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Planar graphs with no incident triangles and no 4- or 5-cycles are (7 : 2)-colorable

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Abstract

Steinberg's conjecture that every planar graph with no 4- or 5-cycles has chromatic number at most 3 has been disproved, but it remains to show the upper bound on the fractional chromatic number of such graphs. An $(s : t)$-coloring of a graph G is a function ϕ which assigns to each vertex of G a t-element subset of $\{1, 2, \ldots, s\}$ such that $\phi(u) \cap \phi(v) = \emptyset$ whenever $uv \in E(G)$, and the fractional chromatic number of G is $\min\{s/t :$ an $(s : t)$-coloring of G exists$\}$. It has recently been shown by Dvořák and Hu that planar graphs with no 4- or 5-cycles have fractional chromatic number at most 11/3. We include a slight relaxation and show that planar graphs with no incident triangles and no 4- or 5-cycles have fractional chromatic number at most 7/2. Specifically, we show that such graphs are (7 : 2)-colorable.

1 Introduction

A famous conjecture of Steinberg from 1976 claims that every planar graph with no cycles of length 4 or 5 is 3-colorable. The conjecture has attracted significant attention in recent decades, and was finally disproved by Cohen-Addad, Hebdige, Král', Li, and Salgado [4] in 2017. While disproved, Steinberg's conjecture is only one aspect of the broader three color problem—a search for sufficient conditions for a planar graph to be 3-colorable. Significant information about the progress on the three color problem can be found in Section 7 of an informative survey by Borodin [1]. We will mention here some results of present interest.

Erdős (see [7]) suggested a relaxation as an approach to solving Steinberg's conjecture by asking for which values of k a planar graph with no cycles of lengths $4, 5, \ldots, k$ is 3-colorable. It has been shown by Borodin, Glebov, Raspaud, and Salavatipour [3]

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that planar graphs without cycles of lengths $4, 5, 6, 7$ are 3-colorable, representing the best current answer. Thus $k = 6$ is the only remaining open case of Erdős' question. As a separate relaxation of Steinberg's conjecture, research has been done with Bordeaux-style restrictions in place, that is prescribing a minimum distance between triangles. Borodin and Glebov [2] proved that any planar graph with no 5-cycles and minimum distance between triangles at least 2 is 3-colorable.

In the present paper, we will study Steinberg's conjecture in the context of fractional coloring. For integers s, t, an $(s : t)$-*coloring* of a graph G is a function ϕ which assigns to each vertex of G a t-element subset of $\{1, 2, \ldots, s\}$ such that $\phi(u) \cap \phi(v) = \emptyset$ whenever $uv \in E(G)$. A graph G is $(s : t)$-*colorable* if an $(s : t)$-coloring of G exists. The *fractional chromatic number* of a graph G, denoted $\chi_f(G)$, is defined as $\chi_f(G) = \min\{s/t : G \text{ is } (s : t)\text{-colorable}\}$. Note that an $(s : 1)$-coloring of a graph G is simply a proper coloring using s colors. Hence every planar graph G satisfies $\chi_f(G) \leq 4$ by the Four Color Theorem. We remark that the counterexample to Steinberg's conjecture found by Cohen-Addad, et al. [4] is $(6 : 2)$-colorable, and so has fractional chromatic number at most 3. Thus the following question is still of interest.

Question 1.1 What is the minimum t such that $\chi_f(G) \leq t$ for every planar graph G with no cycles of length 4 or 5?

Clearly, $3 \leq t \leq 4$, where the lower bound comes from considering the graph K_3 consisting of a single triangle. Recently, Dvořák and Hu [6] became the first to show that $t < 4$. Specifically, they proved that planar graphs with no cycles of length 4 or 5 are $(11 : 3)$-colorable, showing that $t \leq 11/3$. As a relaxation of Question 1.1, one can use the idea of Erdős to also exclude cycles of longer length. Wu, Chen, and Hu [8] showed that planar graphs with no cycles of length 4 or 6 are $(7 : 2)$-colorable. We will examine a separate relaxation of Question 1.1 by instead including a Bordeaux-style restriction. Given an embedding of a planar graph, we say two faces are *incident* if they have at least one common vertex.

