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Diffusion: quiescence and perturbation

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Abstract

Originally introduced by Duffy et al. in 2018, diffusion is a variant of chip-firing in which chips flow from places of high concentration to places of low concentration. In the variant, perturbation diffusion, the first step involves a "perturbation" in which some number of vertices send chips to each of their respective neighbours, regardless of relative stack size, then standard diffusion rules are followed thereafter. In this paper, we ask the question "Given an initial configuration, which vertices, when perturbed, will return the initial configuration after some number of steps in diffusion?" For graphs with an equal number of chips on each vertex, we characterize subsets of vertices such that when perturbed, this initial configuration is eventually obtained and show in such cases it must occur after two time steps of the diffusion process. We provide results for general graphs, and then explore the family of paths in detail.

1 Introduction

Consider a scenario where resources (we will call them chips) are initially distributed in a network and then are shared over time, with those who have more sending chips to those who have less. That is, the chips are 'diffused' throughout the network. In [3] the process of diffusion on graphs was introduced by Duffy et al. We will expand on the work of Duffy et al. by introducing a variant in which a vertex will send chips to a neighbour that is neither poorer nor richer than itself.

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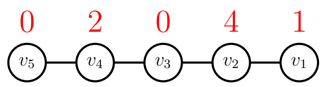
Diffusion is a process defined on a simple finite graph, G, in which each vertex is assigned an integer to represent the size of a stack of chips. At each time step, the vertices of G all *fire* simultaneously, which is how the chips are redistributed. If adjacent vertices have the same stack sizes, there is no exchange of chips between them. Otherwise, when a vertex fires, the chips are redistributed via the following rules: If a vertex v is adjacent to a vertex u with fewer chips, v takes a chip from its stack and adds it to the stack of u. We say that v *sends* a chip and u *receives* a chip. Note that it is possible for a stack size to become negative if a vertex sends more chips than it has. An example of diffusion is provided in Figure 1. In Figure 1, we see at each time step, the vertices of P_5 have a stack size. This assignment of stack sizes to the vertices of a graph is referred to as a *configuration*. Each step of the diffusion process yields a configuration.



	v_5	v_4	v_3	v_2	v_1
Step 0	0	2	0	4	1
Step 1	1	0	2	2	2
Step 2	0	2	1	2	2
Step 3	1	0	3	1	2
Step 4	0	2	1	3	1
Step 5	1	0	3	1	2
Step 6	0	2	1	3	1

Figure 1: Stack sizes during several steps in a diffusion process on P_5.

In this paper we define and analyze the diffusion variant, Perturbation Diffusion. We refer to the configuration in which every vertex has 0 chips as the *0-configuration*. Note that in [3] any configuration in which every stack size is equal is referred to as “fixed.” Let G be a graph with the 0-configuration and H be a subset of $V(G)$. A *perturbation* of H is when the vertices in H send a chip to each of their respective neighbours in G. In this case, we call H a *perturbation subset* (an example of a perturbation is shown in Figure 2).

We define Perturbation Diffusion on a graph G as the variant of diffusion where the first firing (which takes place at time step 0 and is referred to as the *initial firing*) is such that for some $H \subseteq V(G)$, H is a perturbation subset. After the initial firing, so time steps $t \geq 1$, Perturbation Diffusion follows the standard diffusion rules.

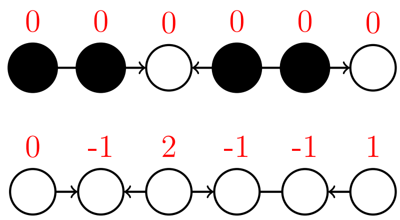


Figure 2: Perturbation of the subset H (marked with black vertices) of $V(P_6)$ with directed edges depicting the flow of chips: first from vertices of H to vertices of $V(P_6) \setminus H$, and then from richer vertices to poorer vertices.

Let $\mathrm{Seq}(C_0) = (C_0, C_1, C_2, \dots)$ be the configuration sequence on a graph G with initial configuration C_0 (so C_0 yields C_1, which yields C_2, etc.). The positive integer p is a *period length* if $C_t = C_{t+p}$ for all $t \geq N$ for some N. In this case, N is a *preperiod length*. For such a value, N, if $k \geq N$, then we say that the configuration, C_k, is *inside* the period. For the purposes of this paper, all references to period length will refer to the *minimum period length* p in a given configuration sequence. Also, all references to preperiod length will refer to the *least preperiod length* that yields that minimum period length p in a given configuration sequence.

