-1}}$, the result follows. □

Theorem 2.2 *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 3$, $c \geq 10$ and $i_g(G_{r,c}) = r(c-1)\beta$, where $0 \leq \beta < \frac{c-2\lceil \frac{c}{4} \rceil-4}{c}$, and $\mu(G_{r,c}) \leq r\left(\frac{(1-\beta)(c-2)}{2\lceil \frac{c}{4} \rceil+2} - 1\right)$. If*

$$2^{c-2} > rc\binom{c-1}{\lceil \frac{c}{4} \rceil+2}\frac{(1+\beta)^{c-2\lceil \frac{c}{4} \rceil-4}\left(\lceil \frac{c}{4} \rceil+2\right)}{c(1-\beta)-2\lceil \frac{c}{4} \rceil-4},$$

then there exists an hc-partition.

Proof. If $G_{r,c}$ does not have an hc-partition, then, by Lemma 2.3, there exists $x_0 \in V(G_{r,c})$ such that

$$T^+_{\lceil \frac{c}{4} \rceil+1}(x_0) \geq \frac{r^{c-1}}{2rc}. \tag{4}$$

Let $Q := rc\binom{c-1}{\lceil\frac{c}{4}\rceil+2}\frac{(1+\beta)^{c-2\lceil\frac{c}{4}\rceil-4}(\lceil\frac{c}{4}\rceil+2)}{c(1-\beta)-2\lceil\frac{c}{4}\rceil-4}$. In order to use Theorem 1.2, we need to prove that $\delta(G_{r,c}) \geq q\,(r+\mu(G_{r,c}))\,\frac{c-1}{c-2}$ with $q = \lceil\frac{c}{4}\rceil+1$. Since $G_{r,c}$ is r-balanced, for every vertex x, $d^+(x)+d^-(x) = r(c-1)$; thus the vertex with maximum out-degree is the same as the one with minimum in-degree. Therefore $i_g(G_{r,c}) = r(c-1)-2\delta(G_{r,c})$, and thus $\delta(G_{r,c}) = \frac{r(c-1)-i_g(G_{r,c})}{2}$; and by hypothesis, $i_g(G_{r,c}) = r(c-1)\beta$.

Therefore $\delta(G_{r,c}) = \frac{r(c-1)}{2}(1-\beta)$. As $\mu(G_{r,c}) \leq r\left(\frac{(1-\beta)(c-2)}{2\lceil\frac{c}{4}\rceil+2}-1\right)$, we see that $(1-\beta) \geq \frac{\left(2\lceil\frac{c}{4}\rceil+2\right)(\mu(G_{r,c})+r)}{r(c-2)}$.

Thus, $\delta(G_{r,c}) \geq \frac{r(c-1)}{2}\frac{\left(2\lceil\frac{c}{4}\rceil+2\right)(\mu(G_{r,c})+r)}{r(c-2)} = \left(\lceil\frac{c}{4}\rceil+1\right)(r+\mu(G_{r,c}))\,\frac{c-1}{c-2}$.

By Theorem 1.2 and Inequality (4), for every $x \in V(G_{r,c})$,

$$\frac{r^{c-1}}{2rc} \leq T^+_{\lceil\frac{c}{4}\rceil+1}(x) \leq \binom{c-1}{\lceil\frac{c}{4}\rceil+2}\left(\frac{r}{2}\right)^{c-1}\frac{(1+\beta)^{c-2\lceil\frac{c}{4}\rceil-4}\left(\lceil\frac{c}{4}\rceil+2\right)}{c(1-\beta)-2\lceil\frac{c}{4}\rceil-4}.$$

Simplifying, we obtain that $2^{c-2} \leq Q$. Therefore, if $2^{c-2} > Q$, there exists an hc-partition. □

3 From hc-partitions to hamiltonian connected balanced multipartite graphs

In this section we prove that having an hc-partition and sufficiently large minimum degree are sufficient conditions for a balanced multipartite tournament to be hamiltonian connected. To do so we define the digraph of a partition as follows.