Theorem 1.2 *If G is a planar graph with no incident triangles, and no* 4- *or* 5-*cycles, then G is* $(7 : 2)$-*colorable.*

The study of Question 1.1 is also of interest in determining the size of a largest independent set in such a graph G, denoted by $\alpha(G)$. For any graph G, it is known that $\alpha(G) \geq |V(G)|/\chi_f(G)$. As a result, we obtain the following immediate corollary of Theorem 1.2.

Corollary 1.3 *If G is a planar graph with no incident triangles, and no* 4- *or* 5-*cycles, then* $\alpha(G) \geq |V(G)|/3.5$.

We conclude this section with some notation and terminology we will use throughout the paper. Let G be a graph with a fixed planar embedding. A subgraph H of G is called *induced* if for all $u, v \in V(H)$ we have $uv \in E(H)$ if and only if $uv \in E(G)$. We use $F(G)$ to denote the set of faces of G (with respect to the fixed embedding).

Two faces are *adjacent* if they have at least one common edge. For any $f \in F(G)$, we use $d(f)$ to denote the length of f, where the length of a face is the number of edges in its facial walk counting multiplicities. For any $v \in V(G)$, we use $d(v)$ to denote the degree of v, and we say that v is a *k-vertex* (respectively, *k^+-vertex*) if $d(v) = k$ (respectively, $d(v) \geq k$). We define k-face and k^+-face analogously. $N_G(v)$ denotes the set of neighbors of v in G. A vertex v is *incident* to a face f if v belongs to the facial walk of f. For brevity, by a $(d_1, \ldots, d_k)$-face $v_1, \ldots, v_k$, we will mean a face with facial walk $v_1, \ldots, v_k$ such that $d(v_i) = d_i$ for $i \in \{1, \ldots, k\}$. A $(d_1, \ldots, d_k)$-path is defined analogously.

2 Reducible Configurations

In this section, we provide reducible configurations that we will need for the proof of Theorem 1.2. We will say a graph G is *minimal* if G is a minimal counterexample to Theorem 1.2. That is, G is a planar graph with no incident triangles and no 4- or 5-cycles that is not $(7:2)$-colorable, but any planar graph on fewer vertices with no incident triangles and no 4- or 5-cycles is $(7:2)$-colorable.

Suppose that ϕ is a function which assigns to each vertex $v \in V(G)$ a subset of $\{1, 2, \ldots, 7\}$ such that $|\phi(v)| \leq 2$ and $\phi(v) \cap \phi(u) = \emptyset$ for every $u \in N_G(v)$. Informally, such a function ϕ can be thought of as a partial $(7:2)$-coloring of G. Then for each $v \in V(G)$ with $|\phi(v)| < 2$, we define $L_\phi(v)$ (or simply $L(v)$ if the coloring ϕ is clear) to be the set

$$L_\phi(v) = \{1, 2, \ldots, 7\} \setminus \bigcup_{u \in N_G(v)} \phi(u) \cup \phi(v).$$

That is, $L(v)$ is the set of available colors at v.

By *assigning a color c to $\phi(v)$*, we mean extending ϕ to a coloring ϕ^* of G by setting $\phi^*(v) = \phi(v) \cup \{c\}$ and $\phi^*(u) = \phi(u)$ for all $u \neq v$. If $|\phi(v)| < 2$, then by *greedily coloring* v, we mean extending ϕ to a coloring ϕ^* of G by arbitrarily selecting a set $C \subseteq L(v)$ with $|C| = 2 - |\phi(v)|$ and setting $\phi^*(v) = \phi(v) \cup C$ and $\phi^*(u) = \phi(u)$ for all $u \neq v$. In either situation, we will assume any such coloring ϕ^* is then renamed to ϕ, replacing the original coloring. Hence we omit the * notation in our proofs. For a sequence of vertices $v_1, \ldots, v_k$, if

$$(2 - |\phi(v_i)|) + \sum_{\{j<i \,:\, v_j \in N_G(v_i)\}} (2 - |\phi(v_j)|) \leq |L(v_i)|$$

for each $i \in \{1, \ldots, k\}$, that is the total number of colors needed at v_i and its neighbors which appear before it in the sequence is at most $|L(v_i)|$, then we may greedily color $v_1, \ldots, v_k$ in order. We will frequently use that observation in the following proofs.