The assigned value of a vertex v in a configuration C is its *stack size in* C and is denoted $|v|^C$. We omit the superscript when the configuration is clear from context. A vertex v is said to be *richer* than another vertex u in configuration C if $|v|^C > |u|^C$. In this instance, u is said to be *poorer* than v in C. If $|v|^C < 0$, we say v is *in debt in* C. In diffusion, the stack size of a vertex, v, at step t, is referred to as its *stack size at time* t. If the initial configuration is C, then the stack size of v at time t is denoted $|v|_t^C$. This implies that $|v|^C = |v|_0^C$. We omit the superscript when the configuration is clear from context. Given a vertex v, $N(v)$ is the open neighbourhood of v. Given a set of vertices $A \subseteq V(G)$, $N(A)$ is the open neighourhood of A and the subgraph induced by A will be denoted as $G|_A$.

Long and Narayanan [4] showed that given any configuration on any graph, the minimum period length is always either 1 or 2. In [5], it is shown that only configurations in which every stack size is equal exist inside periods of length 1. It is worth noting that the total number of chips on the graph is a constant. That is to say, during the (Perturbation) Diffusion process, the total number of chips remains the same at each time step. If the number of chips distributed throughout the graph is divisible by the number of vertices, then it is unknown whether the period length will be 1 or 2, but if the number of chips is not divisible by the number of vertices, then the period length must be 2.

Perturbation Diffusion starting with the 0-configuration on our graph can result in different periodic behaviours based on the perturbation subset. For example, consider the cycle on four vertices. If our perturbation subset is a single vertex, we obtain the configuration sequence $\mathrm{Seq}(C_0) = (C_0, C_1, C_2, C_3, \dots)$ where C_0 is the 0-configuration and $C_1 = C_3$ and we have period 2. To contrast, if two antipodal vertices are in the perturbation subset we obtain the configuration sequence $\mathrm{Seq}(C_0) = (C_0, C_1, C_2, \dots)$

where C_0 is the 0-configuration and $C_0 = C_2 = C_t, t \geq 3$. We note that adding a constant to every stack size does not change the behaviour of the model at any future step [3, 5].

Definition 1.1. *Let G be a graph and let H be a subset of $V(G)$.*

- *H is **0-invoking** in G if a perturbation of H from an initial 0-configuration on G results in the 0-configuration on G after a finite number of firings.*
- *H is 0_2 **invoking** in G if perturbing H from an initial 0-configuration and then firing once results in the 0-configuration on G.*
- *The **perturbation quiescent number** of a graph G, denoted $\boldsymbol{PQ}(G)$, is the size of the smallest nontrivial 0-invoking subset of $V(G)$. So, $PQ(G) = min\{|H| : H \neq \emptyset$ is 0-invoking in $G\}$.*
- *The **2-perturbation quiescent number** or $\boldsymbol{PQ_2}(G)$ is the size of the smallest nontrivial 0_2-invoking subset of $V(G)$. So, $PQ_2(G) = min\{|H| : H \neq \emptyset$ is 0_2-invoking in $G\}$.*

Note that $PQ(G)$ and $PQ_2(G)$ are well-defined because $V(G)$ is itself both a 0-invoking subset and a 0_2-invoking subset of $V(G)$.

In this paper, we study the following question: Which perturbation subsets will yield a period of length 1? We characterize all 0_2-invoking subsets on all graphs in Theorem 2.2 and in Theorem 2.9 we show that all 0-invoking subsets are 0_2-invoking. We then turn our attention to paths in particular. In Theorem 3.3, we show that $PQ_2(P_n) = \lceil \frac{n}{3} \rceil$, the same as the domination number, for all $n \geq 1$ and in Theorem 3.7, we count the number of 0_2-invoking subsets that exist on P_n for all $n \geq 1$.

2 0_2-invoking Sets

In this section, we characterize the 0_2-invoking subsets of all finite, simple graphs and highlight a relationship with dominating sets. We begin by defining a new concept, a *complementary component dominant* subset and then show, with Theorem 2.2, that a subset of vertices in a graph is 0_2-invoking if and only if it is complementary component dominant.

Definition 2.1. *Given a graph G, a subset H of $V(G)$ is **Complementary Component Dominant** or **CCD** if both the following conditions hold:*

(i) For all adjacent pairs of vertices, $x, y \in H$, the number of neighbours of x in $V(G) \setminus H$ is equal to the number of neighbours of y in $V(G) \setminus H$.

(ii) For all adjacent pairs of vertices, $u, v \in V(G) \setminus H$, the number of neighbours of u in H is equal to the number of neighbours of v in H.

Note that this definition implies that if H is complementary component dominant in G, then so is $V(G) \setminus H$.

