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Separating topological recurrence from measurable recurrence: exposition and extension of Kriz's example

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Abstract

We prove that for every infinite set $E \subseteq \mathbb{Z}$, there is a set $S \subseteq E - E$ which is a set of topological recurrence and not a set of measurable recurrence. This extends a result of Igor Kriz, proving that there is a set of topological recurrence which is not a set of measurable recurrence. Our construction follows Kriz's closely, and can be considered an exposition of the original argument.

1 Structure of difference sets

We write $\mathbb{Z}$ for the group of integers. If $A \subseteq \mathbb{Z}$, we write $A - A$ for the *difference set*, defined to be $\{a_1 - a_2 : a_1, a_2 \in A\}$. If $A, B \subseteq \mathbb{Z}$, we write $A + B$ for the *sumset*, $\{a + b : a \in A, b \in B\}$. The *upper asymptotic density* (or just *upper density*) of A is $\bar{d}(A) := \limsup_{n\to\infty} \frac{|A\cap\{1,\dots,n\}|}{n}$. If the limit exists, we write $d(A)$ in place of $\bar{d}(A)$.

1.1 Kriz's theorem

We say that $S \subseteq \mathbb{Z}$ is a *set of density recurrence* (or "S is *density recurrent*") if for all $A \subseteq \mathbb{Z}$, with $\bar{d}(A) > 0$, there exists $a, b \in A$ such that $b - a \in S$. In other words, $(A - A) \cap S \neq \varnothing$ whenever $\bar{d}(A) > 0$. The set $\{n^2 : n \in \mathbb{N}\}$ of perfect squares is density recurrent, as proved by Furstenberg ([9], via ergodic theory) and Sárközy ([27], using the circle method). This result, along with earlier work by Bogolioúboff [4] and Følner [6], motivated further investigation into the structure of difference sets.

Another result influencing this investigation is van der Waerden's theorem on arithmetic progressions [29]: if $\mathbb{Z}$ is partitioned into finitely many sets $A_1, \dots, A_r$, then at least one of the sets A_j contains arithmetic progressions of every finite length.

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The "density version" of van der Waerden's theorem is Szemerédi's theorem [28], which says that if $\bar{d}(A) > 0$, then A contains arithmetic progressions of every finite length. These results suggested a possible deep connection between partition results and density results. To formalize this suggestion, we say a set $S \subseteq \mathbb{Z}$ is a *set of chromatic recurrence* (or "is *chromatically recurrent*") if for every partition $\{A_1, \ldots, A_r\}$ of $\mathbb{Z}$ into finitely many sets, there is a cell A_j of the partition such that $(A_j - A_j) \cap S \neq \varnothing$. Upper density is finitely subadditive, so if $\mathbb{Z}$ is partitioned into finitely many sets, then at least one of those sets has positive upper density, and its difference set therefore intersects every density recurrent set. Thus every density recurrent set is also chromatically recurrent. Bergelson [2], Furstenberg [10], and Ruzsa [25] each asked whether the converse holds: is every chromatically recurrent set also density recurrent? Igor Kriz gave a negative answer in the following theorem from [18]. The main purpose of this article is to give an expository proof.

Theorem 1.1. *There is a set of integers which is chromatically recurrent and not density recurrent.*

Readers familiar with [18] will see that the methods there can be combined with Lemma 6.2 below to prove the following generalization of Theorem 1.1. Our second purpose in this article is to give an explicit proof of this generalization.

Theorem 1.2. *If E is an infinite set of integers, then there is a subset $S \subseteq E - E$ such that S is chromatically recurrent and not density recurrent.*

Theorem 1.2 was motivated by [15, Proposition 2.3]; cf. §8.1 below.

Remark 1.3. In terms of dynamical systems, Theorem 1.1 says "there is a set of topological recurrence which is not a set of measurable recurrence." Likewise, Theorem 1.2 says that if $E \subseteq \mathbb{Z}$ is infinite, then there is a subset of $E - E$ which is a set of topological recurrence and not a set of measurable recurrence; see [3] for a general discussion of the equivalences between various recurrence properties. Since our constructions do not explicitly reference dynamical systems, we use only the terms "chromatic recurrence" and "density recurrence."

There are several proofs of Theorem 1.1 ([26],[7],[22],[23],[30]), all of which follow the same broad outline as [18] and overcome the essential difficulties in a similar way. Hopefully our exposition will help readers solve some related open problems, or find a fundamentally different approach to Theorems 1.1 and 1.2. Our approach is nearly identical to Kriz's [18], except that our proof of Lemma 3.6 is closer to an argument from Ruzsa's construction in [26].

Remark 1.4. It may seem unnatural to use upper density when studying subsets of $\mathbb{Z}$, since $\bar{d}(-\mathbb{N}) = 0$. Intuitively, $\bar{D}(A) := \limsup_{n\to\infty} \frac{|A\cap\{-n,\ldots,n\}|}{2n+1}$ is a more natural notion of density for subsets of $\mathbb{Z}$. However, upper asymptotic density is more convenient for the proof of Lemma 3.6. This suggests that we should work in $\mathbb{N}$ rather than in $\mathbb{Z}$; we work in $\mathbb{Z}$ because we will use group homomorphisms from $\mathbb{Z}$ into other groups.

1.2 Outline of the article

Our proof of Theorems 1.1 and 1.2 consists of two independent steps.

Step I. Identify finite subsets of $\mathbb{Z}$ which approximate, in a precise sense, the property of being chromatically recurrent and not density recurrent. This is split into two substeps:

(A) Find subsets of a finite group G approximating the property of being chromatically recurrent and not density recurrent.

(B) Copy these sets from G into $\mathbb{Z}$ in a way that maintains their recurrence properties.

Step II. Piece together the finite sets found in Step I in a way that maintains their recurrence properties.

In §2 we prove a version of Theorem 1.1 for groups of the form $(\mathbb{Z}/2\mathbb{Z})^d$. This carries out Step I(A) and introduces a main idea in a clearer setting.

In §3 we state and prove Lemma 3.6, completing Step II of the outline. We then state the other main lemmas, Lemmas 3.7 and 3.8, and prove Theorems 1.1 and 1.2. The latter two lemmas are proved in §4, carrying out Part I(B) of the outline.

The proofs of Theorems 1.1 and 1.2 are mostly elementary. The only highly nontrivial result required, for both proofs, is a lower bound for chromatic numbers of Kneser graphs, discussed in §2.2. Theorem 1.2 uses some standard results on subsets of $\mathbb{T}^d$ of the form $\{n\boldsymbol{\alpha} : n \in E\}$ for arbitrary infinite sets $E \subseteq \mathbb{Z}$; we provide proofs in §7 for completeness. We also use, without proof, the elementary binomial estimate $\lim_{n\to\infty} \binom{n}{\lfloor n/2 \rfloor} 2^{-n} = 0$.

Section 8 contains some remarks and open problems.

2 A model setting

While Theorems 1.1 and 1.2 are about sets of integers, one of our main arguments is easier to develop in finite vector spaces over the field with two elements. The examples in this section will be used as building blocks for the main construction in §§3-5. Forrest takes a similar approach in [7] and in [8].

2.1 Finite approximations to recurrence

Let Γ be a finite abelian group and $S \subseteq \Gamma$. We say that S is

- δ-*density recurrent* if $(A - A) \cap S \neq \varnothing$ for all $A \subseteq \Gamma$ having $|A| > \delta|\Gamma|$;
- δ-*nonrecurrent* if there exists $A \subseteq \Gamma$ having $|A| > \delta|\Gamma|$ such that $(A - A) \cap S = \varnothing$;

- *k-chromatically recurrent* if for every partition of Γ into k sets $A_1, A_2, \ldots, A_k$, the intersection $(A_j - A_j) \cap S$ is nonempty for some $j \leq k$.

Note that if $0_\Gamma \in S$, then S is both k-chromatically recurrent for every k and δ-density recurrent for all $\delta \geq 0$, as $0_\Gamma \in A - A$ for every nonempty A. However, our interest is only in sets S not containing 0_Γ.