Let $G_{r,c}$ be an r-balanced c-partite tournament with an hc-partition, $\mathcal{P} = \{T_1,\ldots,T_r\}$ of $G_{r,c}$. We define $D_{\mathcal{P}}$ of the partition $\mathcal{P}$ as a digraph with vertex set $V(D_{\mathcal{P}}) = \{T_1,\ldots,T_r\}$ and such that $T_iT_j \in A(D_{\mathcal{P}})$ if and only if in $G_{r,c}$ there are at least three independent T_iT_j-arcs. Observe that $D_{\mathcal{P}}$ may have symmetric arcs.

Lemma 3.1 *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 3$, $c \geq 5$ and an hc-partition $\mathcal{P} = \{T_1,\ldots,T_r\}$. Then the digraph $D_{\mathcal{P}}$ is semicomplete.*

Proof. Let $T_i, T_j \in V(D_{\mathcal{P}})$ and let $V(T_i) = \{x_0,\ldots,x_{c-1}\}$ and $V(T_j) = \{y_0,\ldots,y_{c-1}\}$. Without loss of generality, suppose that for $i \in \{0,1,\ldots,c-1\}$, x_i and y_i belong to the same partite set of $G_{r,c}$. Let $A = \{x_iy_{i+1} \in A(G_{c,r}) : i \mod c\} \cup \{y_{i+1}x_i \in A(G_{c,r}) : i \mod c\}$. A is a set of c independent arcs in $G_{r,c}$. Since $c \geq 5$, it follows that either $|A\cap(T_i,T_j)| \geq 3$ or $|A\cap(T_j,T_i)| \geq 3$, and the result follows. □

Lemma 3.2 *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 3$, $c \geq 5$. Let $T_i, T_j \in V(D_{\mathcal{P}})$. If $T_iT_j \notin A(D_{\mathcal{P}})$, then $|(T_i,T_j)| \leq 2(c-1)$.*

Proof. Let $T_i, T_j \in V(D_{\mathcal{P}})$ such that $T_iT_j \notin A(D_{\mathcal{P}})$. Thus there is no triple of independent T_iT_j-arcs in $G_{r,c}$. If the maximum size of a set of independent T_iT_j-arcs is one, then all the T_iT_j-arcs are incident to the same vertex, and therefore $|(T_i, T_j)| \leq c-1$. If the maximum size of a set of independent T_iT_j-arcs is two, then let $\{x_1y_1, x_2y_2\}$ be a pair of independent T_iT_j-arcs. Then each of $(T_i - x_1, T_j - y_1)$ and $(T_i - x_2, T_j - y_2)$ contains exactly one independent (T_i, T_j)-arc. Therefore, by the above arguments, each of them contains at most $c-2$ (T_i, T_j)-arcs. Observe that (T_i, T_j) may also contain the arcs x_1y_2 and x_2y_1. Thus, we obtain that $|(T_i, T_j)| \leq 2(c-1)$ and the result follows. □

Theorem 3.1 *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 5$, $c \geq 5$ and an hc-partition. If $\delta(G_{r,c}) \geq \frac{c(r+12)}{4} + \frac{3r}{4}$, then $D_{\mathcal{P}}$ is 5-connected.*

Proof. For a contradiction, suppose that $D_{\mathcal{P}}$ is not 5-connected. Let

$$S = \{T_1, T_2, T_3, T_4\} \subseteq V(D_{\mathcal{P}})$$

such that $D_{\mathcal{P}} - S$ is not strong. Let $\{A, B\}$ be a partition of $V(D_{\mathcal{P}} - S)$ such that $(B, A) = \emptyset$. Without loss of generality suppose that $|A| \leq |B|$; then $|A| \leq \frac{r-4}{2}$. Let $A' = \bigcup_{T\in A} V(T) \subseteq V(G_{r,c})$. By Lemma 3.2, for every $T_i \in A$ and every $T_j \in B$, $|(T_i, T_j)| \leq 2(c-1)$, and since $|A'| = c|A|$, it follows that

$$\sum_{x\in A'} d^-(x) \leq \binom{|A|c}{2} - c\binom{|A|}{2} + |A||B|2(c-1) + 4(c-1)c|A|.$$