Theorem 2.2. *Let G be a graph. A subset H of $V(G)$ is 0_2-invoking in G if and only if H is CCD.*

Proof. (Necessity) Let a graph G have the 0-configuration. Suppose $H \subseteq V(G)$ is CCD. In Figure 3, we see $G|_H$ and $G|_{V(G)\setminus H}$ separated into their respective connected components.

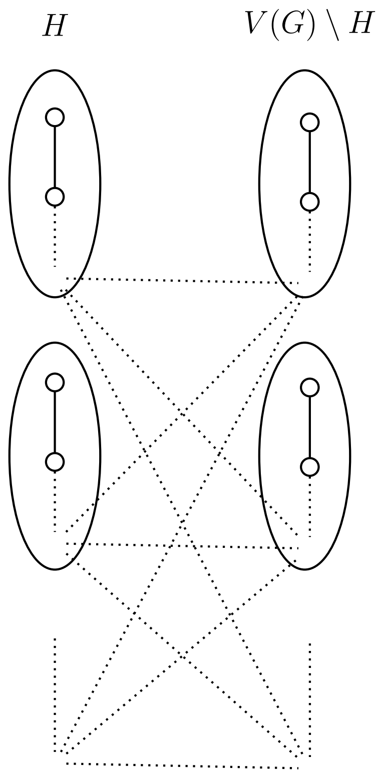


Figure 3: Graph, G, with 0_2-invoking subset, H, of $V(G)$

Remember that when H is perturbed, the edges that have both endpoints in H will have chips travelling along them both ways. We can equivalently view these edges as not having any chips travelling along them. For all vertices h in H, let $\deg_{V(G)\setminus H}(h)$ be the number of vertices in $V(G) \setminus H$ that are adjacent to h, and for all vertices g in $V(G) \setminus H$, let $\deg_H(g)$ be the number of vertices in H that are adjacent to g. Thus, when every vertex in H sends a chip to each of its neighbours as a result of the perturbation, the resulting configuration (at step $t = 1$) leaves every vertex, h, in H with a number of chips equal to $0 - \deg_{V(G)\setminus H}(h)$. Every vertex, g, in $V(G) \setminus H$ would be left with $0 + \deg_H(g)$ chips. We know from the definition of CCD that every pair of adjacent vertices in H must be adjacent to the same number of vertices in $V(G) \setminus H$. By transitivity, this will extend to entire connected components within $G|_H$.

Since the definition of CCD also states that the vertices of $V(G) \setminus H$ follow the same rule with every adjacent pair of vertices being adjacent to the same number of vertices in the complement, we get, by transitivity, that this extends to entire connected components in $G|_{V(G)\setminus H}$. Thus at step 1, each connected component of $G|_H$ will have the same stack size and each connected component of $G|_{V(G)\setminus H}$ will have the same stack size.

At step 1, every vertex in H has a negative stack size and each vertex in the complement has a positive stack size. So, when the vertices fire at step 1, every vertex in H will receive from each of its neighbours in $V(G) \setminus H$ and will not send to or receive from any vertices in H. Likewise, every vertex in $V(G) \setminus H$ will send to each of its neighbours in H and will not send to or receive from any vertices in $V(G) \setminus H$. So for each $h \in H$, we get that

$$\begin{aligned} |h|_2 &= |h|_1 + \deg_{V(G)\setminus H}(h) \\ &= -\deg_{V(G)\setminus H}(h) + \deg_{V(G)\setminus H}(h) \\ &= 0 \end{aligned}$$

and for all $g \in V(G) \setminus H$,

$$\begin{aligned} |g|_2 &= |g|_1 - \deg_H(g) \\ &= \deg_H(g) - \deg_H(g) \\ &= 0. \end{aligned}$$

Thus, the 0-configuration is restored in the first two steps.

(Sufficiency) Let H be a perturbation subset of $V(G)$, suppose H is 0_2-invoking, and suppose that at step 0, every vertex in G has 0 chips. If H is perturbed, then the configuration at step 2 is again, the 0-configuration. This implies that the net effect of two steps of firings on each vertex is $+0$. Every vertex in H will necessarily send a chip to each of its neighbours in $V(G) \setminus H$ as a result of the perturbation (at step 0) and will receive from those same vertices in the firing at step 1. Thus for all vertices h in H, if h receives a chip from a vertex in H during the firing at step 1, then h must also send a chip to a vertex in H at step 1 as well. However following the perturbation, for each connected component H_i in H, there must exist some vertex in H_i that has no poorer neighbours in H_i. So if any chip is sent from a vertex in H to another vertex in H during the firing at step 1, then there will exist at least one vertex h_i that received a chip from a neighbour in H, but did not send a chip to a neighbour in H. This implies that h_i will have a positive stack size at step 2, having received more chips in the firing at step 1 than it sent in the initial firing. This, however, contradicts our assumption that H is 0_2-invoking. Thus, we can conclude that every vertex in a connected component in $G|_H$ has a common stack size after the initial firing. So each vertex belonging to the same connected component in $G|_H$ shares the same number of neighbours in $V(G) \setminus H$. A similar argument will show the result for vertices in $V(G) \setminus H$. Thus, we can conclude that all 0_2-invoking subsets are CCD. □

