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Rainbow numbers for the generalized Schur equation $x_1 + x_2 + \cdots + x_{m-1} = x_m$

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Abstract

We consider the rainbow Schur number $RS_m(n)$, defined to be the minimum number of colors such that every coloring of $\{1, 2, \ldots, n\}$, using all $RS_m(n)$ colors, contains a rainbow solution to the equation $x_1 + x_2 + \cdots + x_{m-1} = x_m$. Recently, the exact values of $RS_3(n)$ and $RS_4(n)$ were determined for all n. In this paper, we expand upon this work by providing a formula for $RS_m(n)$ that holds for all $m \geq 4$ and all n. A weakened version of the rainbow Schur number is also considered, for which one seeks solutions to the above-mentioned linear equation, where, for a fixed $t \leq m$, at least t colors are used.

1 Introduction

Many classical problems in Ramsey theory involve determining the existence, or non-existence, of certain monochromatic structures under finite colorings of a set. The subject known as rainbow Ramsey theory deals with what might be considered the opposite notion; namely, instead of looking for a structure such that all of its members have the same color, we look for a structure where no two elements have the same color.

An *r-coloring* of a set S is a map from S to $\{1, 2, \ldots, r\}$. One of the earliest results in Ramsey theory is due to Schur [16], which states that for any positive integer r, there exists a least positive integer $S(r)$ such that every r-coloring of $\{1, 2, \ldots, S(r)\}$

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contains a monochromatic solution to the equation $x_1 + x_2 = x_3$. More generally, for each $m \geq 3$, one can consider the Schur-type numbers for the equation

$$E_m : x_1 + x_2 + \cdots + x_{m-1} = x_m;$$

the existence of these numbers follows easily from the work of Rado [15]. In this paper, we consider solutions to E_m from the perspective of rainbow Ramsey theory.

If χ is an r-coloring of a set S that uses all r colors (i.e, it is surjective), we say that χ is an *exact* r-coloring. For $m \geq 3$ and $n \geq \frac{m(m-1)}{2}$, define the *rainbow Schur number* $RS_m(n)$ to be the minimum number of colors such that every exact $RS_m(n)$-coloring of $[1, n]$ contains a rainbow solution to E_m. The assumption that $n \geq \frac{m(m-1)}{2}$ guarantees that E_m is solvable using distinct elements from $[1, n]$. Since coloring $[1, n]$ using n colors will always produce a rainbow solution to E_m, we know that $RS_m(n) \leq n$. Note that rather than fix the number of colors and seek an optimal value of n, as in the definition of Schur numbers, rainbow Schur numbers fall under the subject of anti-Ramsey theory (introduced by Erdős, Simonovits, and Sós [6]) by fixing n and seeking an optimal number of colors.

In [3], it was proved that

$$RS_3(n) = \lfloor \log_2(n) \rfloor + 2, \quad \text{for all } n \geq 3.$$

This result was proved independently by Fallon et al. [7], where it was also shown that

$$RS_4(n) = \left\lceil \frac{n+6}{2} \right\rceil, \quad \text{for all } n \geq 6. \tag{1}$$

In addition, [7] provides a general lower bound for $R_m(n)$ by exhibiting a particular coloring of $[1, n]$ that avoids rainbow solutions to E_m. In Section 2, we we determine the exact value of $RS_m(n)$ for all $m \geq 4$ and $n \geq \frac{m(m-1)}{2}$, which has Equation (1) as a special case. In Section 3, we provide a formula for a generalization of rainbow Schur numbers, which we call weakened rainbow Schur numbers, where, for a given $t \leq m$, we seek solutions to E_m that use at least t colors.

2 The Value of $RS_m(n)$

In this section, we give the exact value of $RS_m(n)$ for all $m \geq 4$ and $n \geq \frac{m(m-1)}{2}$. The next theorem shows that this value serves as a lower bound for $RS_m(n)$. In [7], the authors give a proof of this lower bound, but for the sake of completeness we include a proof here, which is similar to, but slightly different from their proof.