Lemma 2.1 *If G is minimal, then $\delta(G) \geq 3$.*

Proof. Suppose for a contradiction that there exists $v \in V(G)$ such that $d(v) \leq 2$. Since G is minimal, there exists a $(7:2)$-coloring ϕ of $G - \{v\}$. Since v has at most two neighbors, $|L(v)| \geq 3$, and so we can greedily color v to extend ϕ to a $(7:2)$-coloring of G, a contradiction. □

Lemma 2.2 *If G is minimal, then G is 2-connected.*

Proof. Clearly, we may assume that G is connected. So suppose that v is a cut-vertex of G. Let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = G[\{v\}]$, and $V(G_i) \neq \{v\}$ for $i \in \{1,2\}$. Since G is minimal, there exists a $(7:2)$-coloring ϕ_i of G_i for $i = \{1,2\}$. By permuting the colors of ϕ_2, say, we may assume that $\phi_1(v) = \phi_2(v)$. But then ϕ_1 and ϕ_2 may be combined to give a $(7:2)$-coloring of G, a contradiction. □

As an immediate consequence of Lemma 2.2, we get the following.

Lemma 2.3 *If G is minimal, then every face in G is bounded by a cycle.*

We remark that the reducible configurations of the following Lemmas 2.4 and 2.5 were also shown in [8]. We provide proofs here for the sake of completeness.

Lemma 2.4 *If G is minimal, then G has no induced $(3,3,3)$-path.*

Proof. Suppose $v_1v_2v_3$ is an induced $(3,3,3)$-path in G such that $d(v_1) = d(v_2) = d(v_3) = 3$. Since G is minimal, there exists a $(7:2)$-coloring ϕ of $G - \{v_1, v_2, v_3\}$. By choosing an arbitrary subset if necessary, we may assume $|L(v_i)| = 3$ for $i \in \{1,3\}$, and $|L(v_2)| = 5$. Since $|L(v_1)| + |L(v_3)| > |L(v_2)|$ either $L(v_1) \cap L(v_3) \neq \emptyset$ or $(L(v_1) \cup L(v_3)) \setminus L(v_2) \neq \emptyset$. In the former case, we assign the same color from $L(v_1) \cap L(v_3)$ to both $\phi(v_1)$ and $\phi(v_3)$. In the latter case, say $L(v_1) \setminus L(v_2) \neq \emptyset$, and we assign a color from $L(v_1) \setminus L(v_2)$ to $\phi(v_1)$ and an arbitrary color from $L(v_3)$ to $\phi(v_3)$. In either case, we now have $|L(v_1)| = |L(v_3)| = 2$ and $|L(v_2)| \geq 4$. We may then greedily color v_1, v_3, v_2 in order to obtain a $(7:2)$-coloring of G, a contradiction. □

Lemma 2.5 *If G is minimal, then G has no induced $(3,3,4,3)$-path.*

Proof. Suppose $v_1v_2v_3v_4$ is an induced $(3,3,4,3)$-path in G such that $d(v_1) = d(v_2) = d(v_4) = 3$ and $d(v_3) = 4$. Since G is minimal, there exists a $(7:2)$-coloring ϕ' of the graph $G - \{v_1, v_2\}$. Consider the restriction ϕ of ϕ' to $G - \{v_1, \ldots, v_4\}$. Clearly, ϕ is a $(7:2)$-coloring of $G - \{v_1, \ldots, v_4\}$. Without loss of generality, we may assume $|L_\phi(v_i)| = 3$ for $i \in \{1,3,4\}$, and $|L_\phi(v_2)| = 5$. Note that by the existence of ϕ', we may further assume that $L_\phi(v_3) \neq L_\phi(v_4)$. Hence we may assign a color from $L_\phi(v_4) \setminus L_\phi(v_3)$ to $\phi(v_4)$ and a color from $L_\phi(v_3) \setminus L_\phi(v_4)$ to $\phi(v_3)$ arbitrarily. Now $|L_\phi(v_1)| = 3$, $|L_\phi(v_2)| \geq 4$, and $|L_\phi(v_i)| = 2$ for $i \in \{3,4\}$. Next, arbitrarily assign a color from $L_\phi(v_2) \setminus L_\phi(v_3)$ to $\phi(v_2)$. Then greedily color v_1, v_2, v_3, v_4 in order to extend ϕ to a $(7:2)$-coloring of G, a contradiction. □