Corollary 2.3. *If H is 0_2-invoking in G, then so is $V(G) \setminus H$.*

Note that not all graphs have a proper non-trivial 0_2-invoking subset. In Figure 4, we see such a graph. This can be justified by first supposing, by way of contradiction, that v_2 were in a such a subset. Note that either v_5 or v_6 must be in such a subset. If this subset contains both v_2 and v_6, then v_5 is adjacent to two vertices in H and v_3 is adjacent to only one vertex in H. So, v_3 must be in H. Now, v_4 must be in H since it is adjacent to only one vertex in H while v_5 is adjacent to 3 vertices in H. Also, v_1 must be in H because v_4 is adjacent to 2 vertices in H while v_3 is only adjacent to 1. Finally we have reached a contradiction as v_1 is adjacent to no vertices in $V(G) \setminus H$. The other cases follow similarly.

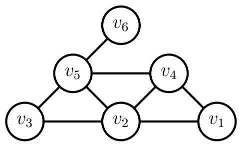


Figure 4: Graph with no proper nontrivial 0_2-invoking subsets

We now provide some results relating dominating sets to 0_2-invoking sets. Given a graph G, a *dominating set* is a subset D of $V(G)$ such that every vertex in $V(G)$ is either in D or adjacent to a vertex in D. The domination number of a graph G, $\gamma(G)$, is the size of the smallest dominating set in G. A *minimal dominating set* is a dominating set M such that if any vertex were removed from M, then the resulting set would not be dominating.

Corollary 2.4. *For all connected graphs G, all nontrivial 0_2-invoking subsets of $V(G)$ are also dominating sets.*

Proof. Let G be a graph and let $H \subseteq V(G)$ be a nontrivial 0_2-invoking subset. By Theorem 2.2, H is CCD. By the definition of CCD, every vertex in the complement of H must be adjacent to at least one vertex in H unless H is empty. Since H is nontrivial, H is dominating. □

Note that the graph in Figure 4 has a domination number of 2, while the size of the smallest nontrivial 0_2-invoking subset of its vertices is 6. This shows us that the 2-perturbation quiescent number of a graph G, $PQ_2(G)$, is not necessarily equal to the domination number $\gamma(G)$. So for an arbitrary graph, a minimal dominating set may not be 0_2-invoking, but for paths they are.

Lemma 2.5. *Every minimal dominating set of P_n, $n \geq 2$, is CCD and thus, 0_2-invoking.*

Proof. Let H be a minimal dominating set of P_n, $n \geq 2$. We will show that H is CCD. Since H is a minimal dominating set, every pair of adjacent vertices in $V(P_n) \setminus H$ are adjacent to exactly one vertex in H each. Similarly, every pair of adjacent vertices in H are adjacent to exactly one vertex in $V(P_n) \setminus H$ each, since H is a minimal dominating set. Thus, H is CCD. □

Given a graph G, an *independent set* is a subset I of $V(G)$ such that no two vertices in I are adjacent in G. From [1], an *efficient dominating set*, or *perfect code*, is an independent subset, A, of the vertex set of a graph, G, such that every vertex in $V(G) \setminus A$ is adjacent to exactly one vertex in A. Thus from the definition of CCD and Theorem 2.2 we obtain the following result.

Corollary 2.6. *Efficient dominating sets (or perfect codes) are CCD and thus* 0_2*-invoking.*

Question 2.7. *Is there a characterization of minimal dominating sets that are also* 0_2*-invoking subsets?*

We will now briefly look at an approach to this problem for graphs with small domination numbers. If $\gamma(G) = 1$, then there must be a dominating vertex. A single vertex in G is itself a 0_2-invoking set. If $\gamma(G) = 2$, with dominating set $\{x, y\}$, then the solution is not so simple. We will break the problem into two cases: x not adjacent to y, and x adjacent to y. Suppose first that x and y are not adjacent. For this pair of vertices to also be a 0_2-invoking set, it must be true that the set $\{x, y\}$ is also complementary component dominant.

