Regular digraphs of diameter 2 and maximum order

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Abstract — It is known that *Moore* digraphs of degree $d > 1$ and diameter $k > 1$ do not exist (see [16] or [4]). For degree 2, it has been shown that for diameter $k \geq 3$ there are no digraphs of order 'close' to, i.e., one less than, the *Moore* bound [14]. For diameter 2, it is known that digraphs close to Moore bound exist for any degree because the line digraphs of complete digraphs are an example of such digraphs. However, it is not known whether these are the only digraphs close to Moore digraphs. In this paper, we shall consider the general case of directed graphs of diameter 2, degree $d \geq 3$ and with the number of vertices $n = d + d^2$, that is, one less than the *Moore* bound. Using the eigenvalues of the corresponding adjacency matrices we give a number of necessary conditions for the existence of such digraphs. Furthermore, for degree 3 we prove that there are no digraphs close to Moore bound other than the line digraph of K_4.

Keywords — digraphs, *Moore* bound, diameter, degree.

1. Introduction

By a *digraph* we mean a structure $G = (V, A)$ where $V(G)$ is a nonempty set of distinct elements called *vertices*; and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices $u, v \in V$ called *arcs*.

The *order* of a digraph G is the number of vertices in G, i.e., $|V(G)|$. An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in G$. Similarly, an *out-neighbour* of a vertex v is a vertex w such that $(v, w) \in G$. For $S \subset V(G)$ denote by $N^-(S)$ (respectively $N^+(S)$) the set of all in-neighbours (respectively out-neighbours) of elements of S. The *in-degree* (respectively *out-degree*) of a vertex $v \in G$ is the number of its in-neighbours (respectively out-neighbours) in G. If in a digraph G, the in-degree equals the out-degree ($= d$) for every vertex, then G is called a *diregular* digraph of degree d.

A v_0-v_k *walk* W of length k in G is an alternating sequence $(v_0 a_1 v_1 a_2 ... a_k v_k)$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i. A *closed* walk has $v_0 = v_k$. If the arcs $a_1, a_2, ..., a_k$ of W are distinct, W is called a *trail*. If, in addition, the vertices $v_0, v_1, ..., v_k$ are also distinct, W is called a *path*. A *cycle* C_k of length k is a closed trail of length $k > 0$ with all vertices distinct (except the first and the last).

The *distance* from vertex u to vertex v in G, denoted by $\delta(u, v)$, is defined as the length of the shortest path from vertex u to vertex v. Note that in general $\delta(u, v)$ is not necessary equal to $\delta(v, u)$. The *diameter* k of a digraph G is the maximum distance between any two vertices in G.

Let one vertex be distinguished in a diregular digraph of degree d and diameter k, having n vertices. Let n_i, $i = 0, 1, ..., k$ be the number of vertices at distance i from the distinguished vertex. Then,

$$n_i \leq d^i \qquad \text{for } i = 0, 1, ..., k \tag{1}$$

Hence,

$$n = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + ... + d^k \tag{2}$$

If the equality sign holds in (2) then such a digraph is called a *Moore* digraph. The right-hand side of (2) is called the *Moore* bound.

It is well known that except for trivial cases (for $d = 1$ or $k = 1$) *Moore* digraphs do not exist (see [16] or [4]). The trivial cases are the cycles C_{k+1} of length $k+1$ and the complete digraphs K_{d+1} on $d+1$ vertices. The problem of how 'close' to the *Moore* bound the order of a diregular digraph of diameter $k \geq 2$ and degree $d \geq 2$ can be is an interesting problem. Several results have been obtained. For instance, in [14] it is proved that for degree 2 there is no diregular digraph of diameter $k \geq 3$ whose order is one less than the *Moore* bound (i.e., the 'defect' is 1). Furthermore, for degree 2 and diameter $k > 2$, it has been shown that diregular digraphs with defect 2 do not exist for most values of k [15]. Digraphs with order close to *Moore* bound arise in the construction of optimal networks [3, 11, 12, 17].

The corresponding problem for undirected graphs has been studied extensively by several authors and many results have been obtained (see [5, 10]).