Lemma 2.6 *If G is minimal, then G has no induced $(3, 4, 3, 4, 3)$-path.*

Proof. Suppose $v_1 \dots v_5$ is an induced $(3, 4, 3, 4, 3)$-path in G such that $d(v_1) = d(v_3) = d(v_5) = 3$ and $d(v_2) = d(v_4) = 4$. Since G is minimal, there exists a $(7:2)$-coloring ϕ' of the graph $G - \{v_3\}$. Consider the restriction ϕ of ϕ' to $G - \{v_1, \dots, v_5\}$. Without loss of generality, we may assume $|L_\phi(v_i)| = 3$ for $i \in \{1, 2, 4, 5\}$, and $|L_\phi(v_3)| = 5$. By the existence of ϕ', we may further assume that $L_\phi(v_1) \neq L_\phi(v_2)$ and $L_\phi(v_4) \neq L_\phi(v_5)$. We now assign a color from $L_\phi(v_1) \setminus L_\phi(v_2)$ to $\phi(v_1)$, and a color from $L_\phi(v_5) \setminus L_\phi(v_4)$ to $\phi(v_5)$. This leaves $|L_\phi(v_1)| = |L_\phi(v_5)| = 2$, $|L_\phi(v_2)| = |L_\phi(v_4)| = 3$, and $|L_\phi(v_3)| = 5$. Since $|L_\phi(v_2)| + |L_\phi(v_4)| > |L_\phi(v_3)|$ either $L_\phi(v_2) \cap L_\phi(v_4) \neq \emptyset$ or $(L_\phi(v_2) \cup L_\phi(v_4)) \setminus L_\phi(v_3) \neq \emptyset$. In the former case, we assign the same color from $L_\phi(v_2) \cap L_\phi(v_4)$ to both $\phi(v_2)$ and $\phi(v_4)$. In the latter case, say $L_\phi(v_2) \setminus L_\phi(v_3) \neq \emptyset$, and we assign a color from $L_\phi(v_2) \setminus L_\phi(v_3)$ to $\phi(v_2)$ and an arbitrary color from $L_\phi(v_4)$ to $\phi(v_4)$. In either case, we now have $|L_\phi(v_i)| \geq 1$ for $i \in \{1, 5\}$, $|L_\phi(v_2)| = |L_\phi(v_4)| = 2$, and $|L_\phi(v_3)| \geq 4$. We can then greedily color v_1, v_2, v_5, v_4, v_3 in order to extend ϕ to a $(7:2)$-coloring of G, a contradiction. □

Lemma 2.7 *If G is minimal, then any 5-vertex of G has at most four neighbors of degree 3.*

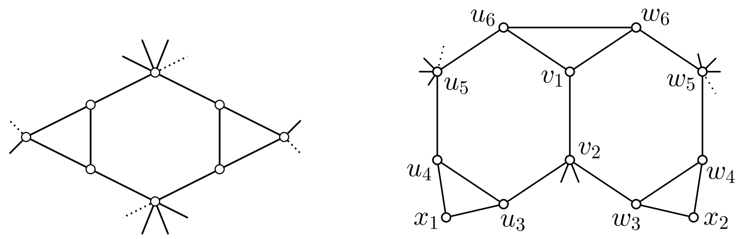
Proof. Suppose v is a 5-vertex with neighbors $v_1, \dots, v_5$, such that $d(v_i) = 3$ for $i \in \{1, \dots, 5\}$. Let ϕ be a $(7:2)$-coloring of $G - \{v, v_1, \dots, v_5\}$. We consider two cases.

First, suppose that v is not incident to a triangle. Without loss of generality, we may assume $|L(v_i)| = 3$ for $i \in \{1, \dots, 5\}$, and $|L(v)| = 7$. By the pigeonhole principle, there must be a common color on three of the lists $L(v_i)$, say $L(v_1) \cap L(v_2) \cap L(v_3) \neq \emptyset$. Assign this common color to $\phi(v_i)$ for $i \in \{1, 2, 3\}$. Then $|L(v)| = 6$, $|L(v_i)| = 2$ for $i \in \{1, 2, 3\}$, and $|L(v_4)| = |L(v_5)| = 3$. Next assign a color from $L(v) \setminus (L(v_3) \cup L(v_4))$ to $\phi(v)$, leaving $|L(v)| = 5$, $|L(v_i)| \geq 1$ for $i \in \{1, 2\}$, $|L(v_i)| \geq 2$ for $i \in \{3, 5\}$, and $|L(v_4)| = 3$. We then greedily color $v_1, v_2, v_5, v, v_3, v_4$ in order to obtain a $(7:2)$-coloring of G, a contradiction.