So, every vertex in a given connected component in $G \setminus \{x, y\}$ must be adjacent to the same number of vertices in $\{x, y\}$ (either 1 or 2). Consider the subset of vertices adjacent to x and not adjacent to y, call it V_x, and the subset of vertices adjacent to y and not adjacent to x, call it V_y, and the subset of vertices adjacent to both x and y, call it V_{xy}. In order for $\{x, y\}$ to be complementary component dominant, it must be true that no edges exist between V_{xy} and $V_x \cup V_y$.

Now, if x and y are adjacent, we must also have an additional rule that $|V_x| = |V_y|$ since both x and y must be adjacent to the same number of vertices. Moving to dominating sets of size 3 or greater appears to be much more difficult.

In $K_{n,n}$, $n \geq 1$, minimal dominating sets come in two forms: either one vertex from each partition, or an entire partition. In both instances, these sets are CCD and thus, 0_2-invoking.

In complete multi-partite graphs, minimal dominating sets come in two forms: either one vertex from two different partitions, or an entire partition. The former is not necessarily CCD, while the latter is necessarily CCD.

Question 2.8. *Is there a graph G such that some subset of $V(G)$ is 0-invoking but not* 0_2*-invoking?*

We now show the answer to this question is no. In [4], Long and Narayanan introduced a potential function, which we will make use of in the next result. Let G be a graph on n vertices labelled $\{v_1, v_2, .., v_n\}$. They define the define the potential function $P(t)$ by:

$$P(t) = \sum_{v \in V(G)} |v|_t \times |v|_{t+1}. \tag{1}$$

Theorem 2.9. *Let G be a finite simple graph with the* 0*-configuration and $H \subset V(G)$ a perturbation subset. If H is* 0*-invoking, then H is* 0_2*-invoking.*

Proof. Let G be a finite simple graph with the 0-configuration and let $H \subset V(G)$ be a 0-invoking perturbation subset.

Let $t = 0$ be the perturbation of H and $t = 1$ be the time step where the standard diffusion rules are thereafter followed. $P(0) = 0$ since every vertex of G begins with a stack size of 0. H is a 0-invoking set, therefore at some time t', $P(t') = 0$. From [4] it is known that Equation 1 is non-increasing with time, therefore $P(t) \geq 0$ for all $t \geq 1$.

At time $t = 1$, consider the stack size of a vertex, v. If v is in H, then $|v|_1 = -\deg_{N(H)}(v)$. If v is in $N(H)$, then $|v|_1 = \deg_H(v)$. Otherwise, $|v|_1 = 0$.

We will show that if H is 0-invoking, but not 0_2-invoking, then $P(1) < 0$. To do so, we will 'partially' fire each vertex, to incrementally update the potential function. We will look at how the potential function changes as chips move along the edges of the graph one by one. Order the edges of G as $e_1, e_2, \ldots, e_m$ and let $|v|_{(t,i)}$ be the stack size of vertex v at time t after chips were moved along the first i edges in our ordering (note it is possible for an edge to experience no chips transferred along it if the endpoints have the same stack size, or equivalently that each endpoint sends and received a chip from each other). Thus for all $v \in V(G)$, $|v|_{(t,m)} = |v|_{(t+1)}$, since after all m edges have experienced the transfer of chips, we obtain the stack sizes for each vertex at time $t + 1$. We will show that as i increases, the potential function decreases after each of these incremental updates, and conclude that $P(1) < 0$, a contradiction.

Order the edges of G as $\{e_1, \ldots e_r, e_{r+1}, \ldots e_\ell, e_{\ell+1}, \ldots e_m\}$, where $e_1, \ldots, e_r$ are the edges that exist between vertices of H and $N(H)$, and $e_{r+1}, \ldots, e_\ell$ are edges within the subgraph induced by H, and $e_{\ell+1}, \ldots, e_m$ are the remaining edges of G.

Consider Equation 1 rewritten to represent the incremental updates to the potential function. We start at time $t = 1$ and take into account the firing process as a chip moves along each edge of the graph. We let $P(1, i)$ denote the potential function value after the first i edges of our edge ordering have had chips travel along them,

$$P(1, i) = \sum_{v \in V(G)} |v|_1 \times |v|_{(1,i)}, \tag{2}$$

with $|v|_1$ representing the stack size of v at $t = 1$ as no vertices have fired.