If $d \in \mathbb{N}$, we write $\mathbb{F}_2^d$ for the group $(\mathbb{Z}/2\mathbb{Z})^d$, the product of d copies of $\mathbb{Z}/2\mathbb{Z}$. Elements of $\mathbb{F}_2^d$ will be written as $\mathbf{x} = (x_1, \ldots, x_d)$, where $x_j \in \{0, 1\}$. The remainder of this section is dedicated to proving the following analogue of Theorem 1.1.

Theorem 2.1. *Let $\delta < \frac{1}{2}$ and $k \in \mathbb{N}$. For all sufficiently large d, there exists $S \subseteq \mathbb{F}_2^d$ such that S is k-chromatically recurrent and δ-nonrecurrent.*

The next subsection summarizes background for the proof of Theorem 2.1.

2.2 Cayley graphs and Kneser graphs

We adopt the usual terminology from graph theory: a *graph* $\mathcal{G}$ is a set V, whose elements are called *vertices*, together with a set E of unordered pairs of elements of V, called *edges*. A *vertex coloring of* $\mathcal{G}$ with k colors (briefly, a *k-coloring*) is a function $f : V \to \{1, \ldots, k\}$. We say that f is *proper* if $f(v_1) \neq f(v_2)$ for every edge $\{v_1, v_2\} \in E$. The *chromatic number* of $\mathcal{G}$ is the smallest k such that there is a proper k-coloring of G.

Let $r \leq n \in \mathbb{N}$. The *Kneser graph* $KG(n, r)$ is the graph whose vertices are the r-element subsets of $\{1, \ldots, n\}$, with two vertices $A, B \subseteq \{1, \ldots, n\}$ joined by an edge if and only if $A \cap B = \varnothing$. Note that $KG(n, r)$ has no edges if $2r > n$. M. Kneser proved that the chromatic number of $KG(n, r)$ is no greater than $n - 2r + 2$, and conjectured that this is the correct value when $2r \leq n$. Lovász proved Kneser's conjecture in [20], and Bárány [1] provided a more elementary proof. Greene [12] gave an even shorter proof. Matousek's book [21] gives a detailed exposition of these proofs and subsequent developments.

Theorem 2.2 ([20])**.** *If $2r \leq n \in \mathbb{N}$, then the chromatic number of $KG(n, r)$ is $n - 2r + 2$.*

Our constructions require only the following corollary of Theorem 2.2.

Corollary 2.3. *If $n_k \to \infty$, $r_k \to \infty$, and $r_k^2/n_k \to 0$, then the chromatic number of $KG(2n_k + r_k, n_k)$ tends to ∞.*

The author is unaware of any proof of nontrivial lower bounds for the chromatic number of $KG(n, r)$, besides those that prove Theorem 2.2. A substantially different proof of Corollary 2.3 could lead to progress on problems in recurrence; cf. Remark 8.4 below.

Given an abelian group Γ and a subset $S \subseteq \Gamma$, the *Cayley graph based on S*, denoted $\mathrm{Cay}(S)$, is the graph whose vertex set is Γ, with two vertices x, y joined by an edge if $x - y \in S$ or $y - x \in S$. It follows immediately from the definitions that S is k-chromatically recurrent if and only if the chromatic number of $\mathrm{Cay}(S)$ is strictly greater than k. To prove S is k-chromatically recurrent, it therefore suffices to prove that the chromatic number of $\mathrm{Cay}(S)$ is at least $k + 1$.

Note that a Cayley graph contains loops (i.e. edges of the form $\{x\}$) if and only if $0_\Gamma \in S$. As mentioned above, we are interested only in recurrence properties of sets not containing 0_Γ, so the Cayley graphs we consider will have no loops.

2.3 Hamming balls in $\mathbb{F}_2^d$

For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{F}_2^d$, we define

$$w(\mathbf{x}) := |\{j \leq d : x_j \neq 0\}|.$$

Given $k \leq d$ and $\mathbf{y} \in \mathbb{F}_2^d$, the *Hamming ball of radius k around* $\mathbf{y}$ is

$$H_k(\mathbf{y}) := \{\mathbf{x} \in \mathbb{F}_2^d : w(\mathbf{y} - \mathbf{x}) \leq k\}.$$

So $H_k(\mathbf{y})$ is the set of $\mathbf{x} = (x_1, \dots, x_d)$ such that $x_j \neq y_j$ for at most k coordinates j.

Let $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{F}_2^d$. The following properties are easy to verify from the definitions:

$$H_k(\mathbf{0}) - H_k(\mathbf{0}) = H_{2k}(\mathbf{0}), \tag{2.1}$$

$$H_k(\mathbf{0}) \cap H_r(\mathbf{1}) = \varnothing \text{ iff } k + r < d. \tag{2.2}$$

Lemma 2.4. *Let $k \in \mathbb{N}$ and $\delta < \frac{1}{2}$. If $d \in \mathbb{N}$ is sufficiently large, then $H_k(\mathbf{1}) \subseteq \mathbb{F}_2^d$ is δ-nonrecurrent.*

We will use the following well known estimates on binomial coefficients: setting $M_d := \max_{0 \leq j \leq d} \binom{d}{j}$, we have $\lim_{d \to \infty} M_d / 2^d = 0$. The identities $\binom{d}{j} = \binom{d}{d-j}$ and $\sum_{j=0}^{d} \binom{d}{j} = 2^d$ then imply that for fixed $k \in \mathbb{N}$ and $\delta < \frac{1}{2}$, we have the following for all large enough d:

$$\sum_{j=0}^{\lfloor d/2 \rfloor - k} \binom{d}{j} > \delta 2^d. \tag{2.3}$$

Proof of Lemma 2.4. Fix $k \in \mathbb{N}$ and $\delta < \frac{1}{2}$. Choose d to be large enough that (2.3) holds. Let $A = H_{\lfloor d/2 \rfloor - k}(\mathbf{0})$, so that A is the set of elements $(x_1, \dots, x_d) \in \mathbb{F}_2^d$ having at most $\lfloor d/2 \rfloor - k$ entries equal to 1. Now $|A|$ is given by the sum in (2.3), so $|A| > \delta |\mathbb{F}_2^d|$. Equation (2.1) implies $A - A = H_{2(\lfloor d/2 \rfloor - k)}(\mathbf{0})$, which is disjoint from $H_k(\mathbf{1})$ by (2.2). So we have shown that $H_k(\mathbf{1})$ is δ-nonrecurrent. □

Lemma 2.5. *Let $k, d \in \mathbb{N}$ with $2k \leq d$. The Cayley graph* $\mathrm{Cay}(H_{2k+1}(\mathbf{1}))$ *in $\mathbb{F}_2^d$ contains a subgraph isomorphic to $KG(d, \lfloor d/2 \rfloor - k)$. Consequently, $H_{2k+1}(\mathbf{1})$ is $2k$-chromatically recurrent.*

Proof. Let $\mathcal{G}$ denote the Cayley graph $\mathrm{Cay}(H_{2k+1}(\mathbf{1}))$ and let $r = \lfloor d/2 \rfloor - k$. To each $C \subseteq \{1, \dots, d\}$ having cardinality r, we will assign an element $\mathbf{x}_C \in \mathbb{F}_2^d$ (without repetition), and we will show that if such C, C' are disjoint (meaning they are joined by an edge in $KG(d, r)$), then $\mathbf{x}_C - \mathbf{x}_{C'} \in H_{2k+1}(\mathbf{1})$ (meaning $\mathbf{x}_C$ and $\mathbf{x}_{C'}$ are joined by an edge in $\mathcal{G}$). The set of $\mathbf{x}_C$ so chosen will thereby determine a subgraph of $\mathrm{Cay}(H_{2k+1}(\mathbf{1}))$ isomorphic to $KG(d, r)$.