Now, suppose v is incident with a triangle, say $v_4 v_5 \in E(G)$. Without loss of generality, $|L(v_i)| = 3$ for $i \in \{1, 2, 3\}$, $|L(v_4)| = |L(v_5)| = 5$, and $|L(v)| = 7$. By the pigeonhole principle, some two of $L(v_1), L(v_2), L(v_3)$ share a common color, say $L(v_1) \cap L(v_2) \neq \emptyset$. Assign this common color to $\phi(v_1)$ and $\phi(v_2)$. Now, $|L(v)| = 6$, while $|L(v_4)| = 5$, so we assign a color from $L(v) \setminus L(v_4)$ to $\phi(v)$. This leaves $|L(v_i)| \geq 1$ for $i \in \{1, 2\}$, $|L(v_3)| \geq 2$, $|L(v_4)| = 5$, $|L(v_5)| \geq 4$, and $|L(v)| = 5$. We then greedily color $v_1, v_2, v_3, v, v_5, v_4$ in order to obtain a $(7:2)$-coloring of G, a contradiction. □

Lemma 2.8 *Suppose ϕ is a $(7:2)$-coloring of $G - \{v_1, v_2, v_3\}$, where $v_1 v_2 v_3$ is a 3-face, and suppose $|L(v_i)| = 5$ for $i \in \{1, 2, 3\}$. Then ϕ can be extended to a $(7:2)$-coloring of G if and only if $L(v_i) \neq L(v_j)$ for some i, j.*

Proof. Clearly, if ϕ can be extended to a $(7:2)$-coloring of G then $|L(v_1) \cup L(v_2) \cup L(v_3)| \geq 6$, so necessarily $L(v_i) \neq L(v_j)$ for some i, j. So to prove the reverse,



(a) A bad face

(b) Two bad faces adjacent along a $(5, 3)$-edge

Figure 2.1

suppose $L(v_1) \neq L(v_2)$, say. Assign a color from $L(v_1) \setminus L(v_2)$ to $\phi(v_1)$ and a color from $L(v_2) \setminus L(v_1)$ to $\phi(v_2)$. Then $|L(v_i)| = 4$ for $i \in \{1, 2\}$, and $|L(v_3)| \geq 3$, and without loss of generality we may assume $|L(v_3)| = 3$. Next assign a color from $L(v_1) \setminus L(v_3)$ to $\phi(v_1)$. Now $|L(v_i)| \geq 3$ for $i \in \{2, 3\}$, so we can greedily color v_2, v_3 to obtain a $(7 : 2)$-coloring of G. □

If a $(3, 5^+, 3, 3, 5^+, 3)$-face f is adjacent to a 3-face along each of its $(3, 3)$-edges, then we say f is a *bad face*, see Figure 2.1(a).

Lemma 2.9 *If G is minimal, then no two bad faces of G can be adjacent along a $(5, 3)$-edge.*

Proof. Suppose $f_1 = v_1v_2u_3u_4u_5u_6$ and $f_2 = v_1v_2w_3w_4w_5w_6$ are adjacent bad faces, where $d(v_1) = d(u_i) = d(w_i) = 3$ for $i \in \{3, 4, 6\}$, $d(v_2) = 5$, $d(u_5) \geq 5$, $d(w_5) \geq 5$, and $u_6w_6 \in E(G)$, and suppose x_1 is the common neighbor of u_3, u_4, and x_2 is the common neighbor of w_3, w_4, see Figure 2.1(b). It is straightforward to verify that all labeled vertices must be distinct because G has no 4- or 5-cycles.