Theorem 2.1. *Let $m \geq 4$ and let $n \geq \frac{m(m-1)}{2}$. Then*

$$RS_m(n) \geq \left\lceil \frac{(m-3)n + \frac{m(m-1)}{2}}{m-2} \right\rceil.$$

Proof. Suppose that $a_1+a_2+\cdots+a_{m-1}=a_m$ is any solution to E_m that lies within $[1,n]$, where $a_i<a_{i+1}$ for all $1\le i\le m-2$. Let

$$k=n+2-\left\lceil\frac{(m-3)n+\frac{m(m-1)}{2}}{m-2}\right\rceil.$$

We claim that $a_2\le k$. For a contradiction, assume $a_2>k$. Then

$$\begin{aligned}\sum_{i=1}^{m-1}a_i&\ge 1+\sum_{i=1}^{m-2}(k+i)\\ &=1+(m-2)(n+2)-(m-2)\left\lceil\frac{(m-3)n+\frac{m(m-1)}{2}}{m-2}\right\rceil\\ &\quad+\frac{(m-1)(m-2)}{2}\\ &>1+(m-2)(n+2)-(m-2)\frac{(m-3)n+\frac{m(m-1)}{2}}{m-2}-(m-2)\\ &\quad+\frac{(m-1)(m-2)}{2}\\ &=1+n+(m-2)\left(2-1+\frac{m-1}{2}\right)-\frac{m(m-1)}{2}\\ &=1+n+m-2+\frac{m-1}{2}(-2)=n,\end{aligned}$$

which contradicts the assumption that the solution lies in $[1,n]$.

To complete the proof, color $[1,n]$ as follows. Color all members of $[1,k]$ the same color, say red, and color each element of $[k+1,n]$ its own unique color, different from red. Note that this is an exact $(n-k+1)$-coloring, that is, it uses

$$\left\lceil\frac{(m-3)n+\frac{m(m-1)}{2}}{m-2}\right\rceil-1$$

colors. By the above claim, a_1 and a_2 are both colored red, and hence, the solution is not rainbow. So, at least

$$\left\lceil\frac{(m-3)n+\frac{m(m-1)}{2}}{m-2}\right\rceil$$

colors are needed to guarantee a rainbow solution to E_m. □

Before presenting the next theorem, we give a definition.

Definition 2.2. Let $n\ge r$ be positive integers, and let χ be an exact r-coloring of $[1,n]$. An integer $x\in[1,n]$ is called a *surplus integer* of χ if there exists a $y<x$ such that $\chi(x)=\chi(y)$.

Remark 2.3. Let s_χ denote the number of surplus integers of χ. From the definition, we see that, for any exact r-coloring χ of an interval $[1, n]$, we have $s_\chi = n - r$.

Theorem 2.4. *Let $m, n \in \mathbb{Z}^+$ with $m \geq 4$ and $n \geq \frac{m(m-1)}{2}$. Assume that one of the following conditions holds:*

$$m \text{ is odd and } n \equiv 1 \pmod{(m-2)}$$

or

$$m \text{ is even and } n \equiv \frac{m}{2} \pmod{(m-2)}.$$

Let

$$c(n, m) = \frac{(m-3)n + m(m-1)/2}{m-2}.$$

Then every exact $c(n, m)$-coloring of $[1, n]$ contains a rainbow solution to E_m.

Proof. First note that the assumptions regarding m and n imply that $c(n, m)$ is a positive integer for all such m and n. We begin with the case in which m is odd. We use induction on $\ell \geq \frac{m+1}{2}$, where $n = (m-2)\ell + 1$. If $\ell = \frac{m+1}{2}$, then

$$n = (m-2)\frac{m+1}{2} + 1 = \frac{m(m-1)}{2}$$

and

$$c(n, m) = \frac{(m-3)\frac{m(m-1)}{2} + \frac{m(m-1)}{2}}{m-2} = \frac{m(m-1)}{2}.$$

Since $c(n, m) = n = \frac{m(m-1)}{2}$, there is only one exact $c(n, m)$-coloring of $[1, n]$, and it contains the rainbow equation $1 + 2 + \cdots + (m-1) = n$. Therefore, the result is true when $\ell = \frac{m+1}{2}$.