Throughout this paper, we shall consider only diregular digraphs of degree $d \geq 2$, diameter 2 and defect 1. Thus, the number of vertices n is one less than the *Moore* bound, i.e., $n = d + d^2$. We shall call such digraphs *(d)-digraphs.* Note that for $d = 1$, there is no digraph with diameter 2 and $n = 2$ vertices. There are exactly 3 nonisomorphic (2)-digraphs; these are described in Section 2.

In a (d)-digraph G, we define the *repeat* of a vertex v, $r(v)$ as a unique vertex w to which there are exactly two v-w paths of lengths not exceeding 2 (Prop. 2 from [1]). Let $r^2(v) = r(r(v))$, etc. The following result was proved in [1]:

Lemma A *For every vertex v of a (d)-digraph we have: (a) $N^+(r(v)) = r(N^+(v))$ and (b) $N^-(r(v)) = r(N^-(v))$.*

The permutation assigning to each vertex v its repeat $r(v)$ consists of permutation cycles. The smallest natural number l_v such that $r^{l_v}(v) = v$ is the length of the permutation cycle containing v.

Let m_l denote the number of permutation cycles of length l $(l = 1, 2, ..., n)$.

$M = \sum_{l=1}^{n} m_l$ (M is the number of all permutation cycles of G).

We shall use the eigenvalues of the adjacency matrix of a (d)-digraph G to derive general necessary conditions for the permutation cycles of G and hence for G itself (Section 3). In particular, this allows us to find restrictions on the set of possible permutation cycles of repeats in a (d)-digraph.

In Section 4, we obtain further results for some special cases of the permutation cycles of equal lengths.

In Section 5, we prove that for $d = 3$ there is only one (3)-digraph (up to isomorphism).

The main results in this paper are Theorems 1 and 2 for $d \geq 2$ (in Section 3) and Theorem 3 for $d = 3$ (in Section 5).

2. Digraphs of order 6

There are three non-isomorphic (2)-digraphs, i.e., with $n = 6$ vertices. These digraphs are presented in Figures 1, 2 and 3. They were found as a by-product of a computer search for a number of certain nonisomorphic digraphs [13]. Each figure shows a particular digraph together with the corresponding information on the repeats, the length of the shortest v-$r(v)$ walk and the number of permutation cycles.

The digraph of Figure 1 was first presented in [16] in a different context. It is the line digraph of the complete digraph K_3. Also in general, for each $d \geq 2$ the line digraph of K_{d+1} is a (d)-digraph [6].

It is worthwhile to notice that in some digraphs every arc is contained in exactly one circuit of length $\leq k+1$. The digraphs of Figures 1 and 3 have this property. On the other hand, the digraph of Figure 2 does not have the property (e.g., arc (5,4) lies in two such cycles.).

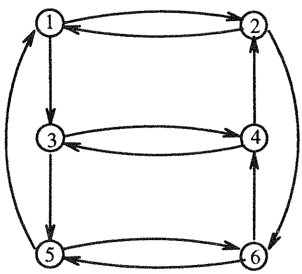
3. Eigenvalues of an adjacency matrix

Let A be the adjacency matrix of a (d)-digraph G (i.e., the (i,j) entry of A is 1 iff (i,j) is an arc of G, 0 otherwise) and P be the permutation matrix of repeats assigned to G such that its (i,j) entry is 1 iff $r(i)=j$.

It can be easily seen that the adjacency matrix of G fulfills the following matrix equation

$$A^2 + A + I = E + P \tag{3}$$

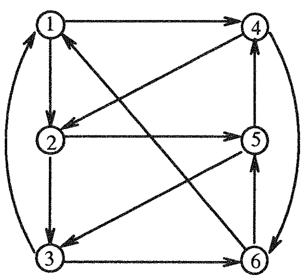
where I is the $n \times n$ identity matrix, E is the $n \times n$ matrix whose entries are all 1's and P is the permutation matrix of repeats.



v	r(v)	v-r(v)
1	1	0
2	2	0
3	3	0
4	4	0
5	5	0
6	6	0

$M = m_1 = 6$

Figure 1:

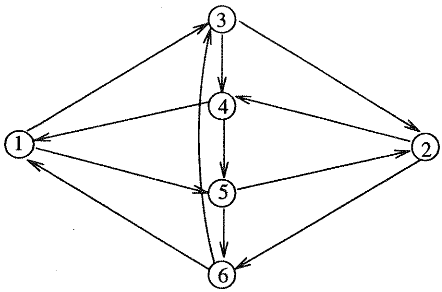


v	r(v)	v-r(v)
1	2	1
2	3	1
3	1	1
4	5	2
5	6	2
6	4	2

$M = m_3 = 2$

Figure 2:

For example, the digraph of Figure 3 has the adjacency matrix A and the permutation matrix of repeats P (Figure 4).



v	r(v)	v-r(v)
1	2	2
2	1	2
3	4	1
4	5	1
5	6	1
6	3	1

$m_2 = 1$

$m_4 = 1$

$M = 2$

Figure 3:

$$A = \begin{bmatrix} & & 1 & & 1 & \\ & & & 1 & & 1 \\ & 1 & & 1 & & \\ 1 & & & & 1 & \\ & 1 & & & & 1 \\ 1 & & 1 & & & \end{bmatrix} \qquad P = \begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & 1 & & & \end{bmatrix}$$

Figure 4:

We will try to obtain some information on G from (3) by considering the eigenvalues of A.

It is well known (see e.g. [8]) that there exists a permutation matrix Q such that $\tilde{P} := Q^{-1}PQ$ is a block diagonal matrix, where each diagonal block of $\tilde{P}$ is a cyclic permutation matrix of the form of Figure 5.

$$\tilde{P}_l = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & \ddots & \\ & & & & \ddots & 1 \\ 1 & & & & & 0 \end{bmatrix}$$

Figure 5:

To compute the spectrum of $E+P$ it is sufficient to compute the spectrum of $Q^{-1}(E+P)Q = E+\tilde{P}$ [8]. It is a routine matter to find an upper triangular form of $E+\tilde{P}-\lambda I$ and thus compute its determinant, which yields that the eigenvalues of $E+P$ are as follows.

1. $n+1$ with multiplicity 1
2. 1 with multiplicity $M-1$
3. $e^{i\frac{2\pi q}{l}}$ $(q = 1, ..., l-1)$ with multiplicity m_l for each $l \geq 2$

We do not know the spectrum of A, but it is well known that each eigenvalue λ of $E+P$ of multiplicity, say m, is associated with some of the roots of the equation

$$x^2 + x + 1 = \lambda. \tag{4}$$

If x_1 and x_2 are the roots of Eq. (4) then x_1 and x_2 are possible eigenvalues of A with the total multiplicity of m. Clearly, if λ is not real, then also the conjugate complex number $\overline{\lambda}$ is an eigenvalue of $E+P$ with the same multiplicity m. Then the roots $\overline{x}_1$ and $\overline{x}_2$ of the equation

$$x^2 + x + 1 = \overline{\lambda} \tag{5}$$

correspond as eigenvalues of A to x_1 and x_2 (note that x_j and $\overline{x}_j$ always have the same multiplicity). Let $\lambda = \lambda_1 + \lambda_2 i$. Then the four roots $x_1, x_2, \overline{x}_1$ and $\overline{x}_2$ of Eqs. (4) and (5) can be expressed by the following (easily checked) formula.

$$\begin{aligned} x_{1,2}, \overline{x}_{1,2} = \; & -\tfrac{1}{2} \pm \tfrac{1}{2\sqrt{2}} \left[\sqrt{\sqrt{(4\lambda_1 - 3)^2 + (4\lambda_2)^2} + 4\lambda_1 - 3} \right. \\ & \left. \pm i \sqrt{\sqrt{(4\lambda_1 - 3)^2 + (4\lambda_2)^2} - (4\lambda_1 - 3)} \right] \end{aligned} \tag{6}$$