Since $c \geq 5$, $|A| = \frac{4|A|}{c} + \frac{(c-4)|A|}{c}$, $|A| + |B| = r-4$, and $|A| \leq \frac{r-4}{2}$, there is $x_0 \in V(G_{r,c})$ such that

$$\begin{aligned}
d^-(x_0) &\leq \frac{\sum_{x\in A'} d^-(x)}{|A|c} = \frac{|A|c-1}{2} - \frac{|A|-1}{2} + 2|B|\frac{c-1}{c} + 4(c-1) \\
&= \frac{|A|(c-1)}{2} + 2|B|\frac{c-1}{c} + 4(c-1) = \frac{c-1}{2}\left(|A| + \frac{4|B|}{c} + 8\right) \\
&= \frac{c-1}{2}\left(\frac{(c-4)|A|}{c} + \frac{4|A|}{c} + \frac{4|B|}{c} + 8\right) \\
&\leq \frac{c-1}{4}\left(\frac{(c-4)(r-4)}{c} + \frac{8(r-4)}{c} + \frac{16c}{c}\right) \\
&= \frac{c-1}{4c}(cr - 4c - 4r + 16 + 8r - 32 + 16c) \\
&= \frac{c-1}{4c}(cr + 12c + 4r - 16) \\
&\leq \frac{1}{4c}(c^2r + 12c^2 + 3cr - 28c - 4r + 16) \\
&= \frac{c}{4c}\left(cr + 12c + 3r - 28 + \frac{-4r+16}{c}\right) \\
&< \frac{1}{4}(cr + 12c + 3r) \\
&< \frac{c}{4}(r+12) + \frac{3r}{4},
\end{aligned}$$

which contradicts the fact that $\delta(G_{r,c}) \geq \dfrac{c(r+12)}{4} + \dfrac{3r}{4}$ and therefore $D_{\mathcal{P}}$ is 5-connected. □

We use the former results and Theorem 3.2 in order to prove Theorem 3.3.

Theorem 3.2 (Corollary 2.11 [6]) *Every 4-connected locally semicomplete digraph is hamiltonian-connected.*

Let Q_1 and Q_2 be paths of a digraph D; the concatenation of $Q_1 = (a_1, a_2, \ldots, a_n)$ and $Q_2 = (a_n = b_1, b_2, \ldots b_k)$ is the walk $Q_1 \circ Q_2 = (a_1, a_2, \ldots, a_n = b_1, \ldots, b_2, \ldots, b_k)$.

Theorem 3.3 *Let $G_{r,c}$ be an r-balanced c-partite tournament with $r \geq 4$, $c \geq 3$, and an hc-partition $\mathcal{P} = \{T_1, \ldots, T_r\}$. If $\delta(G_{r,c}) \geq 2c - 2$ and $D_{\mathcal{P}}$ is 5-connected, then $G_{r,c}$ is hamiltonian connected.*

Proof. Since T_i is hamiltonian connected, for any pair $x, y \in V(T_i)$, there exists a hamiltonian xy-path in T_i, that will be denoted by (xP_iy).

Given a path of $D_{\mathcal{P}}$, say $(T_1, T_2, \ldots, T_s)$, notice that for every vertex $u \in V(T_i)$ with $1 \leq i < s$, there exists a vertex $v \in V(T_i) \setminus \{u\}$ such that $N^+(v) \cap V(T_{i+1}) \neq \emptyset$, because there exist at least three parallel arcs from T_i to T_{i+1}.

