AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **91(1)** (2025), Pages 41–66

Pyramids of segments: nice new formulas with bijective proofs

Svjetlan Feretić

Ksaver 39, HR-10 000 Zagreb
Croatia
svjetlan.feretic@gmail.com

Abstract

In this paper, a segment is a finite row of unit cells. A pyramid of segments is a set of several segments, where one segment is called the base. In every segment other than the base, at least one cell is supported from below by another segment. Pyramids of segments appear in the enumeration of other combinatorial objects (such as directed animals, parallelogram polyominoes and 321-avoiding affine permutations), but are also of independent interest. We shall prove that there are $\binom{n}{r}^2$ pyramids of segments with $n+1$ cells and $r+1$ segments. So there is a total of $\binom{2n}{n}$ pyramids of segments with $n+1$ cells. After obtaining these simple formulas by a generating functions approach, we also give bijective proofs.

1 Introduction

A *pyramid of segments* (see Figure 1) is a plane object made up of *segments*, where a segment is actually a row of unit cells. At its beginning stage, a pyramid of segments is just a single segment. The other segments—either none or a finite number of them—are then added one by one. Each new segment is initially placed so that the new segment's bottom has much greater y-coordinate than the top of the existing pyramid, and so that at least one cell of the new segment stands precisely above some cell of the existing pyramid. Then we let the new segment fall as if gravity acted on it. The fall ends as soon as the bottom side of some cell of the new segment coincides with the top side of some cell of the existing pyramid.

Pyramids of segments are an instance of Viennot's [20, 21] heaps of pieces. Pyramids of two-celled segments (also known as pyramids of dimers) have proved useful in the enumeration of directed animals [2, 9, 20]. Another interesting case is when the leftmost cell of the minimal segment is also a leftmost cell of the whole pyramid. Pyramids of segments having that property are called *semi-pyramids of segments*. They appear in the q-enumeration of *parallelogram polyominoes* (Figure 2,

ISSN: 2202-3518

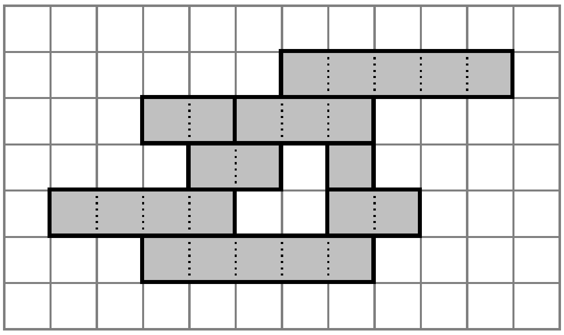

Figure 1: A pyramid of segments.

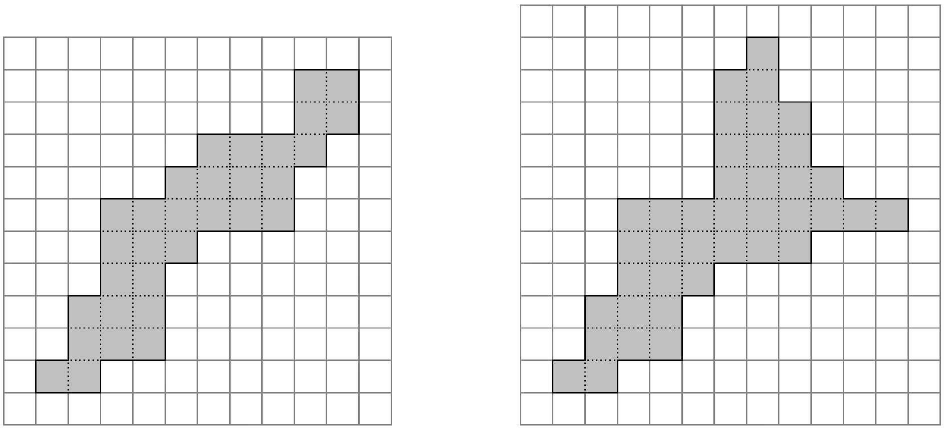

Figure 2: Left: A parallelogram polyomino. Right: A directed convex polyomino.

left) [6, 10]. The whole set of pyramids of segments appears in the q-enumeration of *periodic parallelogram polyominoes*. Periodic parallelogram polyominoes (PPPs) are a certain superset of "ordinary" parallelogram polyominoes. PPPs were introduced recently by Biagioli, Jouhet and Nadeau [3, 4]. In [3, 4], the main goal is to enumerate 321-avoiding affine permutations, and PPPs come in handy because they somehow resemble affine permutations.

When it comes to counting pyramids of segments, one should of course take into account parameters like the number of cells and the number of segments. However, there is one parameter—we shall call it *horizontal spread*—that is less obvious, but not less important. Horizontal spread plays a prominent role in [3, 4, 6, 10]. What is horizontal spread? It is the sum, over all segments of a pyramid, of the distance between the right vertical tangent line of the segment and the left vertical tangent line of the whole pyramid. Let $\hat{A} = \hat{A}(x, s, q)$ be the generating function in which the coefficient of $x^n s^r q^t$ is the number of pyramids of segments that have n cells, r segments and horizontal spread t. From computations of Biagioli, Jouhet and Nadeau [4] it follows at once that

$$\hat{A}(x,s,q) = sx\frac{\sum_{n\geq 1}\frac{n(-sx)^{n-1}q^{\binom{n+1}{2}}}{(q;q)_n(xq;q)_n}}{\sum_{n\geq 0}\frac{(-sx)^n q^{\binom{n+1}{2}}}{(q;q)_n(xq;q)_n}} - sxq\frac{\sum_{n\geq 1}\frac{n(-sxq)^{n-1}q^{\binom{n+1}{2}}}{(q;q)_n(xq;q)_n}}{\sum_{n\geq 0}\frac{(-sxq)^n q^{\binom{n+1}{2}}}{(q;q)_n(xq;q)_n}}\ , \tag{1}$$

where $(a;q)_n$ stands for the product $(1-a)(1-aq)\cdots(1-aq^{n-1})$.

Remark. Biagioli, Jouhet and Nadeau's pyramids of segments [3, 4] are somewhat different from our pyramids of segments. In this paper, pyramids of segments are invariant under all translations. In [3, 4], pyramids of segments are invariant under upward/downward translations, but are not invariant under leftward/rightward translations. Formula (1) holds for this paper's pyramids of segments.

Bousquet-Mélou and Viennot [10] defined a bijection between semi-pyramids of segments and parallelogram polyominoes. That bijection shows that, for all $n \in \{0, 1, 2, \ldots\}$, $r \in \{0, 1, \ldots, n\}$ and $t \in \mathbb{N}$, the number of semi-pyramids of segments with $n+1$ cells, $r+1$ segments and horizontal spread t is equal to the number of parallelogram polyominoes with total perimeter $2n+4$, horizontal perimeter $2r+2$ (that is, $r+1$ columns) and area t.

General pyramids of segments are related to *directed convex polyominoes* (Figure 2, right). This connection is not so tight as the connection between semi-pyramids of segments and parallelogram polyominoes. The number of pyramids of segments with $n+1$ cells, $r+1$ segments and horizontal spread t is generally *not equal* to the number of directed convex polyominoes with total perimeter $2n+4$, horizontal perimeter $2r+2$ and area t. For example, there are two pyramids of segments with 3 cells, 2 segments and horizontal spread 4, but there is only one directed convex polyomino with total perimeter 8, horizontal perimeter 4 and area 4. Still, for $n \in \{0, 1, 2, \ldots\}$, there are $\binom{2n}{n}$ pyramids of segments with $n+1$ cells, and there are also $\binom{2n}{n}$ directed convex polyominoes with perimeter $2n+4$. For $n \in \{0, 1, 2, \ldots\}$ and $r \in \{0, 1, \ldots, n\}$, there are $\binom{n}{r}^2$ pyramids of segments with $n+1$ cells and $r+1$ segments, and there are also $\binom{n}{r}^2$ directed convex polyominoes with perimeter $2n+4$ and $r+1$ columns. In the case of directed convex polyominoes, the results $\binom{2n}{n}$ and $\binom{n}{r}^2$ readily follow from the formula for the perimeter generating function, found by Lin and Chang [15] in 1988. Once Lin and Chang's formula was known, Bousquet-Mélou [5] and Feretić [12] proved the consequent results $\binom{2n}{n}$ and $\binom{n}{r}^2$ via bijections. In the case of pyramids of segments, the results $\binom{2n}{n}$ and $\binom{n}{r}^2$ readily follow from the formula

$$\hat{A}(x,s,1) = \frac{sx}{\sqrt{1-2x-2sx+(1-s)^2x^2}}\ , \tag{2}$$

but formula (2) does not at all readily follow from formula (1). In formula (1) there are an infinity of denominators, and if we set $q = 1$, each of those denominators becomes equal to zero. Fortunately, it is not the case that (2) must be obtained from (1). In Sections 3 and 4 of this paper, formula (2) will be obtained without ever dealing with horizontal spread. We are going to use the Schützenberger methodology

[7, 8], while Biagioli, Jouhet and Nadeau [4] used Viennot's Inversion Lemma [21]. To my best knowledge, formula (2) does not appear in any publication prior to this paper.

This paper continues as follows. In Section 2, we state definitions and introduce notation. In Section 3, we define what are the first, second, third,... segments of a pyramid. Then we encode pyramids of segments. The codes are Motzkin paths satisfying an additional requirement—to put it briefly, returns to level zero and horizontal steps have to be interlaced in a certain specific way. We write $\mathcal{A}$ to denote the set of all codes of pyramids of segments, and $\mathcal{D}$ to denote the set of all Dyck paths. In Section 4, we define two more families of Motzkin paths, denoted $\mathcal{B}$ and $\mathcal{C}$. Then we find lattice path factorizations and compute generating functions. Thus we obtain formula (2) and the consequent results $\binom{2n}{n}$ and $\binom{n}{r}^2$. Taking inspiration from computations of Section 4, in Section 5 we write a new lattice path decomposition, in which the path families $\mathcal{B}$ and $\mathcal{C}$ are bypassed: a path $t \in \mathcal{A}$ is expressed in terms of Dyck paths only. This new decomposition brings to light that every element of $\mathcal{A}$ has an alter ego among bilateral Dyck paths. Mapping the elements of $\mathcal{A}$ to their alter egos, we obtain a bijective proof that there are $\binom{2n}{n}$ pyramids of segments with $n+1$ cells.

The remaining task is to provide a bijective proof of the result $\binom{n}{r}^2$. In Section 6, we formulate that task more precisely: we want to exhibit a bijection from pyramids with $n+1$ cells and $r+1$ segments to bilateral Dyck paths with $2n$ steps, of which r are upward even steps. (By the even steps of a path we mean its 2nd, 4th, 6th,... steps.) We give an example where the bijection of Section 5 (we call it bijection $g \circ f$) maps a pyramid with 35 cells and 12 segments to a bilateral Dyck path with 68 steps, of which 16 (and not 11) are upward even steps. Thus, the bijection $g \circ f$ is no longer adequate and needs to be upgraded. To lay the groundwork for the upgrade, in Section 7 we study the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. In particular, we define a bijection between two familiar Narayana-enumerated objects: Dyck paths having $2n$ steps and k peaks, and Dyck paths having $2n$ steps, of which k are upward odd steps. This bijection is a bit different from bijections between Narayana-enumerated objects that can be found in the literature [16, 18, 19]. Using the results of Section 7, in Section 8 we finally obtain a bijection that does map pyramids with $n+1$ cells and $r+1$ segments to bilateral Dyck paths with $2n$ steps, of which r are upward even steps.