Since G is minimal, there exists a $(7 : 2)$-coloring ϕ' of $G - \{v_1, u_6, w_6\}$. Then $|L_{\phi'}(v_1)| = |L_{\phi'}(u_6)| = |L_{\phi'}(w_6)| = 5$. Since G is not $(7 : 2)$-colorable, it follows from Lemma 2.8 that $L_{\phi'}(u_6) = L_{\phi'}(w_6)$. Since each of u_6, w_6 has only one neighbor in G colored by ϕ', it must be the case that $\phi'(u_5) = \phi'(w_5)$, say $\phi'(u_5) = \phi'(w_5) = \{6, 7\}$. Now, consider the restriction ϕ of ϕ' to $G - \{v_1, v_2, u_3, u_4, u_6, w_3, w_4, w_6\}$. Then $|L_\phi(v_1)| = 7$, $|L_\phi(v_2)| \geq 3$, $|L_\phi(u_3)| = |L_\phi(w_3)| = 5$, $|L_\phi(u_4)| \geq 3$, $|L_\phi(w_4)| \geq 3$, and $L_\phi(u_6) = L_\phi(w_6) = \{1, \ldots, 5\}$. We claim that we may extend ϕ to a $(7 : 2)$-coloring of $G - \{v_1, u_6, w_6\}$ such that $\phi(v_2) \neq \{6, 7\}$. If so, then we will have $|L_\phi(v_1)| = |L_\phi(u_6)| = |L_\phi(w_6)| = 5$ with $L_\phi(v_1) \neq L_\phi(u_6)$, so we will be able to extend ϕ to a $(7 : 2)$-coloring of G by Lemma 2.8, a contradiction which completes the proof. Since $|L_\phi(v_2)| \geq 3$, we may assign a color from $L_\phi(v_2) \setminus \{6, 7\}$ to $\phi(v_2)$. Without loss of generality, we may assume $|L_\phi(u_3)| = |L_\phi(w_3)| = 4$ and $|L_\phi(v_2)| = 2$. To complete our claim, we consider two cases.

First, suppose that $|L_\phi(u_4)| > 3$, say. Without loss of generality, $|L_\phi(u_4)| = 4$ and $|L_\phi(w_4)| = 3$. Assign a color from $L_\phi(w_3) \setminus L_\phi(w_4)$ to $\phi(w_3)$. Then we may greedily color v_2, w_3, w_4, u_3, u_4 in order to obtain a $(7 : 2)$-coloring of $G - \{v_1, u_6, w_6\}$ with $\phi(v_2) \neq \{6, 7\}$, as claimed.

Therefore $|L_\phi(u_4)| = 3$, and by symmetry $|L_\phi(w_4)| = 3$. Since u_4 has only two neighbors u_5 and x_1 colored by ϕ, we must have $\phi(u_5) \cap \phi(x_1) = \emptyset$, that is $\phi(x_1) \cap \{6, 7\} = \emptyset$. Since $\phi(v_2) \cap \{6, 7\} = \emptyset$ as well, it follows that $\{6, 7\} \subset L_\phi(u_3)$. Similarly, $\{6, 7\} \subset L_\phi(w_3)$. We assign the color 7, say, to $\phi(u_3)$ and $\phi(w_3)$. This leaves $|L_\phi(u_3)| = |L_\phi(w_3)| = |L_\phi(u_4)| = |L_\phi(w_4)| = 3$ and $|L_\phi(v_2)| \geq 1$. We then greedily color v_2, u_3, u_4, w_3, w_4 in order to again obtain a $(7 : 2)$-coloring of $G - \{v_1, u_6, w_6\}$ with $\phi(v_2) \neq \{6, 7\}$, as claimed. □

3 Proof of Theorem 1.2

We will now proceed to prove Theorem 1.2 using the discharging method. For more on this proof technique, the reader is referred to an informative survey by Cranston and West [5].

Let G be minimal. To each vertex or face $x \in V(G) \cup F(G)$ we assign an initial charge $\mathrm{ch}(x) = d(x) - 4$. By Euler's formula, it follows that

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8.$$

We will redistribute charge according to the following rules.

(R1) Every 3-face takes $\frac{1}{3}$ charge from each of its adjacent 6^+-faces.

(R2) Suppose v is a 3-vertex. If v is incident to a 3-face, then v takes $\frac{1}{2}$ from each of its incident 6^+-faces. Otherwise, v takes $\frac{1}{3}$ from each of its incident 6^+-faces.

(R3) Suppose v is a 5^+-vertex whose two neighbors u_1 and u_2 on the facial walk of a face f are both 3-vertices. If either edge vu_i belongs to a 3-face adjacent to f, then v gives $\frac{1}{2}$ to f. If f is a bad face, then v gives $\frac{1}{3}$ to f. Otherwise, v gives $\frac{1}{6}$ to f.

We will denote the final charge of each vertex or face x by $\mathrm{ch}^*(x)$. The following lemmas will demonstrate that $\mathrm{ch}^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Lemma 3.1 *If $v \in V(G)$, then* $\mathrm{ch}^*(v) \geq 0$.