It should be noted that for all vertices $|v|_{(1,r)} = 0$. This is because the vertices of $N(H)$ send back the chips they received from each neighbour in H during the perturbation at $t = 0$, and any vertex not in $H \cup N(H)$ has stack size 0 at $t = 1$, thus $P(1, r) = 0$.

If $r = m$, then H is 0-invoking and 0_2-invoking. We are assuming that H is not 0_2-invoking. Thus, there is some edge in either the subgraph induced by H or the subgraph induced by $N(H)$ or between a vertex in $N(H)$ and one in $V(G) \setminus (H \cup N(H))$ that must experience a chip transfer. Note: no chips are transferred along edges in the subgraph induced by $V(G) \setminus (H \cup N(H))$ since the stack size of each vertex in that subgraph is 0 at $t = 1$. Suppose some edge, e_j, which does not have both an endpoint in H and an endpoint in $N(H)$, has a chip move along it. Thus the first j edges have experienced a transfer of chips, for $r < j \leq m$. Let $e_j = uv$ and without loss of generality, assume the chip was transferred from u to v, thus $|u|_1 > |v|_1$. We will compare $P(1, j)$ and $P(1, j-1)$ to see how the potential function has incrementally changed.

$$\begin{aligned}
P(1,j) - P(1,j-1) &= \sum_{v \in V(G)} |v|_1 \times |v|_{(1,j)} - \sum_{v \in V(G)} |v|_1 \times |v|_{(1,j-1)} \\
&= \sum_{v \in V(G)} |v|_1 (|v|_{(1,j)} - |v|_{(1,j-1)}) \\
&= |u|_1 (|u|_{(1,j)} - |u|_{(1,j-1)}) + |v|_1 (|v|_{(1,j)} - |v|_{(1,j-1)}) \\
&= |u|_1(-1) + |v|_1(1) \\
&= |v|_1 - |u|_1 \\
&< 0.
\end{aligned}$$

As the potential function at $t = 1$ is incrementally updated, after the first r edges have fired, the potential function is 0. Each incremental update thereafter obtained by transferring chips along the remaining edges of the graph result in a decrease of the potential function, hence resulting in a negative potential function at $P(1)$. This is a contradiction, so any 0-invoking set of vertices is also 0_2-invoking. □

3 Paths

With a general result characterizing 0_2-invoking subsets on all graphs, we now focus on paths to show $PQ_2(P_n) = \lceil \frac{n}{3} \rceil$, for all $n \geq 1$ (Theorem 3.3), and we determine the number of 0_2-invoking subsets on a path (Theorem 3.7).

3.1 Path Introduction

Before our results on counting 0_2-invoking subsets and calculating $PQ_2(P_n)$, we must introduce some definitions and lemmas regarding diffusion on paths.

Let G be a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $A \subseteq E(G)$. A *graph orientation* of a graph G is a mixed graph obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge xy in $A \subseteq E(G)$. We refer to the edges that are in $E(G) \setminus A$ as *flat*. We refer to the assignment of either $x \to y$, $y \to x$, or flat to an edge xy as xy's *edge orientation*. On a path drawn on a horizontal axis, two directed edges in a graph orientation *agree* if they either both point left or both point right.

Let R be a graph orientation of a graph G. A *suborientation* R' of R is a graph orientation of some subgraph G' of G such that every edge xy in G' is assigned the same edge orientation as in R.

Given two configurations, C and D, of a graph G, in which the vertices are labelled, C and D are *equal* if $|v|^C = |v|^D$ for all $v \in V(G)$.

In Figure 1, the period length is 2 and the preperiod length is 3.

Lemma 3.1. *In diffusion, every configuration induces a graph orientation.*

Proof. Let G be a graph and C_t a configuration on G. For all pairs of adjacent vertices u, v in G at step t, exactly one of the following holds: u gives a chip to v, v gives a chip to u, or the stack sizes of u and v are equal in C_t. Let uv be an edge. Assign directions as follows:

- If u gives a chip to v at time t, assign uv the edge orientation $u \to v$.
- If v gives a chip to u at time t, assign uv the edge orientation $v \to u$.
- If the stack sizes of u and v are equal at time t, do not direct the edge uv.

Thus, a graph orientation on G results. □

We say that this graph orientation is *induced* by C_t, the configuration of G at time t. We see an example of a graph orientation induced by a configuration in diffusion in Figure 5.

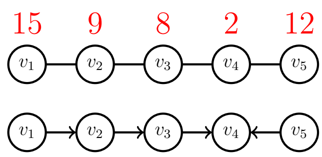


Figure 5: Configuration on P_5 and its induced graph orientation.