2 Definitions and notation

A *unit cell* is a square $[i, i+1] \times [j, j+1]$, where i and j are integers. A *segment* is a finite row of unit cells. Two segments are *cell-disjoint* if they do not have a cell in common. (If two segments are cell-disjoint, they still may have one or more edges in common.) Let H be a finite family of cell-disjoint segments, and let s be a segment of H. If, for every $j \in \mathbb{N}$, the segment $s - (0, j)$ is cell-disjoint with all segments of H, we say that s is a *minimal* segment of H. If, for every $j \in \mathbb{N}$, the segment

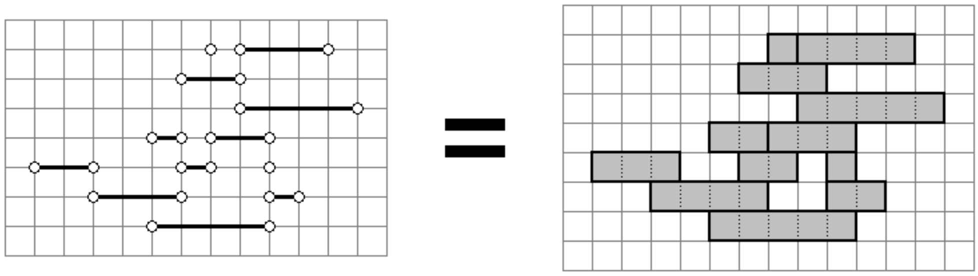

Figure 3: A pyramid of segments can be drawn in two different ways. Left: The segments are really segments (and their endpoints have integer coordinates). Right: The segments are actually rows of unit cells.

$s + (0, j)$ is cell-disjoint with all segments of H, we say that s is a *maximal* segment of H. A segment s is *supported from below* by a segment r if the segment $s - (0, 1)$ *is not* cell-disjoint with r. Notice that, in a given family of cell-disjoint segments, it can happen that a segment is neither minimal nor supported from below by any other segment.

A finite family of cell-disjoint segments H is a *heap of segments* if the following two conditions are satisfied: 1) all minimal segments of H have the same ordinate, and 2) every segment of H is either minimal or is supported from below by at least one other segment of H. A *pyramid of segments* is a heap of segments having only one minimal segment. Thus, our pyramids of segments are practically identical to X. Viennot's [21] pyramids of segments; we only replaced lattice points with unit cells. See Figure 3.

Let P_1 and P_2 be two pyramids of segments. In the upcoming enumeration, if there exists a translation τ such that $P_2 = \tau(P_1)$, we regard P_1 and P_2 as one and the same pyramid of segments.

A *polyomino* is a finite, edge-connected union of unit cells. There is an important class of polyominoes called *partially directed polyominoes* [13, subsection 3.5.2]. Partially directed polyominoes are a subset of pyramids of segments. Namely, a partially directed polyomino is a pyramid of segments in which no two segments share a vertical edge. Next, there is a model called *directed animals on the king's lattice*. (The latter model has been introduced in 2013 by Axel Bacher [1].) A directed animal on the king's lattice is the object that remains when, in a pyramid of segments having no one-celled segments, we delete the leftmost cell of every segment. See Figure 4.

While a pyramid of segments is being constructed, the sizes of segments can be kept under control, but the contacts between segments are practically uncontrollable. Hence, with our method of counting pyramids of segments, one can (*mutatis mutandis*) also count directed animals on the king's lattice, but cannot count partially directed polyominoes.

If a set S has n elements, we write $|S| = n$.

In the lattice paths appearing in this paper, every step is either $(1, 1)$ or $(1, -1)$ or

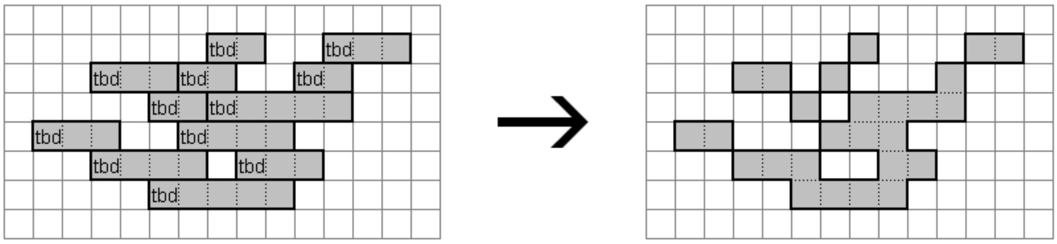

Figure 4: Left: A pyramid of segments in which every segment has at least two cells. The abbreviation *tbd* means "to be deleted". Right: The corresponding directed animal on the king's lattice.

$(1,0)$. We call $(1,1)$ an *upward step*, $(1,-1)$ a *downward step* and $(1,0)$ a *horizontal step*. We denote steps by letters: $x=(1,1)$, $y=(1,-1)$, and $a=(1,0)$. Let w be a lattice path with i upward steps, j downward steps and k horizontal steps. Then we write $|w|_x=i$, $|w|_y=j$, $|w|_a=k$ as well as $|w|=i+j+k$. We denote the empty path by the Greek letter ε. By the product of two lattice paths we mean their concatenation. Given two lattice paths u and w, if there exist lattice paths v and z such that $u=vwz$, we say that w is a *factor* (or *subpath*) of u. If $v=\varepsilon$, w is a *left factor* of u. If $z=\varepsilon$, w is a *right factor* of u. Suppose that, together with $u=vwz$, we have $|v|_x-|v|_y=\ell$ and $|vw|_x-|vw|_y=m$. Then we say that the subpath w starts at level ℓ and ends at level m.

A lattice path u is a *Motzkin path* if it satisfies the following two conditions: 1) $|u|_x=|u|_y$, and 2) if v is a left factor of u, then $|v|_x\geq|v|_y$. A *Dyck path* is a Motzkin path in which horizontal steps do not occur. A *peak* of a Dyck path u is a vertex of u at which an upward step ends and a downward step starts. For example, the path $\widehat{xy}x\widehat{xy}xx\widehat{xy}yyy$ has three peaks. A lattice path u is a *bilateral Dyck path* if $|u|_x=|u|_y$ and $|u|_a$ is zero. A Dyck path cannot go below the horizontal line through its origin, but a bilateral Dyck path can go below that line.

Suppose that a lattice path u has a factorization $u=vwz$ such that: 1) v is either empty or ends with a downward step, 2) w is nonempty and each of its steps is either an upward step or a horizontal one, and 3) z is either empty or starts with a downward step. Then we say that w is a *weakly ascending nest* (*wan*) of u. An *ascending nest* is a weakly ascending nest having no horizontal steps. Now suppose that a lattice path u has a factorization $u=vwz$ such that: 1) v does not end with a downward step, 2) w is nonempty and each of its steps is a downward step, and 3) z does not start with a downward step. Then we say that w is a *descending nest* (*den*) of u. If a lattice path u has ℓ weakly ascending nests and m descending nests, then we write $wan(u)=\ell$ and $den(u)=m$.

Given a lattice path w, we write $\overline{w}$ to denote the mirror image of w in the x-axis. That is, if w has n steps, then $\overline{w}$ also has n steps. For every $i\in\{1,\ldots,n\}$, if the ith step of w is an upward (respectively downward, horizontal) step, then the ith step of $\overline{w}$ is a downward (respectively upward, horizontal) step.

The *odd steps* of a lattice path w are the first, third, fifth, …steps of w. The *even steps* of a lattice path w are the second, fourth, sixth, ... steps of w. If a lattice path w has i upward odd steps and j upward even steps, we write $uo(w) = i$ and $ue(w) = j$.

3 A coding for pyramids of segments

In this section, we define a bijection between pyramids of segments and a certain subset of Motzkin paths.

Let $\mathscr{P}$ denote the set of all pyramids of segments. For $n \in \mathbb{N}$ and $r \in \{1, \ldots, n\}$, let

$$\mathscr{P}_n = \{P \in \mathscr{P} : \; P \text{ has } n \text{ cells}\},$$

$$\mathscr{P}_{n,r} = \{P \in \mathscr{P} : \; P \text{ has } n \text{ cells and } r \text{ segments}\}.$$

Let $\mathscr{A}$ denote the set of all paths u which have the following properties:

1. u is a nonempty Motzkin path,
2. if w is a weakly ascending nest of u, then w begins with an upward step and contains at most one horizontal step,
3. if w is a weakly ascending nest of u, then w contains a horizontal step if and only if the factorization $u = vwz$ starts with a path v that is not empty and ends at level zero.

For $n \in \mathbb{N}$ and $r \in \{1, \ldots, n\}$, let

$$\mathscr{A}_n = \{u \in \mathscr{A} : \; |u|_x = n\},$$

$$\mathscr{A}_{n,r} = \{u \in \mathscr{A} : \; |u|_x = n \;\&\; wan(u) = r\}.$$

Let P be an element of $\mathscr{P}_{n,r}$. We shall soon encode P by an element of $\mathscr{A}_{n,r}$, but first we have to mark the segments of P by numbers. The rth segment of P is the rightmost maximal segment of P. The $(r-1)$st segment of P is the rightmost maximal segment of the pyramid that remains when the rth segment of P is deleted. The $(r-2)$nd segment of P is the rightmost maximal segment of the pyramid that remains when the rth and $(r-1)$st segments of P are deleted, and so on. See Figure 5.

Let s_i stand for the ith segment of P $(i = 1, \ldots, r)$. Let b_i and c_i be the minimal abscissa and the maximal abscissa of s_i, respectively. (If s_i has five cells, then $c_i - b_i = 5$.) Let $\max_i$ denote the maximum element of the set $\{c_1, c_2, \ldots, c_i\}$.

The code of P is $f(P)$, a path that we define as follows. First, $f(P)$ has r weakly ascending nests and r descending nests. The first nest of $f(P)$ is a weakly ascending nest, namely $x^{c_1 - b_1}$. Let $i \in \{2, \ldots, r\}$. If $c_i < \max_{i-1}$ then the $(i-1)$st descending nest of $f(P)$ is $y^{c_i - b_{i-1}}$, and the ith weakly ascending nest of $f(P)$ is $x^{c_i - b_i}$. (It is impossible that $c_i - b_{i-1} \leq 0$ because, at the stage when all but i segments are

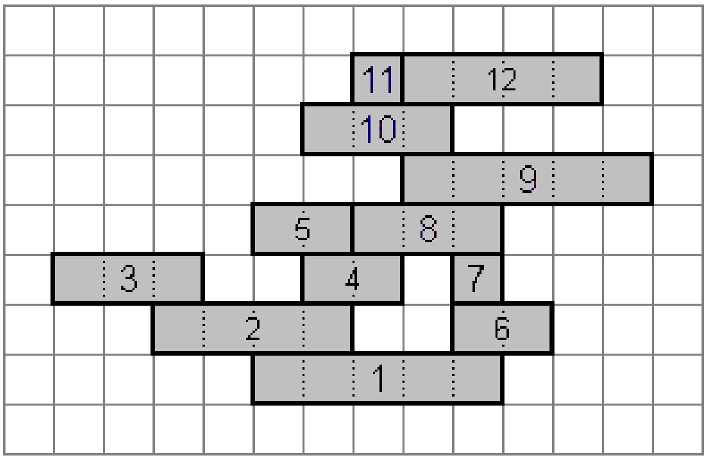

Figure 5: How the segments are numbered.