For each $C \subseteq \{1, \dots, d\}$ having $|C| = r$, we let $\mathbf{x}_C := 1_C$. This is the characteristic function of C, viewed as an element of $\mathbb{F}_2^d$, so $\mathbf{x}_C$ has exactly r entries equal to 1. Now if $C \cap C' = \varnothing$, then $\mathbf{x}_C - \mathbf{x}_{C'}$ has exactly $2r$ entries equal to 1. Since $d - 2r \leq 2k + 1$, this means $\mathbf{x}_C - \mathbf{x}_{C'} \in H_{2k+1}(\mathbf{1})$, so that $\mathbf{x}_C$ and $\mathbf{x}_{C'}$ are joined by an edge of $\mathcal{G}$.

According to Theorem 2.2, the chromatic number of $KG(d, r)$ is $d - 2r + 2$. With $r = \lfloor d/2 \rfloor - k$, we get $d - 2r + 2 \geq 2k + 1$, so $\mathcal{G}$ has chromatic number at least $2k + 1$. This implies that $H_{2k+1}(\mathbf{1})$ is $2k$-chromatically recurrent. $\square$

Proof of Theorem 2.1. Fix $\delta \in (0, \frac{1}{2})$ and $k \in \mathbb{N}$. Combining Lemmas 2.4 and 2.5, we get that for all sufficiently large d, the Hamming ball $H_{2k+1}(\mathbf{1}) \subseteq \mathbb{F}_2^d$ is δ-nonrecurrent and k-chromatically recurrent. $\square$

3 Proof of Theorems 1.1 and 1.2

3.1 Assembling finite pieces

We now return to $\mathbb{Z}$ and prove Theorems 1.1 and 1.2. These involve sets which are not density recurrent, but do have some other recurrence property. Lemma 3.6 provides a general procedure for building such sets from finite pieces. To make this idea precise, we need the following definitions.

Let $S \subseteq \mathbb{Z}$. We say that S is

- *δ-density recurrent* if $(A - A) \cap S \neq \varnothing$ for all $A \subseteq \mathbb{Z}$ having $\bar{d}(A) > \delta$.
- *δ-nonrecurrent* if there exists $A \subseteq \mathbb{Z}$ having $\bar{d}(A) > \delta$ such that $(A - A) \cap S = \varnothing$. In other words, S is not δ-density recurrent.
- *k-chromatically recurrent* if for every partition of $\mathbb{Z}$ into k sets $A_1, A_2, \dots, A_k$, the intersection $(A_j - A_j) \cap S$ is nonempty for some $j \leq k$.

  Equivalently, S is k-chromatically recurrent if for every function $f : \mathbb{Z} \to \{1, \dots, k\}$, there exist $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$ and $b - a \in S$.

Remark 3.1. Note that $(A - A) \cap S = \varnothing$ if and only if $A \cap (A + S) = \varnothing$. This equivalence will be used from time to time without comment. Likewise we will use the following trivial observations, true for all $A, B \subseteq \mathbb{Z}$ and $t \in \mathbb{Z}$:

- $(A - t) - (A - t) = A - A$;
- $(A \cap B) - t = (A - t) \cap (B - t)$.

If $m \in \mathbb{N}$, we write $[m]$ for the interval $\{0, 1, \ldots, m-1\}$ in $\mathbb{Z}$.

Definition 3.2. If $m \in \mathbb{N}$ and $B \subseteq [m]$, we say that (B, m) *witnesses the δ-nonrecurrence of* S if $|B| > \delta m$ and $B \cap (B+S) = \varnothing$, $B+S \subseteq [m]$, and $B+S+S \subseteq [m]$.

Note that if there is an $m \in \mathbb{N}$ and $B \subseteq [m]$ such that (B, m) witnesses the δ-nonrecurrence of S, then S is δ-nonrecurrent: the set $B' := B + m\mathbb{Z}$ has density $d(B') = \frac{1}{m}|B| > \delta$, and the conditions $B, B+S \subseteq [m]$ imply $B' \cap (B'+S) = \big(B \cap (B+S)\big) + m\mathbb{Z}$. Since $B \cap (B+S) = \varnothing$, we have $B' \cap (B'+S) = \varnothing$.

The condition $B+S+S \subseteq [m]$ may seem unmotivated, but it will be used in the proof of Lemma 3.6.

Lemma 3.3. *If $A \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, then there is a $t \in \mathbb{Z}$ such that $|A \cap ([m]+t)| \geq \bar{d}(A)m$.*

Proof. Let $m \in \mathbb{N}$ and let

$$\delta := \sup_{t \in \mathbb{Z}} \frac{|A \cap ([m]+t)|}{m}, \tag{3.1}$$

so that $|A \cap ([m]+t)| \leq \delta m$ for every $t \in \mathbb{Z}$. Given $N \in \mathbb{N}$, write $[N]$ as a union of $\lfloor N/m \rfloor$ mutually disjoint intervals $I_1, \ldots, I_{\lfloor N/m \rfloor}$ of length m, together with another (possibly empty) interval $I_{\lfloor N/m \rfloor + 1}$ of length at most m. Then $|A \cap I_j| \leq \delta m$ for every $j \leq \lfloor N/m \rfloor$, so

$$|A \cap [N]| \leq \sum_{j=1}^{\lfloor N/m \rfloor + 1} |A \cap I_j| \leq \lfloor N/m \rfloor \delta m + m \leq \delta N + m.$$

Dividing the above expressions by N, we see that $\limsup_{N\to\infty} \frac{|A \cap [N]|}{N} \leq \delta$, meaning $\bar{d}(A) \leq \delta$. Since there are only finitely many possible values of $|A \cap ([m]+t)|$, the supremum in (3.1) is attained. □

Lemma 3.4. *If $S \subseteq \mathbb{N}$ is finite and δ-nonrecurrent, then for all sufficiently large m, there is a set $B \subseteq [m]$ such that (B, m) witnesses the δ-nonrecurrence of S.*

Proof. Assuming $\delta > 0$ and that S is finite and δ-nonrecurrent, there is set $A \subseteq \mathbb{Z}$ such that $\bar{d}(A) > \delta$ and $(A-A) \cap S = \varnothing$. Fix such an A. Let $k = \max S$, and choose m_0 sufficiently large that $\bar{d}(A)m - 2k > \delta m$ whenever $m > m_0$.

Let $m > m_0$. We will find a set $B \subseteq [m]$ such that (B, m) witnesses the δ-nonrecurrence of S. By Lemma 3.3, choose $t \in \mathbb{Z}$ so that $|A \cap ([m]+t)| \geq \bar{d}(A)m$. Then $|(A-t) \cap [m]| \geq \bar{d}(A)m$. Let $B = (A-t) \cap [m-2k]$, so that $|B| \geq \bar{d}(A)m - 2k$. Then $|B| > \delta m$, by our choice of m_0. Since B is contained in a translate of A, we have $B - B \subseteq A - A$, meaning $B - B$ is disjoint from S. The containment $B \subseteq [m-2k]$ implies $B+S$ and $B+S+S$ are both contained in $[m]$. □

The next lemma will be used in conjunction with Lemma 3.6 below. If $S \subseteq \mathbb{Z}$ and $m \in \mathbb{N}$, we write mS for $\{mn : n \in S\}$.

Lemma 3.5. *Let $k, m \in \mathbb{N}$. If $S \subseteq \mathbb{Z}$ is k-chromatically recurrent then so is mS.*

Proof. Suppose $m \in \mathbb{N}$, S is k-chromatically recurrent, and let $f : \mathbb{Z} \to \{1, \ldots, k\}$. Form a new coloring $\tilde{f} : \mathbb{Z} \to \{1, \ldots, k\}$ by $\tilde{f}(n) = f(mn)$. Since S is k-chromatically recurrent, there exists $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $\tilde{f}(a) = \tilde{f}(b)$. This means $f(ma) = f(mb)$, and $mb - ma \in mS$. Since f was an arbitrary function into $\{1, \ldots, k\}$, this shows that mS is k-chromatically recurrent. □

Here is the first key lemma for our constructions, a variant of Lemma 3.2 in [18]. We continue to use $[m]$ to denote $\{0, \ldots, m-1\}$.