We now let $\ell \geq \frac{m+1}{2}$ and assume the result holds for $n = (m-2)\ell + 1$. To complete the proof, we will show that the result holds for $(m-2)(\ell+1) + 1 = n + m - 2$, i.e., that every exact $c(n+m-2, m)$-coloring of $[1, n+m-2]$ has a rainbow solution to E_m. For a contradiction, assume χ is an exact $c(n+m-2, m)$-coloring of $[1, n+m-2]$ with no such rainbow solution. By the inductive hypothesis, we may assume that no more than $c(n, m) - 1 = c(n+m-2, m) - (m-2)$ colors are used in $[1, n]$. This implies that each member of $[n+1, n+m-2]$ is the only member of its color class under χ.

For each j such that $0 \leq j \leq m-4$, we define t_j recursively as follows. Let $t_0 = 1$. For each $j = 1, 2, \ldots, m-4$, define t_j to be the least integer greater than t_{j-1} such that $\chi(t_j) \notin \{\chi(t_i) : 0 \leq i \leq j-1\}$. From this definition, we see that the number of surplus integers of χ that lie within $[1, t_{m-4}]$ is $t_{m-4} - (m-3)$, so that (using the notation of Remark 2.3)

$$t_{m-4} \leq s_\chi + m - 3. \tag{2}$$

Note that, by Remark 2.3 and the definition of χ,

$$\begin{aligned} s_\chi &= n + m - 2 - c(n+m-2, m) \\ &= n + m - 2 - \frac{(m-3)(n+m-2) + \frac{m(m-1)}{2}}{m-2}. \end{aligned} \tag{3}$$

By Equations (2) and (3), we have

$$\begin{aligned} t_{m-4} &\leq n + 2m - 5 - \frac{(m-3)(n+m-2) + \frac{m(m-1)}{2}}{m-2} \\ &= \frac{n}{m-2} + m - 2 - \frac{\frac{m(m-1)}{2}}{m-2}. \end{aligned}$$

Since the t_j are strictly increasing, it follows that

$$t_{m-5} \leq \frac{n}{m-2} + m - 3 - \frac{\frac{m(m-1)}{2}}{m-2}$$

and, more generally,

$$t_j \leq \frac{n}{m-2} + j + 2 - \frac{\frac{m(m-1)}{2}}{m-2}, \tag{4}$$

for $1 \leq j \leq m-4$.

Let

$$v = \left\lfloor \frac{n - 3t_{m-4} - \sum_{j=1}^{m-5} t_j + m - 4}{2} \right\rfloor.$$

Note that $v \geq 1$ since, using Inequality (4),

$$\begin{aligned} n - 3t_{m-4} - \sum_{j=1}^{m-5} t_j + m - 4 \geq n &- \frac{3n}{m-2} - 3\left(m - 2 - \frac{\frac{m(m-1)}{2}}{m-2}\right) + m - 4 \\ &- (m-5)\frac{n}{m-2} - \sum_{j=1}^{m-5}\left(j + 2 - \frac{\frac{m(m-1)}{2}}{m-2}\right) \\ &= -4m + 12 + \frac{3\frac{m(m-1)}{2}}{m-2} - \frac{(m-5)(m-4)}{2} \\ &\quad + \frac{(m-5)\frac{m(m-1)}{2}}{m-2} \\ &= -4m + 12 + \frac{m(m-1)}{2} - \frac{(m-5)(m-4)}{2} = 2. \end{aligned}$$

For each i such that $1 \leq i \leq v$, let

$$a_i = t_{m-4} + i$$

and

$$b_i = n + m - 3 - 2t_{m-4} - \sum_{j=1}^{m-5} t_j - i,$$

and let $P_i = \{a_i, b_i\}$. Note that the a_i's are strictly increasing, the b_i's are strictly decreasing, and that $\max\{a_i\}$ and $\min\{b_i\}$ both occur when $i = v$. Also, we have that $a_v < b_v$, since

$$b_v - a_v = n + m - 3 - 2t_{m-4} - \sum_{j=1}^{m-5} t_j - v - (t_{m-4} + v)$$
$$> n + m - 4 - 3t_{m-4} - \sum_{j=1}^{m-5} t_j - 2v \geq 0.$$

From these facts, we see that the sets P_i are pairwise disjoint and that $|P_i| = 2$ for each i.