Let $\overline{\mathrm{Seq}(C_0)} = \{C_t, C_{t+1}, \ldots, C_{t+p-1}\}$ be the ordered set of configurations contained within the period of a configuration sequence $\mathrm{Seq}(C_0)$, where p is the length of the shortest period, and the period begins at step t. A configuration D on a graph G is a *period configuration* if $D \in \overline{\mathrm{Seq}}(C)$ for some configuration C. A configuration D on a graph G is a *p_2-configuration* if $D \in \overline{\mathrm{Seq}}(C)$ for some configuration C and $\overline{\mathrm{Seq}}(C)$ has exactly 2 elements. A *period orientation* is a graph orientation that is induced by a period configuration. A *p_2-orientation* is a graph orientation that is induced by a p_2-configuration. A *0-orientation* is a graph orientation that is induced by a 0-configuration.

A configuration at step t in a configuration sequence is a 0-*preposition* if the configuration at step $t+1$ is the 0-configuration. The underlying orientation, R, of a configuration is a 0-*preorientation* if there exists a 0-preposition which has R as its underlying orientation.

Lemma 3.2. *Given a graph, G, and an orientation R, there is at most one configuration which both induces R and is a 0-preposition.*

Proof. Let G be a graph and R an orientation of G. The orientation R dictates the number of chips that each vertex will give and receive at the next firing. Thus, for each vertex v_k in G, the stack size of v_k following the next firing is equal to the current stack size of v_k plus the number of edges directed toward v_k, A_{v_k}, minus the number of edges directed away from v_k, B_{v_k}. So, if we have that $|v_k| + A_{v_k} - B_{v_k} = 0$, then the stack size of v_k can be determined because it is the only unknown in the equation. □

3.2 Results on Paths

Now, with sufficient background information, we introduce our two main results on paths with Theorems 3.3 and 3.7.

Theorem 3.3. *$PQ_2(P_n) = \lceil \frac{n}{3} \rceil$, $n \geq 1$.*

Proof. We will first prove that $PQ_2(P_n) \geq \lceil \frac{n}{3} \rceil$ and then prove that $PQ_2(P_n) \leq \lceil \frac{n}{3} \rceil$.

From [2], the domination number of a path P_n is $\lceil \frac{n}{3} \rceil$, $n \geq 1$. By Corollary 2.4, we know that every nontrivial 0_2-invoking subset of a graph is also a dominating set. Thus, $PQ_2(P_n) \geq \lceil \frac{n}{3} \rceil$.

Next, by Lemma 2.5, we know that every minimal dominating set of a path, P_n, is also a 0_2-invoking subset of $V(P_n)$. Therefore, $PQ_2(P_n) \leq \lceil \frac{n}{3} \rceil$. □

Let J_n represent the number of 0_2-invoking subsets that exist on P_n. We now count all 0_2-invoking subsets on a path with $n \geq 2$ vertices. Label the vertices v_1, v_2, $\ldots$, v_n.

Lemma 3.4. *Let $H \subset V(P_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, $n \geq 2$, be proper, non-trivial, and 0_2-invoking. Then $v_n \in H$ if and only if $v_{n-1} \in V(G) \setminus H$.*

Proof. Let $H \subset V(P_n) = \{v_1, v_2, \ldots v_{n-1}, v_n\}$ be 0_2-invoking, proper and nontrivial.
(Sufficiency) Suppose first that $v_n \in H$. We know that v_{n-1} is the only neighbour of v_n in P_n. If $v_{n-1} \in H$, then v_n would be adjacent to 0 vertices in $V(P_n) \setminus V(H)$ and thus, since H is 0_2-invoking, every vertex in the same connected component as v_n in $G|_H$ would be adjacent to no vertices in $V(G) \setminus H$. Since P_n is connected, this implies that H is not a proper subset of $V(P_n)$ which is a contradiction. Thus, if $v_n \in H$, then $v_{n-1} \in V(G) \setminus H$.
(Necessity) Suppose now that $v_{n-1} \in V(G) \setminus H$. Then if $v_n \in V(G) \setminus H$, it would be adjacent to 0 vertices in H and thus, since H is 0_2-invoking, every vertex in the same connected component as v_n in $G|_H$ would be adjacent to 0 vertices in H. Since P_n is connected, this implies that H is the trivial subset of $V(G)$ which is a contradiction. Thus, if $v_{n-1} \in V(G) \setminus H$, then $v_n \in H$. □

Corollary 3.5. *Let $H \subset V(P_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, $n \geq 2$, be proper, non-trivial, and 0_2-invoking. Then $v_1 \in H$ if and only if $v_2 \in V(G) \setminus H$.*

Let F_n represent the n-th Fibonacci number.