Lemma 3.6. *Let $\delta, \eta \in (0, \frac{1}{2})$, and let $E, F \subseteq \mathbb{N}$ be finite sets which are δ-nonrecurrent and η-nonrecurrent, respectively. If (A, m) witnesses the δ-nonrecurrence of E, then for all sufficiently large $l \in \mathbb{N}$, there exists $C \subseteq [lm]$ with $A \subseteq C$ such that (C, lm) witnesses the $2\delta\eta$-nonrecurrence of $E \cup mF$. Consequently, for all sufficiently large m, $E \cup mF$ is $2\delta\eta$-nonrecurrent.*

In practice we will apply Lemma 3.6 with η close to $\frac{1}{2}$, so that $2\delta\eta$ will be close to δ.

Proof. Let E, F, A, and m be as in the hypothesis of the lemma. Since (A, m) witnesses the δ-nonrecurrence of E, we have $A \subseteq [m]$ and $|A| > \delta m$. Write $|A|$ as $\delta' m$, so that $\delta' > \delta$. Let $k = \max(E \cup mF)$, and choose l_0 so that for all $l > l_0$, we have

$$2\delta'\eta lm - 2k > 2\delta\eta lm. \tag{3.2}$$

Let $l \in \mathbb{N}$ be greater than l_0 and large enough (by Lemma 3.4) that there is a set $B \subseteq [l]$ such that (B, l) witnesses the η-nonrecurrence of F; fix such a B. We will form C as a union of translates of A. Each such translate $A + t$ will lie in one of the mutually disjoint intervals

$$I_0 = [0, m-1], I_1 = [m, 2m-1], \ldots, I_{l-1} = [(l-1)m, lm-1],$$

and will be arranged so that if $A + t \subseteq I_j$, then $A + t + E \subseteq I_j$. First fix arbitrary elements $e_0 \in E$ and $f_0 \in F$. Let

$$C_1 := \bigcup_{b \in B} A + mb, \qquad C_2 := \bigcup_{b \in B} A + e_0 + m(b + f_0), \qquad C := (C_1 \cup C_2) \cap [lm - 2k].$$

We claim that every element of C can be written uniquely as

$$a + qe_0 + m(b + qf_0), \qquad \text{where } a \in A, b \in B, q \in \{0, 1\}. \tag{3.3}$$

The existence of such a representation is evident from the definition of C. To prove uniqueness, assume that $a, a' \in A$, $b, b' \in B$, $q, q' \in \{0, 1\}$, and

$$a + qe_0 + m(b + qf_0) = a' + q'e_0 + m(b' + q'f_0), \tag{3.4}$$

with the aim of proving $a = a'$, $b = b'$, and $q = q'$. First observe that every element of $[ml]$ can be written uniquely as $s + mt$ with $s \in [m]$ and $t \in [l]$. Since (A, m) witnesses the δ-nonrecurrence of S, we have $a + qe_0, a' + q'e_0 \in [m]$. Likewise $b + qf_0, b' + q'f_0 \in [l]$, so $a + qe_0 = a' + q'e_0$, meaning $a - a' = (q' - q)e_0$. This implies $a - a' \in \{0\} \cup \pm E$, so the assumption $(A - A) \cap E = \varnothing$ implies $a = a'$, whence $q = q'$ as well. Equation (3.4) then simplifies to $a + qe_0 + m(b + qf_0) = a + qe_0 + m(b' + qf_0)$, implying $b = b'$.

We will prove that

$$|C| > 2\delta\eta lm, \tag{3.5}$$

$$C + (E \cup mF), C + (E \cup mF) + (E \cup mF) \subseteq [lm], \tag{3.6}$$

$$C \cap \big(C + (E \cup mF)\big) = \varnothing. \tag{3.7}$$

The unique representation of elements of C in (3.3) implies $|C| \geq 2|A||B| - 2k$ (the subtraction accounts for the containment in $[lm - 2k]$). Our choice of A and B together with (3.2) implies $2|A||B| - 2k > 2\delta'\eta lm - 2k > 2\delta\eta lm$; combined with the lower bound on $|C|$ this proves (3.5). The containment (3.6) follows from the containment $C \subseteq [lm - 2k]$ and our choice of k.

We now prove (3.7) by showing $C \cap (C + E) = \varnothing$ and $C \cap (C + mF) = \varnothing$. First assume, to get a contradiction, that $C \cap (C + E) \neq \varnothing$. Then there are $c, c' \in C$ and $e \in E$ such that $c = c' + e$. Representing c and c' as in (3.3), we have

$$a + qe_0 + m(b + qf_0) = a' + q'e_0 + e + m(b' + q'f_0) \tag{3.8}$$

where $a, a' \in A$, $b, b' \in B$, and $q, q' \in \{0, 1\}$. Since we assumed $A+E, A+E+E \subseteq [m]$, we see that $a + qe_0$ and $a' + q'e_0 + e$ belong to $[m]$, while $b + qf_0$ and $b' + q'f_0$ likewise belong to $[l]$, so (3.8) and uniqueness of the representation in (3.3) implies

$$a + qe_0 = a' + q'e_0 + e, \tag{3.9}$$

$$b + qf_0 = b' + q'f_0. \tag{3.10}$$

Rewriting (3.10), we get $b - b' = (q - q')f_0$. This implies $q = q'$, since otherwise we get $b - b' = \pm f_0$, contradicting our assumption that $(B - B) \cap F = \varnothing$. With $q = q'$ equation (3.9) simplifies to $a = a' + e$. This contradicts our assumption that $A \cap (A + E) = \varnothing$.

To prove that $C \cap (C + mF) = \varnothing$, we argue as above and arrive at the equations $a + qe_0 = a' + q'e_0$ and $b + qf_0 = b' + q'f_0 + f$. As above, the first equation implies $q = q'$, so $b = b' + f$, which contradicts our assumption that $B \cap (B + F) = \varnothing$. This completes the proof of (3.7).

Since C is a union of translates of A, we may replace C with $C - \min(C) + \min(A)$ to get $A \subseteq C$ and maintain the inclusions $C \subseteq [lm]$ and (3.6). Together with (3.5) and (3.7), this shows that (C, lm) witnesses the $2\delta\eta$-nonrecurrence of $E \cup mF$. $\square$

3.2 Proof of Theorem 1.1

Here is our second key ingredient, which we prove in §6.

Lemma 3.7. *For all $k \in \mathbb{N}$ and all $\delta \in (0, \frac{1}{2})$, there is a finite set $S \subseteq \mathbb{Z}$ such that S is k-chromatically recurrent and δ-nonrecurrent.*

We now prove Theorem 1.1 by combining this lemma with Lemma 3.6. The proof will use no additional information about the sets provided by Lemma 3.7.

Proof of Theorem 1.1. Let $\delta \in (0, \frac{1}{2})$. We will find a chromatically recurrent set S and a set $C \subseteq \mathbb{Z}$ with $\bar{d}(C) \geq \delta$ such that $(C - C) \cap S = \varnothing$, meaning S is not density recurrent. To build S and C, we will use Lemmas 3.6 and 3.7 to find increasing sequences of sets $S_1 \subseteq S_2 \subseteq \dots$, $C_1 \subseteq C_2 \subseteq \dots$, and intervals $[m_1], [m_2], \dots$, with $m_k \to \infty$, so that the following conditions hold for all $k \in \mathbb{N}$:

(i) S_k is k-chromatically recurrent,

(ii) $C_k \subseteq [m_k]$, $C_k + S_k \subseteq [m_k]$, $C_k + S_k + S_k \subseteq [m_k]$, and $|C_k| > \delta m_k$,

(iii) $(C_k - C_k) \cap S_k = \varnothing$.

Having constructed these, we let $C := \bigcup_{k=1}^{\infty} C_k$ and $S := \bigcup_{k=1}^{\infty} S_k$. Then $(C-C) \cap S = \varnothing$; otherwise for some $k \in \mathbb{N}$ we would have $c - c' = s$ for some $c, c' \in C_k$ and $s \in S_k$. Item (ii) implies $\bar{d}(C) \geq \delta$, as $\frac{|C_k \cap [m_k]|}{m_k} > \delta$ for every k. Item (i) implies S is chromatically recurrent, being k-chromatically recurrent for every $k \in \mathbb{N}$.