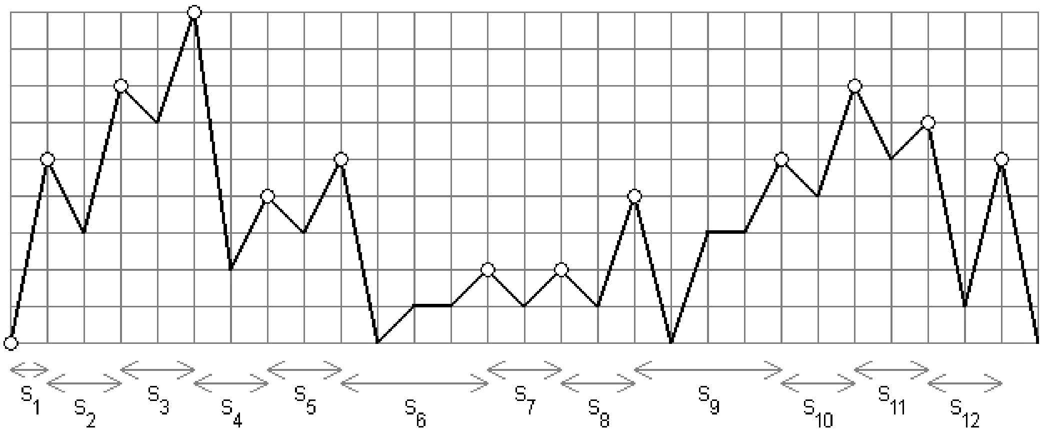

Figure 6: The path $f(P)$, where P is the pyramid of segments of Figure 5. The double arrow labelled s_i shows what section of $f(P)$ is generated by the ith segment of P.

deleted, the rightmost maximal segment is s_i and not s_{i-1}.) If $c_i \geq \max_{i-1}$ then the $(i-1)$st descending nest of $f(P)$ is $y^{\max_{i-1} - b_{i-1}}$, and the ith weakly ascending nest of $f(P)$ is $x^{\max_{i-1} - b_i} a x^{c_i - \max_{i-1}}$. (It is impossible that $\max_{i-1} - b_i \leq 0$ because, as long as s_i is present, the segments that support s_i from below are present too.) Finally, the rth descending nest of $f(P)$ is $y^{\max_r - b_r}$. See Figures 5 and 6 for an example.

The path $f(P)$ is a Motzkin path. This is not obvious from the definition of $f(P)$, but will be proved in the next theorem.

Theorem 3.1. *For $n \in \mathbb{N}$ and $r \in \{1, \ldots, n\}$, the mapping f is a bijection from the set $\mathcal{P}_{n,r}$ to the set $\mathcal{A}_{n,r}$.*

Proof. We shall prove by induction that, if P is an element of $\mathcal{P}_{n,r}$, then, for i from 1 to r, the ith weakly ascending nest (wan) of $f(P)$ ends at level $\max_i - b_i$.

Indeed, the definition of $f(P)$ specifies that the first wan of $f(P)$ ends at level $c_1 - b_1 = \max_1 - b_1$. Suppose that, for some $i \in \{2, \dots, r\}$, the $(i-1)$st wan of $f(P)$ ends at level $\max_{i-1} - b_{i-1}$. The induction step splits into two cases: $c_i < \max_{i-1}$ and $c_i \geq \max_{i-1}$. In the case $c_i < \max_{i-1}$, the $(i-1)$st descending nest (den) of $f(P)$ is $y^{c_i - b_{i-1}}$, and the ith wan of $f(P)$ is $x^{c_i - b_i}$. Hence the ith wan ends at level $\max_{i-1} - b_{i-1} - c_i + b_{i-1} + c_i - b_i = \max_{i-1} - b_i = \max_i - b_i$. In the case $c_i \geq \max_{i-1}$, the $(i-1)$st den of $f(P)$ is $y^{\max_{i-1} - b_{i-1}}$, and the ith wan of $f(P)$ is $x^{\max_{i-1} - b_i} a x^{c_i - \max_{i-1}}$. Once again, it follows that the ith wan ends at level $\max_{i-1} - b_{i-1} - \max_{i-1} + b_{i-1} + \max_{i-1} - b_i + c_i - \max_{i-1} = c_i - b_i = \max_i - b_i$. The induction step is now complete.

The $(i-1)$st den starts at the terminus of the $(i-1)$st wan. In the case $i \in \{2, \dots, r\}$ and $c_i < \max_{i-1}$, the $(i-1)$st den of $f(P)$ is $y^{c_i - b_{i-1}}$ and its final level is $\max_{i-1} - b_{i-1} - c_i + b_{i-1} = \max_{i-1} - c_i > 0$. In the case $i \in \{2, \dots, r\}$ and $c_i \geq \max_{i-1}$, the $(i-1)$st den of $f(P)$ is $y^{\max_{i-1} - b_{i-1}}$ and its final level is $\max_{i-1} - b_{i-1} - \max_{i-1} + b_{i-1} = 0$. Thus, every nonfinal den of $f(P)$ ends at a nonnegative level, and the rth (and last) den of $f(P)$ ends at level $\max_r - b_r - \max_r + b_r = 0$. This means that $f(P)$ is a Motzkin path.

Let $i \in \{2, \dots, r\}$. From the definition of $f(P)$, we see that the ith wan of $f(P)$ contains a horizontal step if and only if $c_i \geq \max_{i-1}$. But the inequality $c_i \geq \max_{i-1}$ holds if and only if the $(i-1)$st den of $f(P)$ ends at level zero. Also, the definition of $f(P)$ ensures that every wan of $f(P)$ starts with an upward step (and not with a horizontal step). Putting the pieces together, we conclude that $f(P)$ is an element of $\mathscr{A}$.

From the definition of $f(P)$ it is soon clear that, for i from 1 to r, the ith wan of $f(P)$ contains exactly $c_i - b_i$ upward steps. So the path $f(P)$ has a total of $\sum_{i=1}^{r}(c_i - b_i)$ upward steps. If the pyramid P has a total of n cells, then $\sum_{i=1}^{r}(c_i - b_i) = n$, and the path $f(P)$ has a total of n upward steps.

So far we have proved that f maps pyramids with n cells and r segments *into* the elements of $\mathscr{A}$ that have n upward steps and r weakly ascending nests. In other words, f is a function from the set $\mathscr{P}_{n,r}$ to the set $\mathscr{A}_{n,r}$. Now we claim that f is an injection. That is, if for some $w \in \mathscr{A}_{n,r}$ there exists a pyramid of segments $P \in \mathscr{P}_{n,r}$ such that $w = f(P)$, then P is uniquely determined. Indeed, the size of the first segment of P is the number of upward steps in the first nest of w. Suppose that, for some $i \in \{2, \dots, r\}$, the sizes and relative positions of the first $i-1$ segments of P have been reconstructed. In the case that the ith wan of w does not contain a horizontal step, in P we have $c_i < \max_{i-1}$, the $i-1$th den of w is $y^{c_i - b_{i-1}}$, and the ith wan of w is $x^{c_i - b_i}$. So c_i can be read off from the $i-1$th den of w, and b_i can then be read off from the ith wan of w. In the case that the ith wan of w contains a horizontal step, in P we have $c_i \geq \max_{i-1}$, the $i-1$th den of w is $y^{\max_{i-1} - b_{i-1}}$, and the ith wan of w is $x^{\max_{i-1} - b_i} a x^{c_i - \max_{i-1}}$. So both b_i and c_i can be read off from the ith wan of w. In both cases, once b_i and c_i are known, we drop a segment of width $c_i - b_i$ from the top of the strip $[b_i, c_i] \times \langle -\infty, \infty \rangle$. The dropped segment will eventually land on some support, the support being one or more of the first $i-1$ segments of P. When the landing is over, the sizes and relative positions of the first

i segments of P are determined uniquely.

The proof that f is a surjection is also not hard. To save space, we leave that proof to the reader. □

4 The generating function A computed and expanded in Taylor series

The coding f is useful because the paths of $\mathcal{A}$ are not hard to enumerate, and once those paths are enumerated, pyramids of segments are enumerated too. We shall enumerate the paths of $\mathcal{A}$ via the *wasp-waist decomposition* [7, subsubsection 3.4.3.1]. The wasp-waist decomposition will involve three additional families of paths. The definitions of those three families follow.

Let $\mathcal{B}$ be the set of all paths u which have the following properties:

1. u is a nonempty Motzkin path,
2. if w is a weakly ascending nest of u, then w begins with an upward step and contains at most one horizontal step,
3. if w is a weakly ascending nest of u, then w contains a horizontal step if and only if the factorization $u = vwz$ starts with a path v that is either empty or is nonempty and ends at level zero.

Remark. The families $\mathcal{A}$ and $\mathcal{B}$ are different because of the following detail. If $u \in \mathcal{A}$, the first wan of u does not contain a horizontal step, whereas if $u \in \mathcal{B}$, the first wan of u contains a horizontal step.

Let $\mathcal{C}$ be the set of all paths u which have the following properties:

1. u is a nonempty Motzkin path,
2. $|u|_a = 1$ and the only horizontal step of u lies in the first wan of u.

Remark. The first wan of $u \in \mathcal{C}$ does not have to begin with an upward step. The horizontal step may appear at any position within that wan, the first position not excluded.

Let $\mathcal{D}$ be the set of all Dyck paths.

We are now ready to write the wasp-waist decompositions. Those decompositions are

$$\mathcal{A} = x\mathcal{D}y(\varepsilon + \mathcal{B}) = xy(\varepsilon + \mathcal{B}) + x(\mathcal{D} - \varepsilon)y(\varepsilon + \mathcal{B}), \tag{3}$$

$$\mathcal{B} = x\mathcal{C}y(\varepsilon + \mathcal{B}), \tag{4}$$

$$\mathcal{C} = a\mathcal{D} + x\mathcal{C}y\mathcal{D} = a + a(\mathcal{D} - \varepsilon) + x\mathcal{C}y\mathcal{D}, \tag{5}$$

$$\mathcal{D} = \varepsilon + x\mathcal{D}y\mathcal{D} = \varepsilon + xy\mathcal{D} + x(\mathcal{D} - \varepsilon)y\mathcal{D}. \tag{6}$$

We are not going to write the proofs of (3)–(6) because these decompositions can be established by inspection. For example, $\mathscr{A} = x\mathscr{D}y(\varepsilon + \mathscr{B})$ means that a path $u \in \mathscr{A}$ can be written either as $u = xvy$, where $v \in \mathscr{D}$, or as $u = xwyz$, where $w \in \mathscr{D}$ and $z \in \mathscr{B}$.