Proof. By Lemma 2.1, $\delta(G) \geq 3$. Note that since G has no 4- or 5-cycle, by Lemma 2.3, v is incident with no 4- or 5-face. Furthermore, since G has no triangles sharing a vertex, any vertex v is incident to at most one 3-face, and all its other incident faces are 6^+-faces. If v is a 3-vertex, then by (R2), $\mathrm{ch}^*(v) = (3-4) + 2 \cdot \frac{1}{2} = 0$ if v is incident to a 3-face, and $\mathrm{ch}^*(v) = (3-4) + 3 \cdot \frac{1}{3} = 0$ otherwise. If v is a 4-vertex, then v is not involved in the discharging procedure, so $\mathrm{ch}^*(v) = \mathrm{ch}(v) = 0$.

Suppose v is a 5-vertex. Note that v has at most four degree 3 neighbors by Lemma 2.7, and so gives charge to at most three faces by (R3). If v is not incident to a 3-face, then by (R3), $\mathrm{ch}^*(v) \geq (5-4) - 3 \cdot \frac{1}{3} = 0$. So suppose v is incident to a 3-face f. Observe that the two faces f_1, f_2 incident to v and adjacent to f along an edge which includes v are not adjacent to f along a $(3, 3)$-edge. Hence by definition of a bad face, neither f_1 nor f_2 can be a bad face. Then by Lemma 2.9, v can be incident to at most one bad face. Therefore by (R3), $\mathrm{ch}^*(v) \geq (5-4) - \frac{1}{2} - \frac{1}{3} - \frac{1}{6} = 0$.

Lastly, suppose v is a 6^+-vertex. If v is not incident to a 3-face, then by (R3), $\mathrm{ch}^*(v) \geq (d(v)-4) - d(v) \cdot \frac{1}{3} = \frac{2}{3}d(v) - 4 \geq \frac{2}{3} \cdot 6 - 4 = 0$. If v is incident to a 3-face, then $\mathrm{ch}^*(v) \geq (d(v)-4) - 2 \cdot \frac{1}{2} - (d(v)-3) \cdot \frac{1}{3} = \frac{2}{3}d(v) - 4 \geq 0$. □

Lemma 3.2 *If $f \in F(G)$, then* $\mathrm{ch}^*(f) \geq 0$.

Proof. If f is a 3-face, then every face adjacent to f is a 6^+-face, so by (R1), $\mathrm{ch}^*(f) \geq (3-4) + 3 \cdot \frac{1}{3} = 0$. Note that G has no 4- or 5-faces since any such face must be bounded by a cycle by Lemma 2.3.

Since no two triangles of G share a vertex, any 6^+-face f is adjacent to at most $\left\lfloor \frac{1}{2}d(f) \right\rfloor$ 3-faces. Furthermore, since G has no induced $(3,3,3)$-path by Lemma 2.4 and every face is bounded by a cycle by Lemma 2.3, it follows that f has at most $\left\lfloor \frac{2}{3}d(f) \right\rfloor$ vertices of degree 3. If f is a 7-face, then f is adjacent to at most three 3-faces and four 3-vertices, so by (R1) and (R2), $\mathrm{ch}^*(f) \geq (7-4) - 3 \cdot \frac{1}{3} - 4 \cdot \frac{1}{2} = 0$. If f is an 8^+-face, then by (R1) and (R2), $\mathrm{ch}^*(f) \geq (d(f)-4) - \left\lfloor \frac{1}{2}d(f) \right\rfloor \cdot \frac{1}{3} - \left\lfloor \frac{2}{3}d(f) \right\rfloor \cdot \frac{1}{2} \geq \frac{1}{2}d(f) - 4 \geq 0$.

Hence, we may assume that $f = v_1 \dots v_6$ is a 6-face. Note that because G has no 4- or 5-cycles, the cycle $v_1 \dots v_6$ has no chords, so any subpath of the facial walk of f is an induced path. Let t denote the number of 3-vertices on f. Then $t \leq 4$. If $t \leq 2$, then since f is adjacent to at most three 3-faces, by (R1) and (R2) we have $\mathrm{ch}^*(f) \geq (6-4) - 3 \cdot \frac{1}{3} - t \cdot \frac{1}{2} \geq 0$.