To find S_k and C_k, we first choose $S_1 = \{1\}$, $m_1 = 2$, and $C_1 = \{0\}$, so that (i)-(iii) are trivially satisfied with $k = 1$.

To perform the inductive step, we assume the sets S_k, C_k, and the integer m_k have been constructed to satisfy (i)-(iii), and choose $\delta_k > \delta$ so that $|C_k| > \delta_k m_k$. Since $\delta_k > \delta$, we may choose $\eta < \frac{1}{2}$ so that $2\delta_k \eta > \delta$. Lemma 3.7 provides a finite set S' which is $(k+1)$-chromatically recurrent and η-nonrecurrent. Apply Lemma 3.6 to find $l \geq 2$ and C_{k+1} such that $C_k \subseteq C_{k+1} \subseteq [lm_k]$, while (C_{k+1}, lm_k) witnesses the $2\delta_k\eta$-nonrecurrence of $S_k \cup m_k S'$. Finally, Lemma 3.5 implies $m_k S'$ is $(k+1)$-chromatically recurrent. Setting $m_{k+1} = lm_k$ and $S_{k+1} = S_k \cup m_k S'$, we get that (i)-(iii) are satisfied with $k+1$ in place of k. □

3.3 Proof of Theorem 1.2

The following modification of Lemma 3.7 is proved in §6.

Lemma 3.8. *Let $k, m \in \mathbb{N}$ and $\delta < \frac{1}{2}$. If $E \subseteq \mathbb{Z}$ is infinite, then there is a finite δ-nonrecurrent set $S \subseteq \mathbb{N}$ such that $mS \subseteq E - E$ and mS is k-chromatically recurrent.*

Theorem 1.2 may be proved by following the proof of Theorem 1.1 verbatim, with two modifications:

- Let $S_1 = \{t\}$ where $t \in E - E$ satisfies $t > (\frac{1}{2} - \delta)^{-1}$, let $C_1 = [t-1]$, and let $m_1 = 2t$. Then $|C_1|/m_1 = (t-1)/(2t) = \frac{1}{2} - \frac{1}{2t} > \delta$, and it is easy to check that (i)-(iii) are satisfied with $k = 1$.

- Cite Lemma 3.8 in place of Lemma 3.7, noting in the inductive step that if $S_k \subseteq E - E$, then $S_{k+1} \subseteq E - E$. □

4 Chromatic recurrence and δ-nonrecurrence in $\mathbb{T}^d$

The remainder of this article contains the proofs of Lemmas 3.7 and 3.8. Very roughly, the proofs proceed by copying the Hamming balls $H_k(\mathbf{1})$ from $\mathbb{F}_2^d$ into $\mathbb{Z}$, passing through $\mathbb{T}^d$ as an intermediate step. To be more precise, we fix some notation.

Let $\mathbb{R}$ denote the real numbers with the usual topology and let $\mathbb{T}$ denote $\mathbb{R}/\mathbb{Z}$ with the quotient topology. For $x \in \mathbb{T}$, let $\tilde{x}$ be the unique element of $[0, 1)$ such that $\tilde{x} + \mathbb{Z} = x$, and write $\|x\|$ for $\min_{n\in\mathbb{Z}} |\tilde{x} - n|$.

When defining subsets of $\mathbb{T}$, we identify subintervals of $\mathbb{R}$ with their images in $\mathbb{T}$ under the quotient map.

Let $G_d := \{0, 1/2\}^d \subseteq \mathbb{T}^d$, so that G_d is a subgroup isomorphic to $\mathbb{F}_2^d$. With the natural identification, $\mathbf{1} \in \mathbb{F}_2^d$ is identified with $\frac{\mathbf{1}}{\mathbf{2}} := (1/2, \dots, 1/2) \in \mathbb{T}^d$. We will use the notation H_k for Hamming balls around elements of G_d, so $H_k(\frac{\mathbf{1}}{\mathbf{2}})$ is, by definition, the set of $(x_1, \dots, x_d) \in G_d$ where at most k entries x_i are not equal to $1/2$.

Let V_ε denote the open box $\{\mathbf{x} \in \mathbb{T}^d : \max \|x_j\| < \varepsilon\}$. For $\boldsymbol{\alpha} \in \mathbb{T}^d$, define

$$\tilde{H}(\boldsymbol{\alpha}; k, \varepsilon) := \{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in H_k(\tfrac{\mathbf{1}}{\mathbf{2}}) + V_\varepsilon\}. \tag{4.1}$$

When $\boldsymbol{\alpha} \in \mathbb{T}^d$ and $E \subseteq \mathbb{Z}$, we write $E\boldsymbol{\alpha}$ for $\{n\boldsymbol{\alpha} : n \in E\}$.

Lemma 4.1. *(i) Let $\delta < \frac{1}{2}$ and $k \in \mathbb{N}$. For all sufficiently large $d \in \mathbb{N}$ there is an $\varepsilon > 0$ such that for all $\boldsymbol{\alpha} \in \mathbb{T}^d$, the set $\tilde{H}(\boldsymbol{\alpha}; k, \varepsilon)$ is δ-nonrecurrent.*

(ii) If $\mathbb{Z}\boldsymbol{\alpha}$ is dense in $\mathbb{T}^d$ and $\varepsilon > 0$, then there is a finite subset of $\tilde{H}(\boldsymbol{\alpha}; 2k + 1, \varepsilon)$ which is $2k$-chromatically recurrent.

(iii) If $E \subseteq \mathbb{Z}$ is such that $E\boldsymbol{\alpha}$ is dense in $\mathbb{T}^d$ and $\varepsilon > 0$, then there is a finite subset of $\tilde{H}(\boldsymbol{\alpha}; 2k + 1, \varepsilon) \cap (E - E)$ which is $2k$-chromatically recurrent.

Parts (ii) and (iii) will be proved in §5. The remainder of this section is dedicated to the proof of Part (i).

Given $\varepsilon > 0$, let $I_\varepsilon = [\varepsilon, \frac{1}{2} - \varepsilon] \subseteq \mathbb{T}$, and let $I_\varepsilon^d \subseteq \mathbb{T}^d$ be its d-fold cartesian power. Observe that

$$\text{the sets } \mathbf{t} + I_\varepsilon^d,\ \mathbf{t} \in G_d, \text{ are mutually disjoint.} \tag{4.2}$$

Write μ for Haar probability measure on $\mathbb{T}^d$. Given $A \subseteq G_d$, let $A_\varepsilon^\square := A + I_\varepsilon^d$, so that (4.2) implies

$$\mu(A_\varepsilon^\square) = |A|\big(\tfrac{1}{2} - 2\varepsilon\big)^d. \tag{4.3}$$

Lemma 4.2. *If $\mathbf{t} \in G_d$ and $A \subseteq G_d$, then $A_\varepsilon^\square \cap (A_\varepsilon^\square + \mathbf{t}) = (A \cap (A + \mathbf{t}))_\varepsilon^\square$.*

Proof. We prove only that $A_\varepsilon^\square \cap (A_\varepsilon^\square + \mathbf{t}) \subseteq (A \cap (A + \mathbf{t}))_\varepsilon^\square$, as the reverse containment is easy to check.

Assuming $\mathbf{x} \in A_\varepsilon^\square$ and $\mathbf{x} \in (A_\varepsilon^\square + \mathbf{t})$, there are $\mathbf{a}, \mathbf{a}' \in A$ such that $\mathbf{x} \in \mathbf{a} + I_\varepsilon^d$ and $\mathbf{x} \in \mathbf{a}' + \mathbf{t} + I_\varepsilon^d$. Thus $\mathbf{x} \in (\mathbf{a} + I_\varepsilon^d) \cap (\mathbf{a}' + \mathbf{t} + I_\varepsilon^d)$, and the mutual disjointness observed in (4.2) implies $\mathbf{a} = \mathbf{a}' + \mathbf{t}$. It follows that $\mathbf{x} \in (A \cap (A + \mathbf{t}))_\varepsilon^\square$, as desired. $\square$

The important consequence of Lemma 4.2 is that if $A, R \subseteq G_d$ with $A \cap (A+R) = \varnothing$, then $A_\varepsilon^\square \cap (A_\varepsilon^\square + R) = \varnothing$.