If there is some pair $P_u = \{a_u, b_u\}$ such that $\chi(a_u) \neq \chi(b_u)$ and

$$\{\chi(a_u), \chi(b_u)\} \cap \{\chi(t_i) : 0 \leq i \leq m - 4\} = \emptyset,$$

then E_m has the rainbow solution

$$1 + t_1 + t_2 + \cdots + t_{m-4} + (t_{m-4} + u) + \left(n+m-3-2t_{m-4} - \sum_{j=1}^{m-5} t_j - u \right) = n+m-2$$

since, as noted previously, $n + m - 2$ is the only member of its color class. Hence, by our assumption about χ, no such P_u exists. Thus, each P_i contributes at least one surplus integer to s_χ. This implies that

$$s_\chi \geq t_{m-4} - (m - 3) + v, \tag{5}$$

because there are exactly $t_m - (m - 3)$ surplus integers contained in $[1, t_{m-4}]$.

By Inequality (4), we have

$$\sum_{j=1}^{m-4} t_j \leq (m - 4) \left(\frac{n - \frac{m(m-1)}{2}}{m - 2} + 2 \right) + \frac{(m - 4)(m - 3)}{2}$$
$$= (m - 4) \frac{2n - m(m - 1) + 4(m - 2) + (m - 3)(m - 2)}{2(m - 2)}$$
$$= \frac{(m - 4)(n - 1)}{m - 2}. \tag{6}$$

Using Inequalities (5) and (6), and the definition of v, we have

$$s_\chi \geq t_{m-4} - (m - 3) + \frac{n - 3t_{m-4} - \sum_{j=1}^{m-5} t_j + m - 5}{2}$$
$$= \frac{n}{2} - \frac{\sum_{j=1}^{m-4} t_j}{2} - \frac{m - 1}{2}$$
$$\geq \frac{n}{2} - \frac{(m - 4)(n - 1)}{2(m - 2)} - \frac{m - 1}{2}$$
$$= \frac{n}{m - 2} - \frac{m^2 - 4m + 6}{2(m - 2)}. \tag{7}$$

Now, from Equation (3) we have

$$\begin{aligned} s_\chi &= \frac{n}{m-2} + \frac{2(m-2)^2 - 3m^2 + 11m - 12}{2(m-2)} \\ &= \frac{n}{m-2} - \frac{m^2 - 3m + 4}{2(m-2)}, \end{aligned}$$

which contradicts Inequality (7). This completes the proof for the case in which m is odd.

The proof for m even is almost identical to the proof for m odd. The only difference is that the induction is done on ℓ where $n = (m-2)\ell + \frac{m}{2}$, and the initial value of ℓ is taken to be $\frac{m}{2}$. Then, just as in the odd case, this initial value of ℓ again gives

$$n = \frac{m(m-1)}{2} = c(n, m).$$

Hence, as explained in the case of m odd, for the initial step of the induction for m even, the result holds for $\ell = \frac{m}{2}$. For the inductive step, we assume that the result holds for $n = \ell(m-2) + \frac{m}{2}$ for some $\ell \geq \frac{m}{2}$, and must then show that it holds for $(\ell+1)(m-2) + \frac{m}{2}$. The rest of the proof is the same as that for the odd case. □

We are now able to give the exact value of $RS_m(n)$ when $m \geq 4$.

Theorem 2.5. *Let $m \geq 4$ and $n \geq \frac{m(m-1)}{2}$. Then*

$$RS_m(n) = \left\lceil \frac{(m-3)n + \frac{m(m-1)}{2}}{m-2} \right\rceil. \tag{8}$$

Proof. Let $m \geq 4$ and $n \geq \frac{m(m-1)}{2}$. Let $a(n, m)$ denote the right-hand side of Equation (8). By Theorem 2.1, we know that $a(n, m)$ is a lower bound for $RS_m(n)$. We claim that $a(n, m)$ is also an upper bound. We begin with the case in which

$$n = \frac{m(m-1)}{2} + i,$$

such that $0 \leq i \leq m - 4$. In this case, we have

$$a(n, m) = \left\lceil \frac{(m-2)\frac{m(m-1)}{2} + i(m-3)}{m-2} \right\rceil = \frac{m(m-1)}{2} + \left\lceil i - \frac{1}{m-2} \right\rceil = n.$$