It is time to define generating functions. That done, decompositions (3)–(6) will give us a system of four algebraic equations, where the generating functions appear as unknowns. Let

$$A = \textstyle\sum_{u\in\mathscr{A}} x^{|u|_x} a^{|u|_a} s^{wan(u)}, \quad B = \sum_{u\in\mathscr{B}} x^{|u|_x} a^{|u|_a} s^{wan(u)},$$

$$C = \textstyle\sum_{u\in C} x^{|u|_x} a s^{wan(u)}, \qquad D = \sum_{u\in\mathscr{D}} x^{|u|_x} s^{wan(u)}.$$

From (3)–(6) we quickly find that

$$A = sx(1+B) + x(D-1)(1+B), \tag{7}$$

$$B = xC(1+B), \tag{8}$$

$$C = as + a(D-1) + xCD, \tag{9}$$

$$D = 1 + sxD + x(D-1)D. \tag{10}$$

Theorem 4.1. *The generating function A is given by the formula*

$$A(x,s,a) = \frac{2sx}{(1-a)(1-x+sx) + (1+a)\sqrt{1-2x-2sx+(1-s)^2x^2}}. \tag{11}$$

Proof. From (9) it follows that

$$C = \frac{a(s+D-1)}{1-xD}. \tag{12}$$

We substitute (12) into (8) to find that

$$B = \frac{ax(s+D-1)}{1-xD-ax(s+D-1)}, \tag{13}$$

and then substitute (13) into (7) to obtain

$$A = x(s+D-1)\left[1 + \frac{ax(s+D-1)}{1-xD-ax(s+D-1)}\right]. \tag{14}$$

Of the two solutions of the quadratic (10), D is the one that does not involve negative powers of x. This means that

$$D = \frac{1+x-sx-\sqrt{1-2x-2sx+(1-s)^2x^2}}{2x}. \tag{15}$$

Plugging (15) into (14) produces a "protoformula" for A, which we rearrange until we get formula (11). $\square$

If we set $a = 1$, formula (11) reduces to

$$A(x, s, 1) = \frac{sx}{\sqrt{1 - 2x - 2sx + (1 - s)^2 x^2}}. \tag{16}$$

The case $s = 1$ of (16) is

$$A(x, 1, 1) = \frac{x}{\sqrt{1 - 4x}}.$$

The Taylor series expansions of $A(x, s, 1)$ and $A(x, 1, 1)$ are

$$A(x, s, 1) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r}^2 x^{n+1} s^{r+1} \tag{17}$$

and

$$A(x, 1, 1) = \sum_{n=0}^{\infty} \binom{2n}{n} x^{n+1}. \tag{18}$$

Corollary 4.1. *For $n \in \{0, 1, 2, \ldots\}$ and $r \in \{0, 1, \ldots, n\}$, there are $\binom{n}{r}^2$ pyramids of segments with $n+1$ cells and $r+1$ segments.*

Proof. From Theorem 3.1 we know that $|\mathscr{P}_{n+1,r+1}| = |\mathscr{A}_{n+1,r+1}|$, and from (17) we see that $|\mathscr{A}_{n+1,r+1}| = \binom{n}{r}^2$. □

Corollary 4.2. *For $n \in \{0, 1, 2, \ldots\}$, there are $\binom{2n}{n}$ pyramids of segments with $n+1$ cells.*

Proof. It is clear that $|\mathscr{P}_{n+1}| = \sum_{r=0}^{n} |\mathscr{P}_{n+1,r+1}|$, and from Theorem 3.1 we know that $\sum_{r=0}^{n} |\mathscr{P}_{n+1,r+1}| = \sum_{r=0}^{n} |\mathscr{A}_{n+1,r+1}|$. It is also clear that $\sum_{r=0}^{n} |\mathscr{A}_{n+1,r+1}| = |\mathscr{A}_{n+1}|$, and from (18) we see that $|\mathscr{A}_{n+1}| = \binom{2n}{n}$. □

In the next four sections, we are going to prove Corollaries 4.1 and 4.2 in a bijective way. We shall do the bijective proof of Corollary 4.2 first, because it is considerably simpler than the bijective proof of Corollary 4.1.

5 A bijective proof of the formula $|\mathscr{P}_{n+1}| = \binom{2n}{n}$

By Theorem 3.1, the mapping f is a bijection from pyramids of segments to the set $\mathscr{A}$; f maps pyramids of segments with $n+1$ cells onto the paths of $\mathscr{A}$ that have $n+1$ upward steps. In this section, we define a new bijection. This new bijection, say g, maps the set $\mathscr{A}$ to the set of all bilateral Dyck paths. The bijection g maps the paths of $\mathscr{A}$ that have $n+1$ upward steps onto bilateral Dyck paths with $2n$ steps. The composition $g \circ f$ is thus a bijection from pyramids of segments with $n+1$ cells to bilateral Dyck paths with $2n$ steps. The number of bilateral Dyck paths with $2n$ steps is obviously $\binom{2n}{n}$. Namely, there is a total of $2n$ steps, and any n of them can be upward steps. This means that, once we define the bijection g and prove that it has the desired properties, the bijective proof of Corollary 4.2 will be complete.

Let $t \in \mathcal{A}$. Suppose that t has p nonempty left factors that end at level zero. If $p = 1$, then t can be written as $t = xuy$, where $u \in \mathcal{D}$. If $p = 2$, then t can be written as $t = xuyxvy$, where $u \in \mathcal{D}$ and $v \in \mathcal{C}$. If $p = 3$, then t can be written as $t = xuyxvyxwy$, where $u \in \mathcal{D}$, $v \in \mathcal{C}$ and $w \in \mathcal{C}$. Altogether, every path of $\mathcal{A}$ can be decomposed into one factor of the form $x \cdot (\textit{an element of } \mathcal{D}) \cdot y$ and either zero or a finite number of factors of the form $x \cdot (\textit{an element of } \mathcal{C}) \cdot y$. By definition of $\mathcal{C}$, a path of the form $x \cdot (\textit{an element of } \mathcal{C}) \cdot y$ has only one horizontal step. The horizontal step lies in the path's first wan. If the horizontal step is preceded by i upward steps, the said path of the form $x \cdot (\textit{an element of } \mathcal{C}) \cdot y$ can be written as $x^i a v_i y v_{i-1} y \cdots v_1 y$, where v_i, $v_{i-1}, \ldots, v_1$ are elements of $\mathcal{D}$. Summing up, we see that every path $t \in \mathcal{A}$ can be written in one of these two ways:

Way 1: $t = xuy$, where $u \in \mathcal{D}$,

Way 2: $t = xuy \cdot (x^i a v_i y v_{i-1} y \cdots v_1 y) \cdot (x^j a w_j y w_{j-1} y \cdots w_1 y) \cdots (x^m a z_m y z_{m-1} y \cdots z_1 y)$, where $i, j, \ldots, m \in \mathbb{N}$ and $u, v_i, v_{i-1}, \ldots, v_1, w_j, w_{j-1}, \ldots, w_1, \ldots, z_m, z_{m-1}, \ldots, z_1 \in \mathcal{D}$.

Let $t \in \mathcal{A}$. If t can be written in Way 1, we set $g(t) = u$. If t can be written in Way 2, we set

$$g(t) = u \cdot (y\overline{v_i}x \cdot xv_{i-1}y \cdots xv_1y) \cdot (y\overline{w_j}x \cdot xw_{j-1}y \cdots xw_1y) \cdots (y\overline{z_m}x \cdot xz_{m-1}y \cdots xz_1y)\,.$$

Comments. The parentheses around $x^i a v_i y v_{i-1} y \cdots v_1 y$, $y\overline{v_i}x \cdot xv_{i-1}y \cdots xv_1y, \ldots$ have been written with the only purpose to make the reading easier. If $i = 1$, then $xv_{i-1}y \cdots xv_1y = \varepsilon$. For the meaning of $\overline{v_i}$, $\overline{w_j}$ and $\overline{z_m}$, see Section 2.

See Figure 7 for an example. In Figure 7, we made use of green and red colours: the wans of the path t are green, and the upward (respectively downward) even steps of the path $g(t)$ are green (respectively red). These colourings are useful because the wans of t and the upward even steps of $g(t)$ thus become easy to count.

Notation. The set of bilateral Dyck paths with $2n$ steps will be denoted by $\mathcal{Z}_n$. Incidentally, the letter $\mathcal{Z}$ is intended to suggest the word "zero": a bilateral Dyck path is only required to end at level zero (and not to make horizontal steps).

Theorem 5.1. *For $n \in \{0, 1, 2, \ldots\}$, the mapping g is a bijection from the set $\mathcal{A}_{n+1}$ to the set $\mathcal{Z}_n$.*

Proof. Let $t \in \mathcal{A}$. If t can be written in Way 1, then $g(t) = u$ is a Dyck path, and every Dyck path is a bilateral Dyck path. It is also clear that $g(t) = u$ has one upward step less than t. Hence, if $|t|_x = n + 1$, then $|g(t)|_x = n$, so that $g(t)$ has $2n$ steps in all.

Now suppose that t can be written in Way 2. Since $|\overline{v_i}|_x = |v_i|_y = |v_i|_x = |\overline{v_i}|_y$,

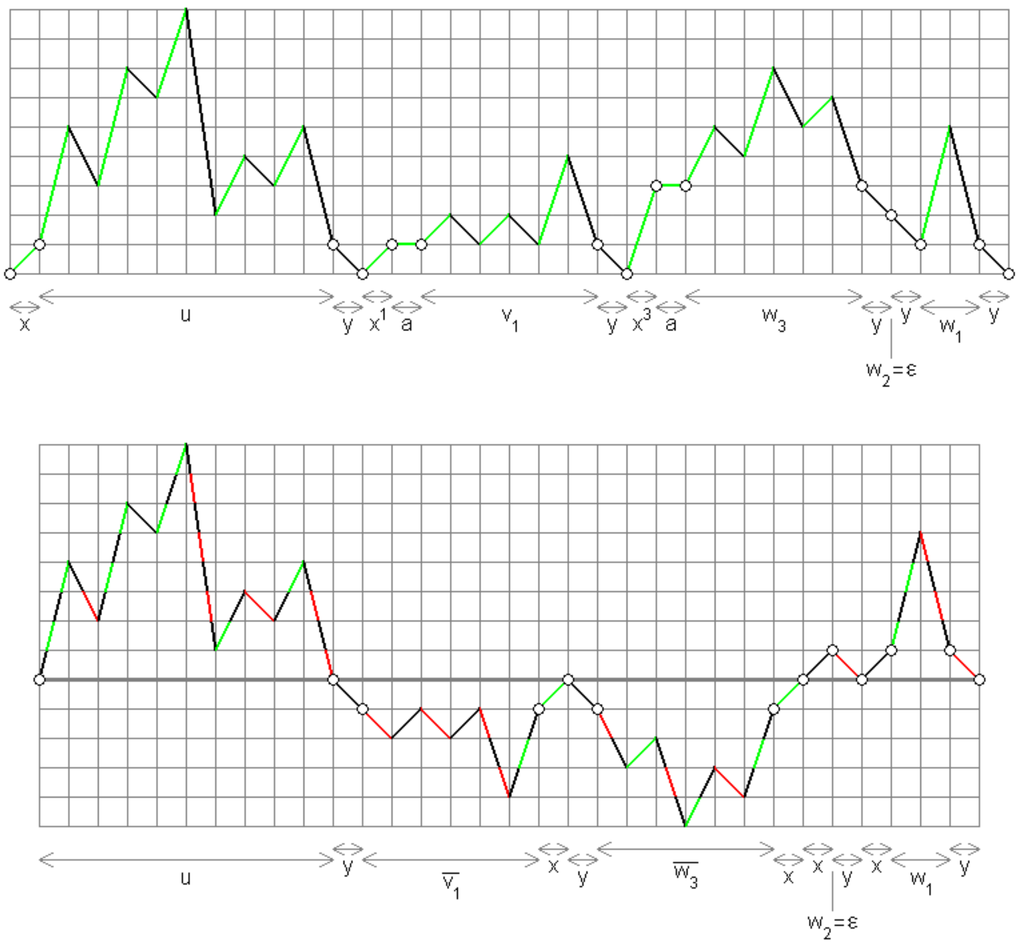

Figure 7: Top: A path $t \in \mathscr{A}$. (The path t can be seen in Figure 6 too.) Bottom: The path $g(t)$. The factorizations of t and $g(t)$ are also shown.