A *box* J in $\mathbb{T}^d$ is a product $J_1 \times \cdots \times J_d$ of intervals $J_i \subseteq \mathbb{T}$; we say that J is open (closed) if each J_i is an open (closed) interval. Note that the sets $A_\varepsilon^\square$ defined above are finite disjoint unions of closed boxes.

The following standard fact about uniform distribution is proved in §7.

Lemma 4.3. *Let $\boldsymbol{\alpha} \in \mathbb{T}^d$ be such that $\mathbb{Z}\boldsymbol{\alpha}$ is dense in $\mathbb{T}^d$. If $A \subseteq \mathbb{T}^d$ is a finite disjoint union of boxes and $C := \{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in A\}$ then $d(C) = \mu(A)$, where μ is Haar probability measure on $\mathbb{T}^d$.*

Lemma 4.4. *Let B, $U \subseteq \mathbb{T}^d$ and assume $\mathbb{Z}\boldsymbol{\alpha}$ is dense in $\mathbb{T}^d$. If B is a finite disjoint union of boxes with $\mu(B) > \delta$ and $B \cap (B+U) = \varnothing$, then $S := \{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in U\}$ is δ-nonrecurrent.*

Proof. Let $C := \{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in B\}$. By Lemma 4.3, $d(C) = \mu(B) > \delta$. To see that $C \cap (C+S) = \varnothing$, note that if $n \in C \cap (C+S)$, we have $n\boldsymbol{\alpha} \in B$ and $n\boldsymbol{\alpha} \in B+U$, meaning $n\boldsymbol{\alpha} \in B \cap (B+U)$, which violates our hypothesis. □

Lemma 4.5. *If $R \subseteq G_d$ is δ-nonrecurrent and $\boldsymbol{\alpha} \in \mathbb{T}^d$, then for sufficiently small ε, $S := \{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in R + V_\varepsilon\}$ is δ-nonrecurrent.*

Proof. Let $A \subseteq G_d$ be such that $A \cap (A+R) = \varnothing$ and $|A| > \delta 2^d$. Choose $\varepsilon' > 0$ small enough that $|A|\left(\frac{1}{2} - 2\varepsilon'\right)^d > \delta$. Let $B := A_{\varepsilon'}^\square$, so that $\mu(B) = |A|\left(\frac{1}{2} - 2\varepsilon'\right)^d > \delta$. Now $B \cap (B+R) = \varnothing$, by Lemma 4.2. Since B is compact, we also have $B \cap (B+R+V_\varepsilon) = \varnothing$, provided ε is sufficiently small. Lemma 4.4 then implies S is δ-nonrecurrent. □

Proof of Lemma 4.1 Part (i). Let $\delta < \frac{1}{2}$ and $k \in \mathbb{N}$. By Lemma 2.4, choose $d \in \mathbb{N}$ so that $H_k(\mathbf{1})$ is a δ-nonrecurrent subset of $\mathbb{F}_2^d$. Then $H_k(\frac{\mathbf{1}}{\mathbf{2}})$ is a δ-nonrecurrent subset of G_d, and Lemma 4.5 provides an $\varepsilon > 0$ so that $\{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in H_k(\frac{\mathbf{1}}{\mathbf{2}}) + V_\varepsilon\}$ is δ-nonrecurrent. This δ-nonrecurrent set is, by definition, $\tilde{H}(\boldsymbol{\alpha}; k, \varepsilon)$. □

5 Lifting chromatic recurrence from topological groups

Here we show how a homomorphism from $\mathbb{Z}$ into a topological abelian group K can be used to copy Cayley graphs from K into $\mathbb{Z}$; we maintain the conventions of §2.2.

For this section we fix a discrete abelian group G, a Hausdorff topological abelian group K (not necessarily compact, not necessarily metrizable), and a homomorphism $\rho : G \to K$ with $\rho(G)$ dense in K. We also fix $E \subseteq G$ with $\rho(E)$ dense in K. The only case we will use in this article is $G = \mathbb{Z}$, $K = \mathbb{T}^d$, and $\rho(n) = n\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{T}^d$, so readers can specialize to this setting at will.

Lemma 5.1. *Assume G, ρ, E, and K are as specified above. If $U \subseteq K$ is open, then every finite subgraph $\mathcal{G}$ of* $\mathrm{Cay}(U)$ *has an isomorphic copy in* $\mathrm{Cay}(\rho^{-1}(U) \cap (E-E))$.

Consequently, if $\mathrm{Cay}(U)$ *has a finite subgraph with chromatic number $\geq k$, then* $\mathrm{Cay}(\rho^{-1}(U) \cap (E-E))$ *has chromatic number $\geq k$.*

Proof. To prove the first statement of the lemma it suffices to prove that if V is a finite subset of K, then there exist $\{g_v : v \in V\} \subseteq E$ such that $g_v \neq g_{v'}$ if $v \neq v'$, and for each $v, v' \in V$, we have

$$v - v' \in U \implies g_v - g_{v'} \in \rho^{-1}(U). \tag{5.1}$$

So let V be a finite subset of K. Let $S := (V - V) \cap U$, and let W be a neighborhood of 0 in K so that $S + W \subseteq U$. Choose a neighborhood W' of 0 so that $W' - W' \subseteq W$, and such that the translates $v + W'$, $v \in V$ are mutually disjoint.

For each $v \in V$, choose $g_v \in E$ so that $\rho(g_v) \in v + W'$; this is possible since $\rho(E)$ is dense in K. Disjointness of the translates $v + W'$ guarantees that $g_v \neq g_{v'}$ if $v \neq v'$. We now prove (5.1) holds with these g_v. Assuming $v - v' \in U$, we have

$$\rho(g_v) - \rho(g_{v'}) \in v + W' - (v' + W') = (v - v') + (W' - W') \subseteq v - v' + W \subseteq U,$$

so $g_v - g_{v'} \in \rho^{-1}(U)$. This proves (5.1).

Since chromatic number is invariant under isomorphism of graphs, the second assertion of the lemma follows immediately from the first. □

Proof of Lemma 4.1 Parts (ii) and (iii). Part (ii) follows from Part (iii), so we only prove Part (iii).

Let $2k \leq d \in \mathbb{N}$, let $E \subseteq \mathbb{Z}$, and let $\boldsymbol{\alpha} \in \mathbb{T}^d$ with $E\boldsymbol{\alpha}$ dense in $\mathbb{T}^d$. We use Lemma 5.1 with the open set $U = H_{2k+1}(\frac{\mathbf{1}}{\mathbf{2}}) + V_\varepsilon$ and $\rho(n) := n\boldsymbol{\alpha}$. Since $H_{2k+1}(\frac{\mathbf{1}}{\mathbf{2}}) \subseteq U$, $\mathrm{Cay}(U)$ contains the finite subgraph $\mathrm{Cay}(H_{2k+1}(\frac{\mathbf{1}}{\mathbf{2}}))$ (with vertex set G_d). The latter graph is isomorphic to $\mathrm{Cay}(H_{2k+1}(\mathbf{1}))$, and by Lemma 2.5, it has chromatic number at least $2k+1$. Now Lemma 5.1 implies $\mathrm{Cay}(\rho^{-1}(U) \cap (E-E))$ has chromatic number at least $2k+1$. Examining the definitions, we see that $\rho^{-1}(U) = \tilde{H}(\boldsymbol{\alpha}; 2k+1, \varepsilon)$, so $\tilde{H}(\boldsymbol{\alpha}; 2k+1, \varepsilon) \cap (E - E)$ is $2k$-chromatically recurrent. □

6 Proof of Lemmas 3.7 and 3.8

Lemma 3.7 is an immediate consequence of Parts (i) and (ii) of Lemma 4.1. For Lemma 3.8, we need an elementary result on recurrence properties of dilates. Given $S \subseteq \mathbb{Z}$ write S/m for the set $\{n \in \mathbb{Z} : mn \in S\}$ $(= \{n/m : n \in S\} \cap \mathbb{Z})$.