Since the only exact n-coloring of $[1, n]$ has no two elements with the same color, we have that $1 + 2 + \cdots + (m-1) = \frac{m(m-1)}{2}$ is a rainbow solution to E_m. This shows that, in this case, $a(n, m)$ is an upper bound on $RS_m(n)$. Thus, we will assume that

$$n \geq \frac{m(m-1)}{2} + m - 3.$$

Consider the case in which m is odd. By Theorem 2.4, we know that the claim is true whenever $n \equiv 1 \pmod{(m-2)}$, so we may assume that $n \equiv i \pmod{(m-2)}$ where $2 \le i \le m-2$. Since m is odd, $\frac{m(m-1)}{2} \equiv 1 \pmod{(m-2)}$, and therefore

$$(m-3)n + \frac{m(m-1)}{2} \equiv -i + 1 \pmod{(m-2)}. \tag{9}$$

Let χ be any exact $a(n,m)$-coloring of $[1,n]$. Since $1 \le i-1 \le m-3$, from Equation (9), we have

$$a(n,m) = \frac{(m-3)n + \frac{m(m-1)}{2} + i - 1}{m-2}. \tag{10}$$

From Equation (10), within the interval $[1, n-i+1]$, there must be at least

$$\begin{aligned} a(n,m) - i + 1 &= \frac{(m-3)n + \frac{m(m-1)}{2} + i - 1 + (m-2)(-i+1)}{m-2} \\ &= \frac{(m-3)(n-i+1) + \frac{m(m-1)}{2}}{m-2} \end{aligned} \tag{11}$$

different colors.

By Theorem 2.4, since $n-i+1 \equiv 1 \pmod{(m-2)}$ and $n-i+1 \ge \frac{m(m-1)}{2}$, it follows that

$$RS_m(n-i+1) \le \frac{(m-3)(n-i+1) + \frac{m(m-1)}{2}}{m-2}.$$

Hence, from Equation (11), under χ there is a rainbow solution to E_m within $[1, a(n,m)-i+1]$, and therefore within $[1, a(n,m)]$, which completes the proof for odd values of m.

Now assume that m is even. Similar to the proof of the odd case, we may assume that $n \equiv i \pmod{(m-2)}$ where $\frac{m}{2}+1 \le i \le \frac{m}{2}+m-3$. Since m is even, we have $\frac{m(m-1)}{2} \equiv \frac{m}{2} \pmod{(m-2)}$, so that

$$(m-3)n + \frac{m(m-1)}{2} \equiv -i + \frac{m}{2} \pmod{(m-2)}.$$

From this and the fact that $1 \le i - \frac{m}{2} \le m-3$, it follows that

$$a(n,m) = \frac{(m-3)n + \frac{m(m-1)}{2} + i - \frac{m}{2}}{m-2}.$$

As in the odd case, if χ is any exact $a(n,m)$-coloring of $[1,n]$, then within the interval $[1, n-i+\frac{m}{2}]$ there must be at least

$$a(n,m) - i + \frac{m}{2} = \frac{(m-3)(n-i+\frac{m}{2}) + \frac{m(m-1)}{2}}{m-2} \tag{12}$$

different colors. By Theorem 2.4, since $n-i+\frac{m}{2} \equiv \frac{m}{2} \pmod{(m-2)}$, it follows that

$$RS_m(n-i+\frac{m}{2}) \le \frac{(m-3)(n-i+\frac{m}{2}) + \frac{m(m-1)}{2}}{m-2}.$$

Hence, from Equation (12), under χ there is a rainbow solution to E_m within $[1, a(n,m)-i+\frac{m}{2}]$, and therefore within $[1, a(n,m)]$, which completes the proof. $\square$

3 Weakened Rainbow Schur Numbers

For $m \geq 3$ and $2 \leq t \leq m$, define the *weakened rainbow Schur number* $RS_{t,m}(n)$ to be the minimum number of colors such that every exact $RS_{t,m}(n)$-coloring of $[1, n]$ contains a solution to E_m that uses at least t of the colors. Here, $RS_{m,m}(n)$ agrees with the rainbow Schur number $RS_m(n)$. Note that if $t < m$ then, in contrast to the situation with rainbow colorings of E_m, the relevant solutions to E_m do not necessarily consist of distinct summands.