Suppose $t = 3$. By Lemma 2.4, at most one pair of 3-vertices of f can be adjacent on the facial walk of f. Suppose first that some pair of 3-vertices is adjacent, say v_1, v_2, and v_4 are 3-vertices. By Lemma 2.5, v_3 must be a 5^+-vertex. If f is adjacent to at most two 3-faces, then by (R1)–(R3), $\mathrm{ch}^*(f) \geq (6-4) - 2 \cdot \frac{1}{3} - 3 \cdot \frac{1}{2} + \frac{1}{6} = 0$. Hence, suppose f is adjacent to three 3-faces. Since no two 3-faces share a vertex, v_3 must be incident to a 3-face. Thus by (R1)–(R3), $\mathrm{ch}^*(f) \geq (6-4) - 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{2} + \frac{1}{2} = 0$.

Thus we may assume no two 3-vertices of f are adjacent, say v_1, v_3, and v_5 are 3-vertices. By Lemma 2.6, at least two of v_2, v_4, v_6 must be 5^+-vertices, say v_2 and v_4 are 5^+-vertices. If f is adjacent to at most two 3-faces, then at most two of its 3-vertices are incident to 3-faces, and so by (R1)–(R3) we have $\mathrm{ch}^*(f) \geq (6-4) - 2 \cdot \frac{1}{3} - \left(2 \cdot \frac{1}{2} + \frac{1}{3}\right) + 2 \cdot \frac{1}{6} > 0$. So suppose f is adjacent to three 3-faces. Then every vertex of f is incident to a 3-face, and so $\mathrm{ch}^*(f) \geq (6-4) - 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} > 0$.

Lastly, suppose $t = 4$. By Lemmas 2.4 and 2.5, f must be a $(3, 5^+, 3, 3, 5^+, 3)$-face. If f is adjacent to at most one 3-face, then at most two 3-vertices of f are incident to a 3-face, so $\mathrm{ch}^*(f) \geq (6-4) - \frac{1}{3} - \left(2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3}\right) + 2 \cdot \frac{1}{6} > 0$. If f is adjacent to two 3-faces, first suppose at least one 5^+-vertex is incident to a 3-face adjacent to f. Then at most three 3-vertices of f are incident with a 3-face, and so $\mathrm{ch}^*(f) \geq (6-4) - 2 \cdot \frac{1}{3} - \left(3 \cdot \frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} + \frac{1}{6}\right) > 0$. Thus neither 5^+-vertex of f is incident to a 3-face adjacent to f, so f is a bad face. In this case, $\mathrm{ch}^*(f) \geq (6-4) - 2 \cdot \frac{1}{3} - 4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} = 0$. Finally, if f is adjacent to three 3-faces, then $\mathrm{ch}^*(f) \geq (6-4) - 3 \cdot \frac{1}{3} - 4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 0$. □

Since no charge is created or destroyed by (R1)–(R3), from Lemmas 3.1 and 3.2, it follows that

$$-8 = \sum_{v \in V(G)} \mathrm{ch}(v) + \sum_{f \in F(G)} ch(f) = \sum_{v \in V(G)} ch^*(v) + \sum_{f \in F(G)} ch^*(f) \geq 0.$$

This contradiction completes the proof of Theorem 1.2.

References

[1] O. V. Borodin, Colorings of plane graphs: A survey, *Discrete Math.* **313** (2013), 517–539.

[2] O. V. Borodin and A. N. Glebov, Planar graphs without 5-cycles and with minimal distance between triangles at least 2 are 3-colourable, *J. Graph Theory* **66 (1)** (2011), 1–31.

[3] O. V. Borodin, A. N. Glebov, A. Raspaud and M. R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, *J. Combin. Theory Ser. B.* **93** (2005), 303–311.

[4] V. Cohen-Addad, M. Hebdige, D. Král', Z. Li and E. Salgado, Steinberg's Conjecture is false, *J. Combin. Theory Ser. B* **122** (2017), 452–456.

[5] D. Cranston and D. West, An introduction to the discharging method via graph coloring, *Discrete Math.* **340** (2017), 766–793.

[6] Z. Dvořák and X. Hu, Planar graphs without cycles of length 4 or 5 are (11 : 3)-colorable, *Europ. J. Combin.* **82** (2019), 102996.

[7] R. Steinberg, The state of the three color problem, quo vadis, graph theory? *Ann. Discrete Math.* **55** (1993), 211–248.

[8] H. Wu, Y. Chen and X. Hu, Planar graphs without 4- and 6-cycles are (7 : 2)-colorable, *J. Combin. Optim.* **40** (2020), 45–58.

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