Lemma 3.6. *The number of binary sequences of length n without three consecutive 0's or three consecutive 1's is $2F_{n+1}$.*

Proof. We will prove this by induction, beginning with base cases of length $n = 1, 2, 3$. There are two binary sequences of length 1 and $2F_{1+1} = 2F_2 = 2$. There are four binary sequences of length 2 and $2F_{2+1} = 2F_3 = 4$. There are $2^3 = 8$ binary sequences of length 3, and two of these have three consecutive 0's or three consecutive 1's. The resulting value, 6, is equal to $2 \times 3 = 2 \times F_4 = 2 \times F_{3+1}$. Now suppose that there exist $2F_{k+1}$ binary sequences of length k which do not have three consecutive 0's or three consecutive 1's. We will call all such binary sequences *satisfactory*. Every satisfactory binary sequence of length $k+1$ must begin with either 01, 001, 10, or 110. Since every satisfactory sequence of length k must begin with either a 0 or a 1, we can conclude that the number of satisfactory sequences of length $k+1$ that begins with either 01 or 10 is equal to the number of satisfactory sequences of length k, $2F_{k+1}$. This is because we append a 1 to the start of the sequence if it starts with a 0 and append a 1 otherwise. Similarly, the number of satisfactory sequences of length $k+1$ that begins with either 001 or 110 is equal to the number of satisfactory sequences of length $k-1$, $2F_k$. So, by induction, the number of satisfactory binary sequences of length $k+1$ is $2F_k + 2F_{k+1} = 2(F_k + F_{k+1}) = 2F_{k+2}$. □

Theorem 3.7. *$J_n = J_{n-1} + J_{n-2} - 2$, for $n \geq 3$, with $J_1 = 2$ and $J_2 = 4$.*

Proof. Note first that we are including the trivial and improper cases, so as to count every 0_2-invoking set on P_n. We begin with the initial values. The path with only one vertex cannot send chips because it has no edges. Thus, whether the lone vertex is in the perturbation subset or not, the chosen set is 0_2-invoking. So, P_1 has two 0_2-invoking subsets: $\emptyset$ and $V(P_1)$. On P_2, a perturbation of any subgraph will return to the fixed configuration after another step. Thus, P_2 has four 0_2-invoking subsets.

Trivially, the empty subset and the entire vertex set are 0_2-invoking in P_n. We will take note of this and move forward counting the 0_2-invoking subgraphs that are both nonempty and have nonempty complement.

Suppose we have P_n partitioned in such a way that H (and thus also, $V(G) \setminus H$) is a 0_2-invoking subset of vertices, where $H \neq \emptyset$ and $H \neq V(P_n)$. We will now count all such possible subsets.

By Lemma 3.4 and Corollary 3.5, we know that v_n and v_{n-1} must be in different partitions and similarly for v_1 and v_2. Note, since H is non-empty, for H to be 0_2-invoking the subgraph it induces cannot contain a path of length 3, as the center vertex of this subpath is not adjacent to any vertices in the other partition, and by Theorem 2.2 would mean $H = V(G)$, a contradiction.

By Lemma 3.6, and remembering to account for the trivial and improper cases, we know that this means that the number of 0_2-invoking sets of P_n is $2F_{n+1-2} + 2 = 2F_{n-1} + 2$. So, $J_n = 2F_{n-1} + 2$.

$$
\begin{aligned}
J_n &= 2F_{n-1} + 2 \\
&= 2(F_{n-2} + F_{n-3}) + 2
\end{aligned}
$$

$$
\begin{aligned}
&= 2F_{n-2} + 2F_{n-3} + 2 \\
&= J_{n-1} + 2F_{n-3} \\
&= J_{n-1} + J_{n-2} - 2
\end{aligned}
$$

□

4 Conclusion

The results in this paper revolve around the broad question "How can we categorize those perturbation subsets that lead to the 0-configuration being eventually restored after some amount of steps?" We proved that all 0-invoking subsets are indeed 0_2-invoking and then characterized all 0_2-invoking subsets as CCD (Theorem 2.2). The study of CCD in families of graphs would be an avenue for future research, such as counting the number of such sets, as was done for paths in this work.

In its most general form, a perturbation is a kind of disruption to the stack sizes of a configuration. In this paper, we analyzed when a perturbation of the 0-configuration eventually returned to the 0-configuration. However, a more general question would be "Which initial configurations can return after being perturbed?" Does it matter how many vertices are in a perturbation set when answering this question? Starting from the 0-configuration, what configurations can exist within the period of perturbation diffusion?

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