When $t_1 \leq t_2$, observe that every solution to E_m that uses at least t_2 colors necessarily uses at least t_1 colors. It follows that

$$RS_{t_1,m}(n) \leq RS_{t_2,m}(n),$$

for all $m \geq 3$ and n for which a solution to E_m exists that can use at least t_2 colors. From this observation, we see that $RS_{t,m}(n)$ is defined for all t such that $2 \leq t < m$ whenever $RS_m(n)$ is defined.

While a range of values for n was not specified in the definition of $R_{t,m}$, it is natural to only consider values of n for which there can exist a solution to E_m that uses at least t colors. For any $m \geq 3$ and $2 \leq t \leq m$, the equation

$$\underbrace{1 + 1 + \cdots + 1}_{m-t+1 \ terms} + 2 + 3 + \cdots + (t-1) = \frac{t(t-1)}{2} + m - t$$

has the least sum among all equations in E_m that can be colored using at least t colors. For this reason, we assume $n \geq \frac{t(t-1)}{2} + m - t$ when considering $RS_{t,m}(n)$. In the evaluations of $RS_{t,m}(n)$ that follow, we often restrict the values of n beyond this natural bound.

Theorem 3.1. *For all $m \geq 3$ and $n \geq 2m - 4$, we have $RS_{2,m}(n) = 2$.*

Proof. At least two colors are required in order to have a 2-colored solution to E_3, and hence $RS_{2,3}(n) \geq 2$. Now consider an exact 2-coloring of $[1, n]$. Without loss of generality, assume that 1 is red and $i \in [2, n]$ is the least positive integer that is colored blue. We consider two cases, based on the value of i.

Case 1 If $i \leq n - m + 2$, then the equation

$$\underbrace{1 + 1 + \cdots + 1}_{m-2 \ terms} + i = i + m - 2 \leq n$$

is in E_m and uses at least two colors.

Case 2 If $i > n - m + 2$, then consider the equation

$$\underbrace{1 + 1 + \cdots + 1}_{m-2 \ terms} + (i - (m-2)) = i.$$

This equation uses at least two colors and is in E_m whenever $i \geq m - 1$, which for this case, occurs when

$$n - m + 3 \geq m - 1.$$

This is equivalent to $n \geq 2m - 4$, as assumed in the statement of the theorem.

In both cases, we find that there exists an equation in E_m that uses two colors, from which it follows that $RS_{2,m}(n) \leq 2$. □

To demonstrate the need for the assumption $n \geq 2m-4$ in the previous theorem, consider the case where $m = 6$, $t = 2$, and $n = 6$. The only solutions to E_6 contained in $[1, n]$ are

$$1 + 1 + 1 + 1 + 1 = 5 \quad \text{and} \quad 1 + 1 + 1 + 1 + 2 = 6.$$

Using the color classes

$$C_1 = \{1, 2, 5, 6\}, \quad C_2 = \{3\}, \quad \text{and} \quad C_3 = \{4\},$$

we find that $RS_{2,6}(6) \geq 4$.

Theorem 3.2. *Let $m \geq 4$ and $3 \leq t \leq m$. Then for all $n \geq \frac{t(t-1)}{2} + m - t$, we have*

$$RS_{t,m}(n) = \left\lceil \frac{(t-3)n + \frac{t(t-1)}{2} + m - t}{t-2} \right\rceil.$$

Proof. Let

$$k = k(n, m, t) = \left\lceil \frac{(t-3)n + \frac{t(t-1)}{2} + m - t}{t-2} \right\rceil$$

and note that

$$k(t-2) \leq (t-3)(n) + \frac{t(t-1)}{2} + m - 3. \tag{13}$$

Let α be the exact $(k-1)$-coloring of $[1, n]$ having the following color classes:

$$C_1 = [1, n+2-k],\ C_2 = \{n+3-k\},\ C_3 = \{n+4-k\},\ \ldots,\ C_{k-1} = \{n\}.$$