we have

$$\begin{aligned}
|y\overline{v_i}x \cdot xv_{i-1}y \cdots xv_1y|_x &= i + |\overline{v_i}|_x + |v_{i-1}|_x + \ldots + |v_1|_x \\
&= i + |v_i|_x + |v_{i-1}|_x + \ldots + |v_1|_x \\
&= |x^i av_i yv_{i-1}y \cdots v_1 y|_x \\
&= |x^i av_i yv_{i-1}y \cdots v_1 y|_y \\
&= |v_i|_y + |v_{i-1}|_y + \ldots + |v_1|_y + i \\
&= |\overline{v_i}|_y + |v_{i-1}|_y + \ldots + |v_1|_y + i \\
&= |y\overline{v_i}x \cdot xv_{i-1}y \cdots xv_1y|_y.
\end{aligned}$$

In the same way, one can show that

$$\begin{aligned}
|y\overline{w_j}x \cdot xw_{j-1}y \cdots xw_1y|_x &= |y\overline{w_j}x \cdot xw_{j-1}y \cdots xw_1y|_y, \ldots, \\
|y\overline{z_m}x \cdot xz_{m-1}y \cdots xz_1y|_x &= |y\overline{z_m}x \cdot xz_{m-1}y \cdots xz_1y|_y.
\end{aligned}$$

The path $g(t)$ is the product of the Dyck path u and of parenthesized paths, and each of the parenthesized paths is a bilateral Dyck path. Therefore $g(t)$ is a bilateral Dyck path too. From the above computation, we also see that $|y\overline{v_i}x \cdot xv_{i-1}y \cdots xv_1y|_x = |x^i av_iyv_{i-1}y \cdots v_1y|_x$. Similarly, we have

$$\begin{aligned}|y\overline{w_j}x \cdot xw_{j-1}y \cdots xw_1y|_x &= |x^j aw_jyw_{j-1}y \cdots w_1y|_x, \ldots,\\ |y\overline{z_m}x \cdot xz_{m-1}y \cdots xz_1y|_x &= |x^m az_myz_{m-1}y \cdots z_1y|_x.\end{aligned}$$

Hence the numbers $|t|_x$ and $|g(t)|_x$ differ only because the initial factor of t is xuy, and the initial factor of $g(t)$ is u alone. If $|t|_x = n+1$, then $|g(t)|_x = n$, which means that $g(t)$ has $2n$ steps in all.

To sum up, if $t \in \mathscr{A}$ and $|t|_x = n+1$, then, regardless of how t can be written (in Way 1 or Way 2), $g(t)$ is always a bilateral Dyck path with $2n$ steps.

So far we have proved that g is a function from $\mathscr{A}_{n+1}$ to $\mathscr{Z}_n$. Now we want to prove that g is a surjection. Let p be a bilateral Dyck path with $2n$ steps. If p is a Dyck path, then xpy is an element of $\mathscr{A}$, and we have $|xpy|_x = n+1$ as well as $g(xpy) = p$. Now suppose that p is not a Dyck path. Then p has one or more y-steps that start at level zero. Between any two such y-steps, as well as before the first and after the last such y-step, p may (but does not need to) have one or more x-steps that start at level zero. This means that p can be written as

$$p = u \cdot (y\overline{v_i}x \cdot xv_{i-1}y \cdots xv_1y) \cdot (y\overline{w_j}x \cdot xw_{j-1}y \cdots xw_1y) \cdots (y\overline{z_m}x \cdot xz_{m-1}y \cdots xz_1y),$$

where $i, j, \ldots, m$ are positive integers and $u, v_i, v_{i-1}, \ldots, v_1, w_j, w_{j-1}, \ldots, w_1, \ldots, z_m, z_{m-1}, \ldots, z_1$ are Dyck paths. Let

$$t = xuy \cdot \left(x^i av_iyv_{i-1}y \cdots v_1y\right) \cdot \left(x^j aw_jyw_{j-1}y \cdots w_1y\right) \cdots \left(x^m az_myz_{m-1}y \cdots z_1y\right).$$

It is not hard to see that t is an element of $\mathscr{A}$. In addition, we have $|t|_x = n+1$ and $g(t) = p$.

Thus, g is a surjection from $\mathscr{A}_{n+1}$ to $\mathscr{Z}_n$. The sets $\mathscr{A}_{n+1}$ and $\mathscr{Z}_n$ are equinumerous: each of them has $\binom{2n}{n}$ elements. (The cardinality of $\mathscr{A}_{n+1}$ was found in Corollary 4.2, and the cardinality of $\mathscr{Z}_n$ is obvious.) For this reason, in addition to being a surjection from $\mathscr{A}_{n+1}$ to $\mathscr{Z}_n$, g is also a bijection from $\mathscr{A}_{n+1}$ to $\mathscr{Z}_n$. □

Now that Theorem 5.1 is proved, we can easily do an extra enumeration. Let $P \in \mathscr{P}_{n+1}$, where $n \in \mathbb{N}$. From the definitions of f and g it is soon clear that, if the first segment of P has only one cell, then the path $(g \circ f)(P)$ begins with a downward step. If instead the first segment of P has two or more cells, then the path $(g \circ f)(P)$ begins with an upward step. So the restriction of $g \circ f$ to the set $\{P \in \mathscr{P}_{n+1} :$ the first segment of P has only one cell$\}$ is a bijection from that set to the set $\{z \in \mathscr{Z}_n :\ z$ begins with a downward step$\}$. Also, the restriction of $g \circ f$ to the set $\{P \in \mathscr{P}_{n+1} :$ the first segment of P has two or more cells$\}$ is a bijection from the latter set to the set $\{z \in \mathscr{Z}_n :\ z$ begins with an upward step$\}$. Each of the sets $\{z \in \mathscr{Z}_n :\ z$ begins with a downward step$\}$ and $\{z \in \mathscr{Z}_n :\ z$ begins with an upward step$\}$ obviously has $\binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}$ elements. Hence we have this corollary:

Corollary 5.1. *For $n \in \mathbb{N}$, the sets*

$$\{P \in \mathscr{P}_{n+1} : \textit{the first segment of } P \textit{ has only one cell}\}$$

and

$$\{P \in \mathscr{P}_{n+1} : \textit{the first segment of } P \textit{ has two or more cells}\}$$

have $\frac{1}{2}\binom{2n}{n}$ elements each.

6 Why is it hard to prove the formula $|\mathscr{P}_{n+1,r+1}| = \binom{n}{r}^2$ bijectively?

Notation. The set $\{p \in \mathscr{Z}_n : \ p \textit{ has } r \textit{ upward even steps}\}$ will be denoted by $\mathscr{Z}_{n,r}$.

By Theorem 3.1, the mapping f is a bijection from the set $\mathscr{P}_{n+1,r+1}$ to the set $\mathscr{A}_{n+1,r+1}$. Now we want to find a bijection (say h) from the set $\mathscr{A}_{n+1,r+1}$ to the set $\mathscr{Z}_{n,r}$. Once we find h, the bijective proof of Corollary 4.1 will be very nearly complete. Namely, the composition $h \circ f$ is a bijection from $\mathscr{P}_{n+1,r+1}$ to $\mathscr{Z}_{n,r}$, and this latter set is easily seen to have $\binom{n}{r}^2$ elements. Indeed, there are n even steps, and any r of them can be upward steps; there are n odd steps, and any $n-r$ of them can be upward steps. Altogether there are $\binom{n}{r} \cdot \binom{n}{n-r} = \binom{n}{r}^2$ choices.

However, the bijection h is not easy to define. We already have a bijection $g : \mathscr{A}_{n+1} \to \mathscr{Z}_n$, but g does not map every path with $r+1$ wans into a path with r upward even steps. For example, in Figure 7, the path t has 12 wans, whereas the path $g(t)$ has 16 upward even steps. Therefore h cannot be equal to g.

The formula for g involves a number of Dyck paths. To obtain a formula for h, one has to replace each of those Dyck paths by another, more suitable Dyck path. To be specific, we are going to replace each of those Dyck paths by its image under φ or, if appropriate, by its image under ψ. Here, φ and ψ are two bijections between Narayana-enumerated objects. Of course, to revise the formula for g in such a way, one has to know something about the Narayana numbers. Therefore, in Section 7 we shall study the Narayana numbers. The formula for h will then be stated in Section 8.

7 Bijections between three Narayana-enumerated objects

For $n \in \{1, 2, 3, \ldots\}$ and $k \in \{1, 2, \ldots, n\}$, the *Narayana number* $N(n,k)$ is given by the formula

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

The Narayana numbers occur in various combinatorial enumerations. For our purposes, it is most relevant that:

1. $N(n,k)$ is the number of Dyck paths with $2n$ steps and k peaks,

2. $N(n,k)$ is the number of Dyck paths with $2n$ steps, of which k are upward odd steps,
3. $N(n,k)$ is the number of Dyck paths with $2n$ steps, of which $k-1$ are upward even steps.

The Narayana numbers are named after a Canadian mathematician Tadepalli Venkata Narayana. Narayana's paper [17], published in 1959, contains a result that is very close to the first of the three facts listed above. The discoveries of the facts 2) and 3) cannot be dated quite accurately. Anyway, the references provided by Kreweras [14] indicate that 2) and 3) were found in the early 1980s.

For $n \in \{1, 2, 3, \ldots\}$ and $k \in \{1, 2, \ldots, n\}$, let

$$\begin{aligned} \hat{\mathscr{P}}_{n,k} &= \{u \in \mathscr{D} :\ u \text{ has } 2n \text{ steps and } k \text{ peaks}\}, \\ \mathscr{O}_{n,k} &= \{u \in \mathscr{D} :\ u \text{ has } 2n \text{ steps, } k \text{ of them being upward odd steps}\}, \\ \mathscr{E}_{n,k-1} &= \{u \in \mathscr{D} :\ u \text{ has } 2n \text{ steps, } k-1 \text{ of them being upward even steps}\}. \end{aligned}$$

Remark. In Section 3 we defined a set $\mathscr{P}_{n,r}$ and here we defined a set $\hat{\mathscr{P}}_{n,k}$. The sets $\mathscr{P}_{n,r}$ and $\hat{\mathscr{P}}_{n,k}$ are not as closely related as their names might suggest: $\mathscr{P}_{n,r}$ is a set of pyramids of segments, whereas $\hat{\mathscr{P}}_{n,k}$ is a set of Dyck paths.

As stated above, each of the sets $\hat{\mathscr{P}}_{n,k}$, $\mathscr{O}_{n,k}$ and $\mathscr{E}_{n,k-1}$ has $N(n,k)$ elements. Hence each two of these sets can be placed in a one-to-one correspondence. However, of the one-to-one correspondences $\hat{\mathscr{P}}_{n,k} \leftrightarrow \mathscr{O}_{n,k}$, $\hat{\mathscr{P}}_{n,k} \leftrightarrow \mathscr{E}_{n,k-1}$ and $\mathscr{O}_{n,k} \leftrightarrow \mathscr{E}_{n,k-1}$, only the third one is easy to see. Therefore we start with the third one-to-one correspondence.