For every $y_1 \in V(T_1)$ such that $N^+(y_1) \cap V(T_2) \neq \emptyset$, there exist $x_s \in V(T_s)$ and a y_1x_s-path $Q_{T_1,T_s}(y_1, x_s)$ of $G_{r,c}$ with vertex set $\cup_{i=2}^{s-1} V(T_i) \cup \{y_1, x_s\}$. Now $Q_{T_1,T_s}(y_1, x_s)$ is constructed as follows. Let $x_2 \in N^+(y_1) \cap V(T_2)$. As noticed above, there is a vertex $y_2 \in V(T_2) \setminus \{x_2\}$ such that there exists $x_3 \in N^+(y_2) \cap V(T_3)$. From the fact that T_2 is hamiltonian strongly connected, we can concatenate the arc (y_1, x_2) with a hamiltonian path $x_2P_2y_2$ and the arc (y_2, x_3), obtaining a y_1x_3-path with vertex set $V(T_2) \cup \{y_1, x_3\}$. Following this procedure, there exists $Q_{T_1,T_s}(y_1, x_s) = ((y_1, x_2) \circ (x_2P_2y_2) \circ (y_2, x_3) \circ (x_3P_3y_3) \circ (y_3, x_4) \circ \cdots \circ (y_{s-2}, x_{s-1}) \circ (x_{s-1}P_{s-1}y_{s-1}) \circ (y_{s-1}, x_s))$ such that $N^+(y_t) \cap T_{t+1} \neq \emptyset$ and $x_{t+1} \in N^+(y_t) \cap T_{t+1}$, for every $t \in \{1, 2 \ldots s-1\}$.

Let $a, b \in V(G_{r,c})$. It is sufficient to prove that there is a hamiltonian ab-path. Without loss of gererality we have two cases.

Case 1. a and b are vertices in different tournaments of the hc-partition.

Without loss of generality, we can assume that $a \in V(T_1)$, $b \in V(T_r)$. By Lemma 3.1 and Theorem 3.2, $\mathcal{D}_P$ is hamiltonian connected. Therefore there is a hamiltonian path, $(T_1, \ldots, T_r)$.

Let y_1 be a vertex in $V(T_1) \setminus \{a\}$ such that $N^+(y_1) \cap V(T_2) \neq \emptyset$. There exists $x_{r-1} \in V(T_{r-1})$ such that $Q_{T_1,T_{r-1}}(y_1, x_{r-1})$ is a y_1x_{r-1}-path with vertex set $\cup_{i=2}^{r-1} V(T_i) \cup \{y_1, x_{r-1}\}$. Since T_1 is hamiltonian connected, in T_1 there is a hamiltonian ay_1-path, say Q_1. There are at least three parallel arcs from $V(T_{r-1})$ to $V(T_r)$; therefore at least one of them is an arc uv such that $u \neq x_{r-1}$ and $v \neq b$. Since T_{r-1} and T_r are hamiltonian connected, there exists a hamiltonian $x_{r-1}u$-path of T_{r-1}, say Q_{r-1}, and a hamiltonian vb-path of T_r, say Q_r. Then $Q_1 \circ Q_{T_1T_{r-1}}(y_1, x_r) \circ Q_{r-1} \circ Q_r$ is a hamiltonian ab-path of $G_{c,r}$.

Case 2. a and b are vertices of the same tournaments of the hc-partition.

We can assume that $a, b \in V(T_r)$. Since $d^+(a) \geq 2c - 2 \geq c + 1$, we can assume that there exists $u \in V(T_1)$ such that $au \in A(G_{r,c})$. As T_r is hamiltonian connected, there exists a hamiltonian path, $(a, a_1, \ldots a_{c-1} = b)$ in T_r. By hypothesis, $d^-(a_1) \geq 2c - 2 \geq c + 1$; thus there exists a partite set $T_{r-1} \neq T_1$ and a vertex $w \in V(T_{r-1})$ such that $w \in N^-(a_1)$. By hypothesis, $D_{\mathcal{P}}$ is 5-connected; therefore $D_{\mathcal{P}} - T_r$ is 4-connected and by Theorem 3.2, hamiltonian connected. Let $(T_1, \ldots, T_{r-1})$ be a hamiltonian path of $D_{\mathcal{P}} - T_r$.