To show that $RS_{t,m}(n) \geq k$, it suffices to show that α does not have a solution to E_m that uses at least t colors. For a contradiction, assume that

$$a_1 + a_2 + \cdots a_{m-1} = a_m, \quad \text{where } a_1 \leq a_2 \leq \cdots \leq a_{m-1},$$

is such a solution. Since at least t colors occur among the a_i, no color class contains more than $m - t + 1$ of the a_i. Hence, by Inequality (13),

$$\begin{aligned}
a_1 + a_2 + \cdots + a_{m-1} \\
&\leq \underbrace{1 + 1 + \cdots + 1}_{m-t+1 \text{ terms}} + (n+3-k) + (n+4-k) + \cdots + (n+t-k) \\
&= (m-t+1) + 3 + 4 + \cdots + t + n(t-2) - k(t-2) \\
&= (m-t-2) + \frac{t(t+1)}{2} + n(t-2) - k(t-2) \\
&\geq (m-t-2) + \frac{t(t+1)}{2} + n(t-2) \\
&\qquad - \left((t-3)n + \frac{t(t-1)}{2} + m - 3\right) \\
&= n + 1,
\end{aligned}$$

which contradicts the fact that $a_m \leq n$.

To prove that

$$RS_{t,m}(n) \leq k(n,m,t), \tag{14}$$

we use induction on $m+n$, where $m \geq 4$ and $n \geq \frac{t(t-1)}{2} + m - t$. To establish the base cases of the induction, we will show that the Inequality (14) holds for each of the following two cases: (a) all $n \geq \frac{t(t-1)}{2} + m - t$, when $m = 4$ and $3 \leq t \leq 4$; and (b) $n = \frac{t(t-1)}{2} + m - t$, for all $m \geq 4$ and $3 \leq t \leq m$.

To establish Inequality (14) in case (a), note that for $m = 4$, we have either $t = 3$ or $t = 4$. When $m = t = 4$, Inequality (14) holds by Equation (1). Now assume that $m = 4$ and $t = 3$. We will show that $RS_{3,4}(n) \leq k(n,4,3) = 4$ by induction on $n \geq 4$. When $n = 4$, an exact 4-coloring of $[1,4]$ has every number receiving a unique color, and hence $1+1+2 = 4$ is a solution to E_4 that uses (at least) 3 colors. Now assume that $RS_{3,4}(n-1) \leq 4$ for some $n-1 \geq 4$ and let β be an exact 4-coloring of $[1,n]$. If n is a surplus integer under β, then $[1,n-1]$ uses all 4 colors and, since $RS_{3,4}(n-1) \leq 4$, it contains a solution to E_4 that uses at least 3 colors.

If n is not a surplus integer under β, then it receives its own unique color and $[1,n-1]$ uses 3 colors. In this latter situation, let $i \in [2,n]$ be the least integer such that $\beta(i) \neq \beta(1)$, and consider the equation

$$1 + i + (n-i-1) = n. \tag{15}$$

Note that $n-i-1$ is a positive integer, since otherwise $i \geq n-1$, which would imply that β uses at least 5 colors. It follows that Equation (15) is a solution to E_4 that uses at least 3 colors.

To show that Inequality (14) holds in case (b), we have

$$\begin{aligned} k(n,m,t) &= \left\lceil \frac{(t-3)\left(\frac{t(t-1)}{2} + m - t\right) + \frac{t(t-1)}{2} + m - t}{t-2} \right\rceil \\ &= \left\lceil \frac{(t-2)\left(\frac{t(t-1)}{2} + m - t\right)}{t-2} \right\rceil \\ &= \frac{t(t-1)}{2} + m - t = n, \end{aligned}$$

and hence, in any exact $\left(\frac{t(t-1)}{2} + m - t\right)$-coloring of $[1,n]$, each element is the only member of its color class. So, in this case, every solution to E_m is necessarily rainbow, establishing Inequality (14) in this case.

Having taken care of the base cases in the inductive proof of Inequality (14), we now let $m \geq 5$, $t \geq 3$, and $n \geq \frac{t(t-1)}{2} + m - t + 1$. Assume that for all $m' \geq 4$ and $n' \geq \frac{t(t-1)}{2} + m - t$ with $m' + n' < m + n$, we have

$$RS_{t,m'}(n') \leq k(n', m', t) \tag{16}$$

for all $3 \le t \le m'$. To complete the proof, it suffices to prove that for every t such that $3 \le t \le m$, every exact $k(n,m,t)$-coloring of $[1,n]$ contains a solution to E_m using at least t colors. Let χ be an exact $k(n,m,t)$-coloring of $[1,n]$. By the division algorithm, let

$$(t-3)n + \frac{t(t-1)}{2} + m - t = (t-2)\ell + j, \tag{17}$$

where $\ell, j \in \mathbb{Z}$ and $0 \le j \le t-3$. The remainder of the proof is separated into two cases.