Lemma 6.1. *If $m \in \mathbb{N}$ and $S \subseteq \mathbb{Z}$ is δ-nonrecurrent, then S/m is also δ-nonrecurrent.*

Proof. Let $S \subseteq \mathbb{Z}$ be δ-nonrecurrent, and choose $A \subseteq \mathbb{Z}$ such that $(A - A) \cap S = \varnothing$ and $\bar{d}(A) > \delta$. The subadditivity of $\bar{d}$ implies that for some $t \in \{0, \ldots, m-1\}$, the set $A_t := A \cap (t + m\mathbb{Z})$ satisfies $\bar{d}(A_t) > \delta/m$. The elements of $A_t - t$ are all divisible by m, so $A' := (A_t - t)/m$ satisfies $\bar{d}(A') > \delta$. Furthermore, $m(A' - A') \subseteq A - A$, so $m(A' - A')$ is disjoint from S, meaning $A' - A'$ is disjoint from S/m. □

We recall the statement of Lemma 3.8: if $k, m \in \mathbb{N}$, $\delta < \frac{1}{2}$, and $E \subseteq \mathbb{Z}$ is an infinite set, then there is a finite δ-nonrecurrent set $S \subseteq \mathbb{N}$ such that mS is k-chromatically recurrent and $mS \subseteq E - E$. We need the following lemma, which we prove in §7.

Lemma 6.2. *Let $E \subseteq \mathbb{Z}$ be an infinite set and $d \in \mathbb{N}$. Then there is an $\boldsymbol{\alpha} \in \mathbb{T}^d$ such that $E\boldsymbol{\alpha} := \{n\boldsymbol{\alpha} : n \in E\}$ is topologically dense in $\mathbb{T}^d$.*

Proof of Lemma 3.8. Fix $k, m \in \mathbb{N}$ and $\delta < \frac{1}{2}$. Let $E \subseteq \mathbb{Z}$ be infinite. Let $E' \subseteq E$ be an infinite subset with $m|(b-a)$ for all $a, b \in E'$ (i.e. the elements of E' are mutually congruent mod m).

By Part (i) of Lemma 4.1, choose $d \in \mathbb{N}$ large enough and $\varepsilon > 0$ small enough that for all $\boldsymbol{\alpha} \in \mathbb{T}^d$, $\tilde{H}_{\boldsymbol{\alpha}} := \tilde{H}(\boldsymbol{\alpha}; 2k+1, \varepsilon)$ is δ-nonrecurrent. Using Lemma 6.2, we fix $\boldsymbol{\alpha} \in \mathbb{T}^d$ such that $E'\boldsymbol{\alpha}$ is dense in $\mathbb{T}^d$. We then apply Part (iii) of Lemma 4.1 to find a finite subset $S_0 \subseteq \tilde{H}_{\boldsymbol{\alpha}} \cap (E' - E')$ which is $2k$-chromatically recurrent.

Since the elements of E' are mutually congruent mod m, we have $S_0 \subseteq m\mathbb{Z}$, and the previous paragraph shows that S_0 is $2k$-chromatically recurrent and δ-nonrecurrent. Let $S := S_0/m$. By Lemma 6.1, S is also δ-nonrecurrent. Now S is the desired set: we have $mS = S_0 \subseteq E' - E' \subseteq E - E$, S is δ-nonrecurrent, and mS is $2k$-chromatically recurrent. □

7 Uniform distribution and denseness in $\mathbb{T}^d$

The material in this section is due to Kronecker and to Weyl; the standard reference is [17]. We present it here to make the proof of Lemmas 4.3 and 6.2 self-contained. For this section we fix $d \in \mathbb{N}$ and write μ for Haar probability measure on $\mathbb{T}^d$.

7.1 Uniform distribution

A sequence $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$ of points in $\mathbb{T}^d$ is called *uniformly distributed* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\boldsymbol{\alpha}_n) = \int f \, d\mu$$

for every continuous $f : \mathbb{T}^d \to \mathbb{C}$. A straightforward approximation argument shows that a sequence $(\boldsymbol{\alpha}_n)_{n\in\mathbb{N}}$ is uniformly distributed if and only if for every box $R := \prod_{j=1}^{d} [a_j, b_j]$ in $\mathbb{T}^d$, we have

$$\lim_{N\to\infty} \frac{|\{n \in \{1, \ldots, N\} : \boldsymbol{\alpha}_n \in R\}|}{N} = \mu(R).$$

Consequently, the terms of a uniformly distributed sequence form a dense subset of $\mathbb{T}^d$.

The above criterion immediately yields the following lemma.

Lemma 7.1. *If $A \subseteq \mathbb{T}^d$ is a finite disjoint union of boxes and $(\boldsymbol{\alpha}_n)_{n\in\mathbb{N}}$ is uniformly distributed then $C := \{n : \boldsymbol{\alpha}_n \in A\}$ satisfies $d(C) = \mu(A)$.*

We write $\mathcal{S}^1$ for the group of complex numbers of modulus 1; the group operation on $\mathcal{S}^1$ is multiplication and the topology is the one inherited from $\mathbb{C}$. A *character* of

$\mathbb{T}^d$ is a continuous homomorphism $\chi : \mathbb{T}^d \to \mathcal{S}^1$; the set of characters of $\mathbb{T}^d$ is denoted by $\widehat{\mathbb{T}}^d$. A well known fact from basic harmonic analysis is that every character of $\mathbb{T}^d$ has the form $\chi(x_1, \dots, x_d) := \exp(2\pi i(n_1x_1 + \cdots + n_dx_d))$, where $n_1, \dots, n_d$ are integers. The *trivial character* satisfies $\chi(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{T}^d$. It is easy to check that if χ is a nontrivial character and $m \in \mathbb{Z} \setminus \{0\}$ then $\mathbf{x} \mapsto \chi(m\mathbf{x})$ is another nontrivial character. Furthermore, the characters of $\mathbb{T}^d$ are mutually orthogonal in $L^2(\mu)$.

The *Weyl criterion* [17, p. 62] for uniform distribution says that a sequence of points $(\boldsymbol{\alpha}_n)_{n\in\mathbb{N}}$ in $\mathbb{T}^d$ is uniformly distributed if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(\boldsymbol{\alpha}_n) = 0 \quad \text{for every nontrivial character } \chi \in \widehat{\mathbb{T}}^d. \tag{7.1}$$

We now use Weyl criterion to prove Lemma 4.3, which says that if $(n\boldsymbol{\alpha})_{n\in\mathbb{N}}$ is dense in $\mathbb{T}^d$, $A \subseteq \mathbb{T}^d$ is a finite disjoint union of boxes, and $C := \{n \in \mathbb{Z} : n\boldsymbol{\alpha} \in A\}$, then $d(C) = \mu(A)$.