Since there are at least three parallel arcs from T_1 to T_2, there exists $y_1 \neq u$ and $x_{r-2} \in V(T_{r-2})$ such that $Q_{T_1,T_{r-2}}(y_1, x_{r-2})$ is a y_1x_{r-2}-path containing vertices $\cup_{i=2}^{r-3} V(T_i)$.

Since there are three parallel arcs from T_{r-2} to T_{r-1}, there exists at least one of these arcs cd such that $c \neq x_{r-2}$ and $d \neq w$. Since T_1 is hamiltonian connected, let Q be a hamiltonian uy_1-path in T_1; analogously there exists S, a hamiltonian $x_{r-2}c$-path of T_{r-2}, and U, a hamiltonian dw-path in T_{r-1}; see Figure 1.

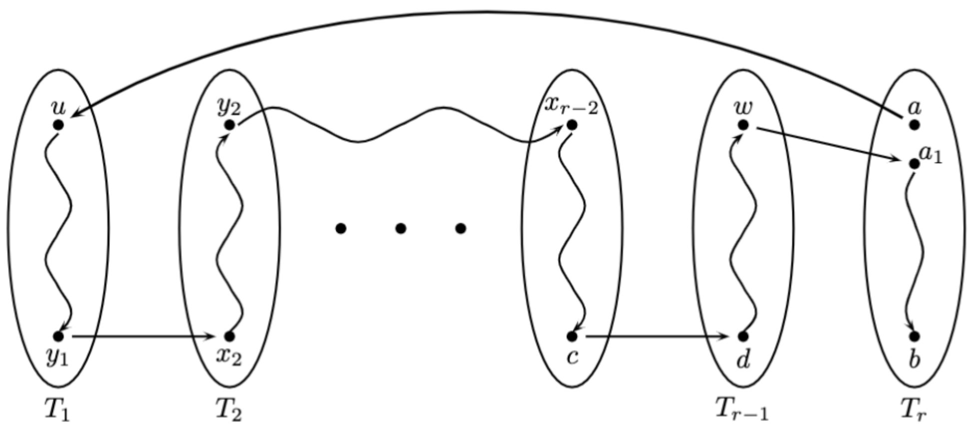

Figure 1

Therefore

$$(Q \circ Q_{T_1,T_{r-2}}(y_1, x_{r-2}) \circ S \circ U \circ (a_1, \ldots, a_{c-1} = b))$$

is an ab-hamiltonian path of $G_{r,c}$. □

Corollary 3.1 *Let $G_{r,c}$ be a regular r-balanced c-partite tournament, with $c \geq 10$, $r \geq 5$ and $cr \geq 12c + 5r$ and $\mu(G_{r,c}) \leq r\left(\frac{c-2}{2\lceil\frac{c}{4}\rceil+2} - 1\right)$. Then $G_{r,c}$ is hamiltonian connected if*

$$2^{c-2} > rc \min\left\{\binom{c-1}{\lceil\frac{c}{4}\rceil + 2}\left(\frac{\lceil\frac{c}{4}\rceil + 2}{c - 2\lceil\frac{c}{4}\rceil - 4}\right), \sum_{k=0}^{\lceil\frac{c}{4}\rceil+1}\binom{c-1}{k}\right\}.$$

Proof. Since $G_{r,c}$ is regular, $\delta(G_{r,c}) = \frac{r(c-1)}{2}$, and using $cr \geq 12c + 5r$ it is not difficult to check that $\frac{r(c-1)}{2} \geq \frac{c(r+12)}{4} + \frac{3r}{4}$. Therefore, by Theorems 2.1, 3.1 and 3.3, $G_{r,c}$ is hamiltonian connected. □

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