Case 1 Assume that $1 \le j \le t-3$. By Equation (17),

$$k(n,m,t) = \left\lceil \frac{(t-3)n + \frac{t(t-1)}{2} + m - t}{t-2} \right\rceil = \left\lceil \ell + \frac{j}{t-2} \right\rceil = \ell + 1.$$

Therefore, the interval $[1, n-1]$ uses at least ℓ colors under χ. Now, by Equation (16),

$$\begin{aligned} RS_{t,m}(n-1) &\le k(n-1,m,t) \\ &= \left\lceil \frac{(t-3)n + \frac{t(t-1)}{2} + m - t}{t-2} - \frac{t-3}{t-2} \right\rceil \\ &= \left\lceil \ell - \frac{(t-3)-j}{t-2} \right\rceil = \ell. \end{aligned}$$

Therefore, in $[1, n-1]$ (and hence in $[1,n]$) there is a solution to E_m that uses at least t colors.

Case 2 Assume that $j = 0$. Then $k(n,m,t) = \ell$ and

$$k(n-1,m,t) = \left\lceil \frac{(t-3)(n-1) + \frac{t(t-1)}{2} + m - t}{t-2} \right\rceil = \left\lceil \ell - \frac{t-3}{t-2} \right\rceil = \ell.$$

If $[1, n-1]$ uses all ℓ colors of χ, then by Equation (16) there exists a solution to E_m that uses at least t colors. Otherwise, $[1, n-1]$ uses only $\ell - 1$ colors and the integer n is the only member of its color class. We may assume that $t \le m-1$ since the $t = m$ case corresponds with Theorem 2.5. For $3 \le t \le m-1$, by Equation (16) we obtain

$$\begin{aligned} RS_{t,m-1}(n-1) &\le k(n-1,m-1,t) \\ &= \left\lceil \frac{(t-3)(n-1) + \frac{t(t-1)}{2} + (m-t-1)}{t-2} \right\rceil \\ &= \left\lceil \ell - \frac{t-3}{t-2} - \frac{1}{t-2} \right\rceil \\ &= \ell - 1. \end{aligned}$$

It follows that there exists a solution $a_1 + a_2 + \cdots + a_{m-2} = a_{m-1}$ to E_{m-1} that uses at least t colors, where $a_{m-1} \le n-1$. Then

$$a_1 + a_2 + \cdots + a_{m-2} + (n - (a_1 + a_2 + \cdots + a_{m-2})) = n$$

is a solution to E_m that uses at least t colors.

In both cases, χ contains a solution to E_m that uses at least t colors. It follows that $RS_{t,m}(n) \leq k(n, m, t)$, completing the proof. □

4 Final Remarks

Results involving rainbow Ramsey numbers for 3-term and 4-term arithmetic progressions may be found in [13]. In [5], the authors provide a rainbow version of Rado's work on systems of linear homogeneous equations. Work on rainbow solutions to 3-variable linear equations in the group $\mathbb{Z}_n$ appears in [1], [2], [11], [12], and [14]. Related work arises in [4], [9], and [10], where additional restrictions are placed on the number of times each color occurs. The work in [8] deals with rainbow solutions to E_3 in the rectangular grid $[1, m] \times [1, n]$, with coordinate-wise addition. As far as we know, weakened versions of the results contained in these related papers have not yet been studied, but are worthy of investigation.

Future research may also consider whether or not the techniques used in this paper can be applied to some of the variations mentioned above. For example, one could consider rainbow solutions in $\mathbb{Z}_n$ to the equations

$$x_1 + x_2 + \cdots + x_{m-1} = kx_m \qquad \text{or} \qquad a_1x_1 + a_2x_2 + \cdots + a_mx_m = b,$$

for values of m larger than those considered in [1] and [2]. Our techniques may also be considered for rainbow numbers for E_n in the rectangular grid $[1, m] \times [1, n]$, where $n \geq 4$, extending the numbers introduced in [8].

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