Proof of Lemma 4.3. By Lemma 7.1, it suffices to prove that if $(n\boldsymbol{\alpha})_{n\in\mathbb{N}}$ is dense in $\mathbb{T}^d$, then (7.1) is satisfied by $\boldsymbol{\alpha}_n = n\boldsymbol{\alpha}$. To prove this, we need only show that $\chi(\boldsymbol{\alpha}) \neq 1$, for then we may apply the identity $\chi(n\boldsymbol{\alpha}) = \chi(\boldsymbol{\alpha})^n$ and simplify the partial sums $S_N := \sum_{n=1}^{N} \chi(n\boldsymbol{\alpha})$ as $\chi(\boldsymbol{\alpha})\frac{1-\chi(\boldsymbol{\alpha})^N}{1-\chi(\boldsymbol{\alpha})}$. Since $|\chi(\boldsymbol{\alpha})| = 1$, the latter quantity is bounded in modulus by a number independent of N, whence $\lim_{N\to\infty} \frac{1}{N} S_N = 0$. To see that $\chi(\boldsymbol{\alpha}) \neq 1$, note that the denseness of $(n\boldsymbol{\alpha})_{n\in\mathbb{N}}$ in $\mathbb{T}^d$ implies that $\{\chi(n\boldsymbol{\alpha}) : n \in \mathbb{N}\}$ is dense in the image of χ (since χ is continuous), and in particular $\chi(n\boldsymbol{\alpha}) \neq 1$ for some n. This implies $\chi(\boldsymbol{\alpha})^n \neq 1$, so $\chi(\boldsymbol{\alpha}) \neq 1$. $\square$

7.2 Denseness

For a set $D \subseteq \mathbb{T}^d$ and $n \in \mathbb{Z}$, let $n^{-1}D := \{x \in \mathbb{T}^d : nx \in D\}$. For example, with $D = (0.5, 0.6) \subseteq \mathbb{T}$, we have $3^{-1}(0.5, 0.6) = (\frac{1}{6}, \frac{1}{5}) \cup (\frac{1}{2}, \frac{8}{15}) \cup (\frac{5}{6}, \frac{13}{15})$.

In the remainder of this section, we use "open box" to mean "nonempty open box."

Lemma 7.2. *Let I and J be open boxes in $\mathbb{T}^d$. For all sufficiently large n, $n^{-1}I \cap J$ contains an open box.*

Proof. We may assume, without loss of generality, that $I = \prod_{m=1}^{d}(a_m, b_m)$ and $J = \prod_{m=1}^{d}(c_m, d_m)$, where $0 \leq a_m, b_m, c_m, d_m \leq 1$. Now $n^{-1}I$ can be written as the union of boxes $L(k_1, \dots, k_d) := L_0 + \frac{1}{n}(k_1, \dots, k_d)$, $0 \leq k_m \leq n-1$, $k_m \in \mathbb{Z}$, where $L_0 := \prod_{m=1}^{d}(a_m/n, b_m/n)$. We see that for n sufficiently large, there are $k_m \in [0, n-1]$ such that $c_m \leq (a_m+k_m)/n, (b_m+k_m)/n \leq d_m$ for each m, meaning $n^{-1}I \cap J$ contains the box $L(k_1, \dots, k_d)$. $\square$

Lemma 7.3. *Let $I \subseteq \mathbb{T}^d$ be an open box and let $E \subseteq \mathbb{Z}$ be an infinite set. Then there is a dense open set of $\boldsymbol{\alpha} \in \mathbb{T}^d$ such that $E\boldsymbol{\alpha} \cap I \neq \varnothing$.*

Proof. We first observe that $\{\boldsymbol{\alpha} \in \mathbb{T}^d : E\boldsymbol{\alpha} \cap I \neq \varnothing\} = \bigcup_{n\in E} n^{-1}I$. So it suffices to prove that $\bigcup_{n\in E} n^{-1}I$ contains a dense open set. To see this, note that Lemma 7.2 implies that for all open boxes J, there is an $n \in E$ such that $n^{-1}I \cap J$ contains an open box. □

Proof of Lemma 6.2. Let $d \in \mathbb{N}$ and let $E \subseteq \mathbb{Z}$ be infinite. Let $\{I^{(k)} : k \in \mathbb{N}\}$ enumerate the boxes in $\mathbb{T}^d$ formed by open intervals with rational endpoints, and for each k let $Q_k := \{\boldsymbol{\alpha} : E\boldsymbol{\alpha} \cap I^{(k)} \neq \varnothing\}$. By Lemma 7.3, each Q_k is dense and open in $\mathbb{T}^d$, so $Q := \bigcap_{k\in\mathbb{N}} Q_k$ is nonempty, by the Baire category theorem. For each box $I^{(k)}$ and each $\boldsymbol{\alpha} \in Q$, we have $E\boldsymbol{\alpha} \cap I^{(k)} \neq \varnothing$, so $E\boldsymbol{\alpha}$ is dense in $\mathbb{T}^d$. □

8 Questions and remarks

A proof or disproof of the following conjecture would be very interesting in relation to the results of [15].

Conjecture 8.1. *If $S \subseteq \mathbb{Z}$ is a set of density recurrence, then there is a set $S' \subseteq S$ such that S' is a set of chromatic recurrence and not a set of density recurrence.*

Perhaps the proof of Theorem 1.2 can be extended to prove Conjecture 8.1. If so, the key lemma to generalize is Part (iii) of Lemma 4.1. Specifically, the required generalization would replace $E - E$ with an arbitrary set of chromatic recurrence, resulting in the following conjecture. We use the notation $\tilde{H}(\boldsymbol{\alpha}; k, \varepsilon)$ defined in (4.1).

Conjecture 8.2. *Assume $S \subseteq \mathbb{Z}$ is a set of chromatic recurrence, $k \in \mathbb{N}$, and $C > 0$. Then there exist infinitely many $d \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{T}^d$, such that for all $\varepsilon > 0$, the intersection*

$$S \cap \tilde{H}(\boldsymbol{\alpha}; C\sqrt{d}, \varepsilon)$$

is a set of k-chromatic recurrence.

See the remarks following Theorem 2.2 above to motivate the appearance of $C\sqrt{d}$ here.

Remark 8.3. There are many constructions of sets with prescribed recurrence properties, such as [5], [8], [14], [13], [16] [24]. All of these use one or both of the main ideas of Kriz's construction (i.e. they use Step I or Step II of the outline in §1.2 above).

Among the constructions of sets demonstrating that some recurrence property does not imply density recurrence, *every* such construction fulfills Step I of our outline using Hamming balls (or approximate Hamming balls) in $\mathbb{F}_p^d$ (or in $\mathbb{T}^d$); such constructions are often called *niveau sets* in the additive combinatorics literature. However, Lemma 2.3 of [13] provides a very different way to implement Step II. Unlike the present article, the implementation of Step II in [13] depends on the precise form of the pieces from Step I.

8.1 Bohr recurrence

We say $S \subseteq \mathbb{Z}$ is a *set of Bohr recurrence* if for all $d \in \mathbb{N}$, all $\boldsymbol{\alpha} \in \mathbb{T}^d$, and all $\varepsilon > 0$, there is an $n \in S$ such that $\|n\boldsymbol{\alpha}\| < \varepsilon$. "Approximative set" is the term used for "set of Bohr recurrence" in [26]. It is easy to verify that every set of chromatic recurrence is a set of Bohr recurrence – see §2 of [16], for instance. Whether the converse holds is a problem made famous by [16]. This problem is surveyed in [11] and discussed in [15]. A consequence of [15, Proposition 2.3] is that every subset of $\{2^n - 2^m : n, m \in \mathbb{N}\}$ which is a set of Bohr recurrence is also a set of chromatic recurrence. This follows from [19, Lemma 7.2], as well. In contrast, Theorem 1.2 above provides a subset of $\{2^n - 2^m : n, m \in \mathbb{N}\}$ which is a set of chromatic recurrence but not a set of measurable recurrence.

Remark 8.4. As observed in [26], Kneser graphs (and hence Theorem 2.2) can be avoided if instead of Theorem 1.1 one aims to prove the weaker statement "there is a set of Bohr recurrence which is not a set of density recurrence." This can be done by following our proof of Theorem 1.1, replacing Step I with "**Step I′**: find finite sets approximating the property of being Bohr recurrent and not density recurrent." Finite subsets of $\widetilde{H}(\boldsymbol{\alpha}; k, \varepsilon)$ will suffice for this purpose; elementary proofs of their Bohr recurrence properties appear in §2.2 of [16] and in §8 of [13]. Step II can then be followed exactly as in our of Theorem 1.1. However, we do not have a proof of the following corollary of Theorem 1.2 that does not prove Theorem 1.2 itself. Hence, we do not have a proof of Corollary 8.5 avoiding the use of Theorem 2.2. Such a proof might be useful to the general theory of recurrence.

Corollary 8.5. *If $E \subseteq \mathbb{Z}$ is infinite, then there is an $S \subseteq E - E$ such that S is a set of Bohr recurrence but not a set of density recurrence.*

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