Let $\tau(\varepsilon) = \varepsilon$ and $\tau(xy) = xy$. If u is a Dyck path with $n \geq 2$ upward steps, we define $\tau(u)$ to be the path obtained by swapping the $2i$th and $(2i+1)$st steps of u, for all i from 1 to $n-1$. With a_i denoting the ith step of u, we have $u = a_1a_2a_3 \ldots a_{2n-2}a_{2n-1}a_{2n}$ and $\tau(u) = a_1a_3a_2 \ldots a_{2n-1}a_{2n-2}a_{2n}$. For example, if $u = x\,\underbrace{xy}\,\overbrace{xx}\,\underbrace{yy}\,\overbrace{xy}\,\underbrace{yx}\,y$, then $\tau(u) = x\,\underbrace{yx}\,\overbrace{xx}\,\underbrace{yy}\,\overbrace{yx}\,\underbrace{xy}\,y$.

The mapping τ is both a bijection from $\mathscr{O}_{n,k}$ to $\mathscr{E}_{n,k-1}$ and a bijection from $\mathscr{E}_{n,k-1}$ to $\mathscr{O}_{n,k}$. In the next proposition, we shall prove the first part of this claim.

Proposition 7.1. *For $n \in \{1, 2, 3, \ldots\}$ and $k \in \{1, 2, \ldots, n\}$, the mapping τ is a bijection from $\mathscr{O}_{n,k}$ to $\mathscr{E}_{n,k-1}$.*

Proof. We have $\mathscr{O}_{1,1} = \{xy\}$, $\mathscr{E}_{1,0} = \{xy\}$ and $\tau(xy) = xy$. So it is clear that τ is a bijection from $\mathscr{O}_{1,1}$ to $\mathscr{E}_{1,0}$. Now let $n \geq 2$ and $k \in \{1, 2, \ldots, n\}$. For $u = a_1 \ldots a_{2n} \in \mathscr{O}_{n,k}$ and $i \in \{1, 2, \ldots, n\}$, the first $2i-1$ steps of $\tau(u)$ form a path (say v_i) with strictly more upward steps than downward steps. Indeed, since u is a Dyck path, we have

$$\begin{aligned} |v_i|_x - |v_i|_y &= |a_1a_3a_2 \ldots a_{2i-1}a_{2i-2}|_x - |a_1a_3a_2 \ldots a_{2i-1}a_{2i-2}|_y \\ &= |a_1a_2a_3 \ldots a_{2i-2}a_{2i-1}|_x - |a_1a_2a_3 \ldots a_{2i-2}a_{2i-1}|_y > 0. \end{aligned}$$

Hence, the first $2i$ steps of $\tau(u)$ form a path (say w_i) such that $|w_i|_x - |w_i|_y \geq 0$. In addition, we have

$$\begin{aligned} |\tau(u)|_x - |\tau(u)|_y &= |a_1a_3a_2 \dots a_{2n-1}a_{2n-2}a_{2n}|_x - |a_1a_3a_2 \dots a_{2n-1}a_{2n-2}a_{2n}|_y \\ &= |a_1a_2a_3 \dots a_{2n-2}a_{2n-1}a_{2n}|_x - |a_1a_2a_3 \dots a_{2n-2}a_{2n-1}a_{2n}|_y \\ &= |u|_x - |u|_y \;=\; 0. \end{aligned}$$

Altogether, this means that $\tau(u)$ is a Dyck path. How does $ue(\tau(u))$ relate to $uo(u)$? (Recall from Section 2 that $ue(\tau(u))$ stands for the number of upward even steps of $\tau(u)$, while $uo(u)$ stands for the number of upward odd steps of u.) Having in mind that $a_1 = x$ and $a_{2n} = y$, we obtain

$$\begin{aligned} ue(\tau(u)) &= ue(a_1a_3a_2 \dots a_{2n-1}a_{2n-2}a_{2n}) \;=\; ue(a_1a_3a_2 \dots a_{2n-1}a_{2n-2}) \\ &= |a_3a_5 \dots a_{2n-1}|_x \;=\; |a_1a_3a_5 \dots a_{2n-1}|_x - 1 \\ &= uo(a_1a_2a_3 \dots a_{2n-2}a_{2n-1}a_{2n}) - 1 \;=\; uo(u) - 1 \;=\; k - 1. \end{aligned}$$

Thus, $\tau(u)$ is a Dyck path with $2n$ steps, of which $k-1$ are upward even steps. This means that $\tau(u)$ is an element of $\mathcal{E}_{n,k-1}$.

Now that we know that τ is a function from $\mathcal{O}_{n,k}$ to $\mathcal{E}_{n,k-1}$, we can prove at once that τ is a bijection. Let $v = b_1b_2b_3 \dots b_{2n-2}b_{2n-1}b_{2n}$ be an element of $\mathcal{E}_{n,k-1}$. If there exists a path $u \in \mathcal{O}_{n,k}$ such that $v = \tau(u)$, then u cannot be anything but $b_1b_3b_2 \dots b_{2n-1}b_{2n-2}b_{2n}$. Thus, τ is an injection. Since $\mathcal{O}_{n,k}$ and $\mathcal{E}_{n,k-1}$ are finite sets of equal cardinality, any injection from $\mathcal{O}_{n,k}$ to $\mathcal{E}_{n,k-1}$ is also a bijection. So τ is also a bijection. This completes the proof. $\square$

Let us move on to another one-to-one correspondence. An obvious bijection from $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{O}_{n,k}$ is out of the question, but an elegant bijection still exists. The definition follows.

For $n \in \{1, 2, 3, \dots\}$ and $k \in \{1, 2, \dots, n\}$, let w be an element of $\hat{\mathcal{P}}_{n,k}$. Since w has k peaks, there exist positive integers $a_1, \dots, a_k$ and $b_1, \dots, b_k$ such that $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2} \cdots x^{a_k}y^{b_k}$. Let $c = xy^{b_1-1}xy^{b_2-1} \cdots xy^{b_k-1}$ and $d = x^{a_1-1}yx^{a_2-1}y \cdots x^{a_k-1}y$. Each of the paths c and d has n steps. Indeed, since $c = \prod_{i=1}^{k} xy^{b_i-1}$, the total number of steps of c is $\sum_{i=1}^{k} b_i = |w|_y = n$. Since $d = \prod_{i=1}^{k} x^{a_i-1}y$, the total number of steps of d is $\sum_{i=1}^{k} a_i = |w|_x = n$. Let c_j (respectively d_j) stand for the jth step of c (respectively d). We define φ to be the function that maps the path w to the path $\varphi(w) = c_1d_1c_2d_2 \cdots c_nd_n$. We also let $\varphi(\varepsilon) = \varepsilon$.

Example 7.1. Let $w = x^3yxy^2xy^2$. The descending nests of w are y, y^2 and y^2. Hence $c = x \cdot xy \cdot xy = xxyxy$. The ascending nests of w are x^3, x and x. Thus $d = x^2y \cdot y \cdot y = xxyyy$. By alternating the steps of c with those of d, we obtain $\varphi(w) = (xx)(xx)(yy)(xy)(yy) = x^4y^2xy^3$. $\square$

Proposition 7.2. *For $n \in \{1, 2, 3, \dots\}$ and $k \in \{1, 2, \dots, n\}$, the mapping φ is a bijection from $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{O}_{n,k}$.*

Proof. Let $w \in \hat{\mathscr{P}}_{n,k}$. Suppose that w can be written as $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_k}y^{b_k}$. Let z be a left factor of $\varphi(w)$ having an odd number of steps. If z has $2j-1$ steps, then $z = c_1d_1\cdots c_{j-1}d_{j-1}c_j$. If $|c_1\cdots c_{j-1}c_j|_x = i$, then the path $c_1\cdots c_{j-1}c_j$ ends at latest at the end of the ith factor of $c = (xy^{b_1-1})(xy^{b_2-1})\cdots(xy^{b_k-1})$. This means that $j \le b_1+b_2+\ldots+b_i$. Since $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_k}y^{b_k}$ is a Dyck path, we have $b_1+b_2+\ldots+b_i \le a_1+a_2+\ldots+a_i$. Consequently, $j \le a_1+a_2+\ldots+a_i$ and $j-1 < a_1+a_2+\ldots+a_i$. Because of this latter inequality, the path $d_1\cdots d_{j-1}$ ends strictly before the end of the ith factor of $d = (x^{a_1-1}y)(x^{a_2-1}y)\cdots(x^{a_k-1}y)$. Hence $|d_1\cdots d_{j-1}|_y \le i-1$, and this implies that $|d_1\cdots d_{j-1}|_x \ge (j-1)-(i-1) = j-i$. Altogether,

$$\begin{aligned} |z|_x &= |c_1d_1\cdots c_{j-1}d_{j-1}c_j|_x \\ &= |c_1\cdots c_{j-1}c_j|_x + |d_1\cdots d_{j-1}|_x \ge i+(j-i) \\ &= j. \end{aligned}$$

Since z has a total of $2j-1$ steps, $|z|_x \ge j$ implies that z ends at a positive level. Now let $\tilde{z}$ be a left factor of $\varphi(w)$ having an even number of steps. If we delete the last step of $\tilde{z}$, the remaining path (say z) is again a left factor of $\varphi(w)$. Since z has an odd number of steps, the end level of z is at least one. Therefore, whether $\tilde{z} = zx$ or $\tilde{z} = zy$, the end level of $\tilde{z}$ is at least zero. Thus every left factor of $\varphi(w)$ ends at a nonnegative level. In addition, we have

$$\begin{aligned} |\varphi(w)|_x &= |c_1c_2\cdots c_n|_x + |d_1d_2\cdots d_n|_x = |c|_x + |d|_x = |c|_x + n - |d|_y \\ &= \left|\prod_{i=1}^{k} xy^{b_i-1}\right|_x + n - \left|\prod_{i=1}^{k} x^{a_i-1}y\right|_y = k+n-k = n. \end{aligned}$$

Having n upward steps and n downward steps, the path $\varphi(w)$ ends at level $n-n = 0$. In summary, $\varphi(w)$ is a Dyck path with $2n$ steps. How many upward odd steps does $\varphi(w)$ have? The answer is

$$uo(\varphi(w)) = |c_1c_2\cdots c_n|_x = |c|_x = \left|\prod_{i=1}^{k} xy^{b_i-1}\right|_x = k.$$

Being a Dyck path with $2n$ steps, of which k are upward odd steps, $\varphi(w)$ is an element of $\mathscr{O}_{n,k}$.

Now that we know that φ is a function from $\hat{\mathscr{P}}_{n,k}$ to $\mathscr{O}_{n,k}$, we can readily prove that φ is a bijection. Let s be an element of $\mathscr{O}_{n,k}$. Suppose that there exists a path $w \in \hat{\mathscr{P}}_{n,k}$ such that $s = \varphi(w)$. Let c (respectively d) denote the path formed by the odd (respectively even) steps of s. Inspecting the path c (respectively d), one gets to know the number and sizes of the descending (respectively ascending) nests of w. Once these numbers and sizes are all known, the path w is determined uniquely. Hence, φ is an injection. Since $\hat{\mathscr{P}}_{n,k}$ and $\mathscr{O}_{n,k}$ are finite sets of equal cardinality, any injection from $\hat{\mathscr{P}}_{n,k}$ to $\mathscr{O}_{n,k}$ is also a bijection. So φ is also a bijection. This completes the proof. $\square$

From Propositions 7.1 and 7.2 we obtain the following corollary.

Corollary 7.1. *Let $\psi = \tau \circ \varphi$. For $n \in \{1, 2, 3, \ldots\}$ and $k \in \{1, 2, \ldots, n\}$, the composition ψ is a bijection from $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{E}_{n,k-1}$.*

Comments. Our bijection φ is new up to a certain measure. Namely, φ has strong ties with some known bijections. Let $\mathcal{Pap}_{n,k}$ stand for the set of parallelogram polyominoes with perimeter $2n$ and k rows. Sulanke [19] observes that a bijective proof that $|\mathcal{E}_{n,k-1}| = |\hat{\mathcal{P}}_{n,k}|$ can be obtained by combining two results that are already known. One of those results is a bijection, essentially due to Narayana [17], that maps $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{Pap}_{n+1,k}$. The other result is a bijection, due to Delest and Viennot [11], that maps $\mathcal{E}_{n,k-1}$ to $\mathcal{Pap}_{n+1,k}$. Composing the former bijection with the inverse of the latter bijection, one obtains a bijection (let us call it α) from $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{E}_{n,k-1}$. Our bijection $\psi = \tau \circ \varphi$ also maps $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{E}_{n,k-1}$. Moreover, for every $u \in \hat{\mathcal{P}}_{n,k}$, we have $\psi(u) = \alpha(u)$. The mathematical definition of functions now says that $\psi = \alpha$. Still, the bijection ψ goes directly from Dyck paths to Dyck paths, whereas α first goes from Dyck paths to parallelogram polyominoes, and then returns from parallelogram polyominoes to Dyck paths. Therefore I think that ψ is a simpler bijection than α.

Osborn [18] gives a bijection (let us call it β) from $\mathcal{E}_{n,k-1}$ to $\hat{\mathcal{P}}_{n,k}$. The bijection β is defined graphically: given a path $u \in \mathcal{E}_{n,k-1}$, we draw the lines $y = x - \ell$ and $y = -x + m$ ($\ell, m \in \mathbb{N} \cup \{0\}$), and then write checkmarks at certain intersections of $y = x$ and $y = -x$ with the other lines. Then we draw the path $\beta(u)$. By contrast, to find the path $\psi^{-1}(u) = (\varphi^{-1} \circ \tau)(u)$, we just write sequences of x's and y's, without drawing anything. So the first impression is that ψ^{-1} and β are two very different bijections. However, when I recast the definition of β in non-graphical terms, it turned out that there is no difference at all: we have $\psi^{-1} = \beta$! I leave it open for discussion whether it is more insightful to draw a path physically, or to produce a sequence of x's and y's from another such sequence.

Mortimer and Prellberg [16] give a bijection (let us call it γ) from $\hat{\mathcal{P}}_{n,k}$ to $\mathcal{O}_{n,k}$. Our bijection φ is different from γ. For example, for $u = x^3yxy^2xy^2$, we have $\varphi(u) = x^4y^2xy^3$ (recall Example 7.1) whereas $\gamma(u) = x^5y^5$. In addition, the bijection γ is defined recursively.

It should be mentioned that the papers [18] and [16] are not only concerned with the "ordinary" Dyck paths. Along with β (respectively γ), the paper [18] (respectively [16]) presents an extension of that bijection, where the domain and codomain are sets of bilateral Dyck paths.

8 A bijective proof of the formula $|\mathcal{P}_{n+1,r+1}| = \binom{n}{r}^2$

We are now ready to define a bijection from $\mathcal{A}_{n+1,r+1}$ to $\mathcal{Z}_{n,r}$. Recall from Section 5 that the set $\mathcal{A}_{n+1,r+1}$ has two kinds of elements: those that can be written in Way 1, and those that can be written in Way 2. Let $t \in \mathcal{A}_{n+1,r+1}$. If t can be written in

Way 1, we set $h(t) = \psi(u)$. If t can be written in Way 2, we set

$$h(t) = \psi(u) \cdot \left[y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y\right]$$

$$\cdot \left[y\overline{\psi(w_j)}x \cdot x\varphi(w_{j-1})y \cdots x\varphi(w_1)y\right] \cdots \left[y\overline{\psi(z_m)}x \cdot x\varphi(z_{m-1})y \cdots x\varphi(z_1)y\right].$$

Here is an example.

Example 8.1. Figure 7 shows a path $t \in \mathscr{A}$, together with its factorization. The factorization being $t = xuy \cdot (xav_1y) \cdot (x^3aw_3yw_2yw_1y)$, the path $h(t)$ has the form

$$h(t) = \psi(u) \cdot \left[y\overline{\psi(v_1)}x\right] \cdot \left[y\overline{\psi(w_3)}x \cdot x\varphi(w_2)y \cdot x\varphi(w_1)y\right].$$

The only way to make this expression concrete is to compute the paths $\psi(u)$, $\overline{\psi(v_1)}$, $\overline{\psi(w_3)}$, $\varphi(w_2)$ and $\varphi(w_1)$.

Let us begin with the path $\psi(u)$. Since $\psi(u) = \tau(\varphi(u))$, the first thing to do is to compute $\varphi(u)$. By definition, $\varphi(u)$ is a combination of two paths, called c and d. To obtain c, in each of the descending nests of u, we replace the first step by an upward step. Then we multiply the resulting paths, and the product is c. Since the descending nests of u are y^2, y, y^7, y and y^4, the paths to be multiplied are xy, x, xy^6, x and xy^3. So $c = xy \cdot x \cdot xy^6 \cdot x \cdot xy^3 = xyx^2y^6x^2y^3$. To obtain d, in each of the ascending nests of u, we replace the last step by a downward step. Then we multiply the resulting paths, and the product is d. Since the ascending nests of u are x^4, x^4, x^3, x^2 and x^2, the paths to be multiplied are x^3y, x^3y, x^2y, xy and xy. Thus $d = x^3yx^3yx^2yxyxy$. With c_i denoting the ith step of c and d_i denoting the ith step of d, we have

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
c_i	x	y	x	x	y	y	y	y	y	y	x	x	y	y	y
d_i	x	x	x	y	x	x	x	y	x	x	y	x	y	x	y

By definition, $\varphi(u)$ is the product $c_1d_1c_2d_2\cdots c_{15}d_{15}$. From the table above, we see that

$$\begin{aligned}\varphi(u) &= (xx)_1(yx)_2(xx)_3(xy)_4(yx)_5(yx)_6(yx)_7 \\ &\quad \cdot(yy)_8(yx)_9(yx)_{10}(xy)_{11}(xx)_{12}(yy)_{13}(yx)_{14}(yy)_{15} \\ &= x^2yx^4y^2xyxyxy^3xyx^2yx^2y^3xy^2.\end{aligned}$$

(The indexed parentheses just help us to associate each step pair with the corresponding column of the table.)

To obtain the path $\psi(u) = \tau(\varphi(u))$, we need to swap the 2nd and 3rd steps of $\varphi(u)$, the 4th and 5th steps of $\varphi(u)$,..., the 28th and 29th steps of $\varphi(u)$. With helpful parentheses adjusted, we have

$$\varphi(u) = x(xy)_{1\to}(xx)_{2\to}(xx)_{3\to}(yy)_{4\to}(xy)_{5\to}(xy)_{6\to}(xy)_{7\to}$$

$$\cdot(yy)_{8\to}(xy)_{9\to}(xx)_{10\to}(yx)_{11\to}(xy)_{12\to}(yy)_{13\to}(xy)_{14\to}y,$$

whence we easily find that

$$\begin{aligned}\psi(u) &= x(yx)_{1\to}(xx)_{2\to}(xx)_{3\to}(yy)_{4\to}(yx)_{5\to}(yx)_{6\to}(yx)_{7\to} \\ &\quad \cdot(yy)_{8\to}(yx)_{9\to}(xx)_{10\to}(xy)_{11\to}(yx)_{12\to}(yy)_{13\to}(yx)_{14\to}y \\ &= xyx^5y^3xyxyxy^3x^4y^2xy^3xy.\end{aligned}$$

We proceed to the path $\overline{\psi(v_1)}$. The descending nests of v_1 are y, y and y^3. So this time we have $c = x \cdot x \cdot xy^2 = x^3y^2$. The ascending nests of v_1 are x, x and x^3. Thus $d = y \cdot y \cdot x^2y = y^2x^2y$. By alternating the steps of c with those of d, we get $\varphi(v_1) = (xy)(xy)(xx)(yx)(yy) = x(yx)(yx)(xy)(xy)y$. Therefore $\psi(v_1) = \tau(\varphi(v_1)) = x(xy)(xy)(yx)(yx)y = x^2yxy^2xyxy$, whence $\overline{\psi(v_1)} = y^2xyx^2yxyx$.

The next path to find is $\overline{\psi(w_3)}$. The descending nests of w_3 are y, y^2 and y^3. So we have $c = x \cdot xy \cdot xy^2 = x^2yxy^2$. The ascending nests of w_3 are x^2, x^3 and x. Thus $d = xy \cdot x^2y \cdot y = xyx^2y^2$. By alternating the steps of c with those of d, we obtain $\varphi(w_3) = (xx)(xy)(yx)(xx)(yy)(yy) = x(xx)(yy)(xx)(xy)(yy)y$. Therefore $\psi(w_3) = \tau(\varphi(w_3)) = x(xx)(yy)(xx)(yx)(yy)y = x^3y^2x^2yxy^3$, whence $\overline{\psi(w_3)} = y^3x^2y^2xyx^3$.

The remaining two paths, $\varphi(w_2)$ and $\varphi(w_1)$, are both easy to find. Since $w_2 = \varepsilon$, we have $\varphi(w_2) = \varepsilon$. Since $w_1 = x^4y^4$, we find at once that $c = xy^3$ and $d = x^3y$. Hence $\varphi(w_1) = (xx)(yx)(yx)(yy) = x^2yxyxy^2$.

Putting the pieces together, we find that $h(t)$ is the path shown in Figure 8.

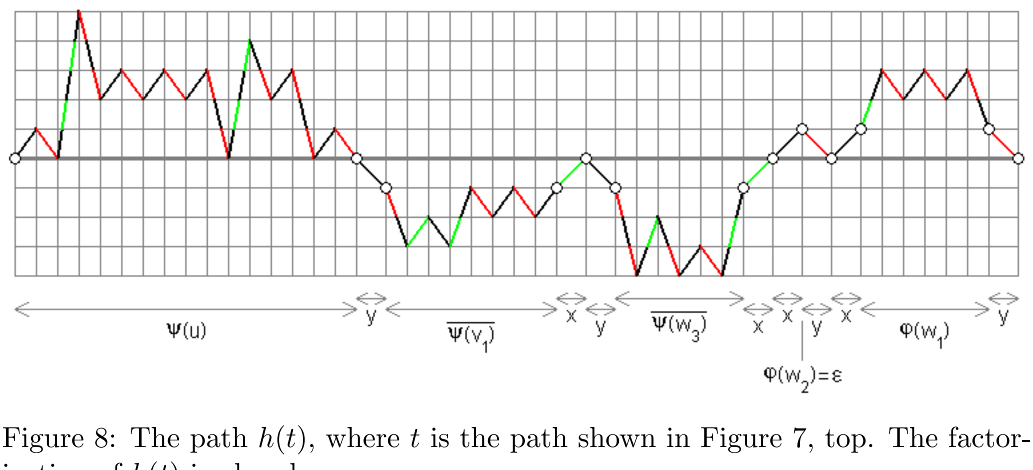

Figure 8: The path $h(t)$, where t is the path shown in Figure 7, top. The factorization of $h(t)$ is also shown.

Figure 8 is coloured in the same way as Figure 7, bottom: the upward even steps of $h(t)$ are green and the downward even steps of $h(t)$ are red. Thanks to the colourings, it is easy to see that the path t has 12 weakly ascending nests, while the path $h(t)$ has 11 upward even steps. With some more effort, it can also be seen that t has 35 upward steps, while $h(t)$ has a total of 68 steps (of which 34 are upward and 34 are downward). Thus, t is an element of $\mathscr{A}_{35,12}$ and $h(t)$ is an element of $\mathscr{Z}_{34,11}$. $\square$

In Example 8.1, the mapping h has mapped an element of $\mathscr{A}_{35,12}$ to an element of $\mathscr{Z}_{34,11}$. That was not a mere coincidence, but an instance of a general rule, stated in the following theorem.

Theorem 8.1. *For $n \in \{0, 1, 2, \ldots\}$ and $r \in \{0, 1, \ldots, n\}$, the mapping h is a bijection from $\mathscr{A}_{n+1,r+1}$ to $\mathscr{Z}_{n,r}$.*

In the proof of Theorem 8.1, one part is new, and the other part is an adaptation—or upgrade—of the proof of Theorem 5.1. The new part of the proof amounts to showing that h maps paths with $r+1$ wans into paths with r upward even steps. Here we shall write only the new part of the proof.

Proof of Theorem 8.1. Let $t \in \mathscr{A}_{n+1,r+1}$. Suppose that t can be written in Way 1. That is, $t = xuy$, where $u \in \mathscr{D}$. If $u \neq \varepsilon$, then $ue(h(t)) = ue(\psi(u)) = wan(u) - 1 = wan(xuy) - 1 = wan(t) - 1$. If $u = \varepsilon$, then again $ue(h(t)) = ue(\psi(\varepsilon)) = ue(\varepsilon) = 0 = wan(xy) - 1 = wan(t) - 1$. So in both cases, $wan(t) = r + 1$ implies that $ue(h(t)) = r$.

Now suppose that t can be written in Way 2. First we note that

$$\begin{aligned} ue(h(t)) &= ue(\psi(u)) + ue\left(y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y\right) \\ &\quad + ue\left(y\overline{\psi(w_j)}x \cdot x\varphi(w_{j-1})y \cdots x\varphi(w_1)y\right) + \ldots \\ &\quad + ue\left(y\overline{\psi(z_m)}x \cdot x\varphi(z_{m-1})y \cdots x\varphi(z_1)y\right). \end{aligned}$$

Much as before, we have $ue(\psi(u)) = wan(xuy) - 1$. Indeed, if $u \neq \varepsilon$, then $ue(\psi(u)) = wan(u) - 1 = wan(xuy) - 1$. If $u = \varepsilon$, then again $ue(\psi(u)) = ue(\varepsilon) = 0 = 1 - 1 = wan(xy) - 1 = wan(xuy) - 1$.

Suppose that $v_i \neq \varepsilon$. The path $\psi(v_i)$ has $ue(\psi(v_i))$ upward even steps. Hence it has $\frac{|\psi(v_i)|}{2} - ue(\psi(v_i))$ upward odd steps and $\frac{|\psi(v_i)|}{2} - \left[\frac{|\psi(v_i)|}{2} - ue(\psi(v_i))\right] = ue(\psi(v_i))$ downward odd steps. That is, $ue(\psi(v_i))$ is both the number of upward even steps of $\psi(v_i)$ and the number of downward odd steps of $\psi(v_i)$. The contribution of $\overline{\psi(v_i)}$ to

$$ue\left(y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y\right)$$

is the number of upward odd steps of $\overline{\psi(v_i)}$, which is equal to the number of downward odd steps of $\psi(v_i)$, which is in its turn equal to the number of upward even steps of $\psi(v_i)$. Therefore,

$$\begin{aligned} ue\left(y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y\right) &= ue(\psi(v_i)) + 1 + uo(\varphi(v_{i-1})) + \cdots + uo(\varphi(v_1)) \\ &= wan(v_i) - 1 + 1 + wan(v_{i-1}) + \cdots + wan(v_1) \\ &= wan(v_i) + wan(v_{i-1}) + \cdots + wan(v_1) \\ &= wan(x^i av_i yv_{i-1}y \cdots v_1 y). \end{aligned}$$

If $v_i = \varepsilon$, then $\psi(v_i) = \varepsilon$ and $\overline{\psi(v_i)} = \varepsilon$. Hence,

$$\begin{aligned} ue\left(y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y\right) &= ue(yx \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y) \\ &= 1 + uo(\varphi(v_{i-1})) + \ldots + uo(\varphi(v_1)) \\ &= 1 + wan(v_{i-1}) + \ldots + wan(v_1) \\ &= wan(x^i ayv_{i-1}y \cdots v_1y) \\ &= wan(x^i av_iyv_{i-1}y \cdots v_1y). \end{aligned}$$

Thus, in either of the cases $v_i \neq \varepsilon$ and $v_i = \varepsilon$, the conclusion is that

$$ue\left(y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y\right) = wan(x^i av_iyv_{i-1}y \cdots v_1y).$$

Similarly, we have

$$\begin{aligned} ue\left(y\overline{\psi(w_j)}x \cdot x\varphi(w_{j-1})y \cdots x\varphi(w_1)y\right) &= wan(x^j aw_jyw_{j-1}y \cdots w_1y), \ldots, \\ ue\left(y\overline{\psi(z_m)}x \cdot x\varphi(z_{m-1})y \cdots x\varphi(z_1)y\right) &= wan(x^m az_myz_{m-1}y \cdots z_1y). \end{aligned}$$

The above findings add up to

$$\begin{aligned} ue(h(t)) &= ue(\psi(u)) + ue(y\overline{\psi(v_i)}x \cdot x\varphi(v_{i-1})y \cdots x\varphi(v_1)y) \\ &\quad + ue\left(y\overline{\psi(w_j)}x \cdot x\varphi(w_{j-1})y \cdots x\varphi(w_1)y\right) + \ldots \\ &\quad + ue\left(y\overline{\psi(z_m)}x \cdot x\varphi(z_{m-1})y \cdots x\varphi(z_1)y\right) \\ &= wan(xuy) - 1 + wan(x^i av_iyv_{i-1}y \cdots v_1y) \\ &\quad + wan(x^j aw_jyw_{j-1}y \cdots w_1y) + \ldots + wan(x^m az_myz_{m-1}y \cdots z_1y) \\ &= wan\left(xuy \cdot (x^i av_iyv_{i-1}y \cdots v_1y)\right. \\ &\quad \left. \cdot (x^j aw_jyw_{j-1}y \cdots w_1y) \cdots (x^m az_myz_{m-1}y \cdots z_1y)\right) - 1 \\ &= wan(t) - 1 = r + 1 - 1 = r. \end{aligned}$$

Thus, the path $h(t)$ has r upward even steps, as required. □

Theorems 3.1 and 8.1 imply the following corollary.

Corollary 8.1. *For $n \in \{0, 1, 2, \ldots\}$ and $r \in \{0, 1, \ldots, n\}$, the composition $h \circ f$ is a bijection from $\mathscr{P}_{n+1,r+1}$ to $\mathscr{Z}_{n,r}$.*

References

[1] A. Bacher, Directed and multi-directed animals on the King's lattice, in: J. Nešetřil, M. Pellegrini (Eds.), *The Seventh European Conference on Combinatorics, Graph Theory and Applications* (Pisa 2013), *CRM Series*, Vol. 16, pp. 535–541, Edizioni della Normale, Pisa, 2013.

[2] J. Bétréma and J.-G. Penaud, Modèles avec particules dures, animaux dirigés, et séries en variables partiellement commutatives, Technical report No. 93-18, LaBRI, Université Bordeaux I, 1993; arXiv:math/0106210v1 [math.CO].

[3] R. Biagioli, F. Jouhet and P. Nadeau, 321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes, *Electron. Notes Discrete Math.* 59 (2017), 115–130.

[4] R. Biagioli, F. Jouhet and P. Nadeau, 321-avoiding affine permutations and their many heaps, *J. Combin. Theory Ser. A* 162 (2019), 271–305.

[5] M. Bousquet-Mélou, Une bijection entre les polyominos convexes dirigés et les mots de Dyck bilatères, *RAIRO Inform. Théor. Appl.* 26 (3) (1992), 205–219.

[6] M. Bousquet-Mélou, Convex polyominoes and heaps of segments, *J. Phys. A: Math. Gen.* 25 (7) (1992), 1925–1934.

[7] M. Bousquet-Mélou, Rapport Scientifique pour obtenir l'habilitation à diriger des recherches, Report No. 1154-96, LaBRI, Université Bordeaux I, 1996.

[8] M. Bousquet-Mélou, Rational and algebraic series in combinatorial enumeration, in: M. Sanz-Solé, J. Soria, J. L. Varona, J. Verdera (Eds.), *Int. Congress of Math'ns.* (Madrid 2006), Vol. III, pp. 789–826, Eur. Math. Soc., Zürich, 2006.

[9] M. Bousquet-Mélou and A. Rechnitzer, Lattice animals and heaps of dimers, *Discrete Math.* 258 (1-3) (2002), 235–274.

[10] M. Bousquet-Mélou and X. G. Viennot, Empilements de segments et q-énumération de polyominos convexes dirigés, *J. Combin. Theory Ser. A* 60 (2) (1992), 196–224.

[11] M.-P. Delest and G. Viennot, Algebraic languages and polyominoes enumeration, *Theoret. Comput. Sci.* 34 (1-2) (1984), 169–206.

[12] S. Feretić, A bijective perimeter enumeration of directed convex polyominoes, *J. Statist. Plann. Inf.* 101 (1-2) (2002), 81–94.

[13] A. J. Guttmann (Ed.), *Polygons, Polyominoes and Polycubes, Lec. Notes in Physics*, Vol. 775, Springer-Verlag, Berlin, 2009.

[14] G. Kreweras, Joint distributions of three descriptive parameters of bridges, in: G. Labelle, P. Leroux (Eds.), *Colloque de Combinatoire Énumérative* (Montréal, 1985), *Lec. Notes in Math.* Vol. 1234, pp. 177–191, Springer-Verlag, Berlin, 1986.

[15] K. Y. Lin and S. J. Chang, Rigorous results for the number of convex polygons on the square and honeycomb lattices, *J. Phys. A: Math. Gen.* 21 (11) (1988), 2635–2642.

[16] P. R. G. Mortimer and T. Prellberg, A bijection on bilateral Dyck paths, *Australas. J. Combin.* 59 (2014), 72–80.

[17] T. V. Narayana, A partial order and its applications to probability theory, *Sankhyā* 21 (1959), 91–98.

[18] J. Osborn, Bi-banded paths, a bijection and the Narayana numbers, *Australas. J. Combin.* 48 (2010), 243–252.

[19] R. A. Sulanke, Constraint-sensitive Catalan path statistics having the Narayana distribution, *Discrete Math.* 204 (1-3) (1999), 397–414.

[20] G. Viennot, Problèmes combinatoires posés par la physique statistique, *Séminaire Bourbaki* Vol. 1983/84, *Astérisque* No. 121-122 (1985), 225–246.

[21] G. X. Viennot, Heaps of pieces, I: Basic definitions and combinatorial lemmas, in: G. Labelle, P. Leroux (Eds.), *Colloque de Combinatoire Énumérative* (Montréal, 1985), *Lec. Notes in Math.* Vol. 1234, pp. 321–350, Springer-Verlag, Berlin, 1986.

(Received 21 Oct 2023; revised 12 Nov 2024)