Since for $\lambda = e^{i\frac{2\pi q}{l}}$ we have $\lambda_1 = cos\ \frac{2\pi q}{l}$ and $\lambda_2 = sin\ \frac{2\pi q}{l}$, the above formula can be transformed as follows (because $\lambda_1^2 + \lambda_2^2 = 1$):

$$\begin{aligned} x_{1,2}, \overline{x}_{1,2} = \; & -\tfrac{1}{2} \pm \tfrac{1}{2\sqrt{2}} \left[\sqrt{\sqrt{25 - 24c_{l_q}} + 4c_{l_q} - 3} \right. \\ & \left. \pm i \sqrt{\sqrt{25 - 24c_{l_q}} - (4c_{l_q} - 3)} \right] \end{aligned} \tag{7}$$

where $c_{l_q} = cos\ \frac{2\pi q}{l}$. In what follows these four roots will be denoted by $x_1(l,q)$, $x_2(l,q)$, $\overline{x}_1(l,q)$ and $\overline{x}_2(l,q)$ and the corresponding λ and $\overline{\lambda}$ by $\lambda(l,q)$ and $\overline{\lambda}(l,q)$, respectively.

If l is even, then for $q = l/2$ we have $c_{l_q} = cos\ \pi = -1$. Then Eq. (4) is the same as Eq. (5) and provides two mutually conjugate roots

$$x_1 = -\frac{1}{2} + \frac{\sqrt{7}}{2} i, \quad x_2 = -\frac{1}{2} - \frac{\sqrt{7}}{2} i \tag{8}$$

The following table (Fig. 6) is a schema for the eigenvalues of $E+P$, their multiplicities, the corresponding candidates for eigenvalues of A and their multiplicities. Note that if an eigenvalue occurs m times then its conjugate eigenvalue occurs m times too.

eig(E+P)	*multiplicity*	*eig(A)*	*multiplicity*
$n+1$	1	d	1
1	$M-1$	$-1,0$	$u, M-1-u$
each odd $l \geq 3$:			
$\lambda(l,1)$	m_l	$x_1(l,1), x_2(l,1)$	$v_l, m_l - v_l$
$\overline{\lambda}(l,1)$	m_l	$\overline{x}_1(l,1), \overline{x}_2(l,1)$	$v_l, m_l - v_l$
$\vdots$	$\vdots$		
$\lambda(l,(l-1)/2)$	m_l	$x_1(l,(l-1)/2), x_2(l,(l-1)/2)$	$v_l, m_l - v_l$
$\overline{\lambda}(l,(l-1)/2)$	m_l	$\overline{x}_1(l,(l-1)/2), \overline{x}_2(l,(l-1)/2)$	$v_l, m_l - v_l$
each even $l \geq 2$:			
$\lambda(l,1)$	m_l	$x_1(l,1), x_2(l,1)$	$v_l, m_l - v_l$
$\overline{\lambda}(l,1)$	m_l	$\overline{x}_1(l,1), \overline{x}_2(l,1)$	$v_l, m_l - v_l$
$\vdots$	$\vdots$		
$\lambda(l,l/2-1)$	m_l	$x_1(l,l/2-1), x_2(l,l/2-1)$	$v_l, m_l - v_l$
$\overline{\lambda}(l,l/2-1)$	m_l	$\overline{x}_1(l,l/2-1), \overline{x}_2(l,l/2-1)$	$v_l, m_l - v_l$
all even $l \geq 2$:			
-1	$\sum_{l\ even} m_l$	$-\frac{1}{2}+\frac{\sqrt{7}}{2}i, -\frac{1}{2}-\frac{\sqrt{7}}{2}i$	$\sum_{l\ even} \frac{m_l}{2}, \sum_{l\ even} \frac{m_l}{2}$

Figure 6:

eig(E+P)	*multiplicity*	*eig(A)*	*multiplicity*
7	1	2	1
1	1	-1	1
i	1	i	1
$-i$	1	$-i$	1
-1	2	$-0.5+1.3229i$, $-0.5-1.3229i$	1, 1

Figure 7:

For example, the digraph of Figure 3 (with A and P in Figure 4) has the spectrum of $E+P$ and A shown in Figure 7.

For the reader's convenience we give also the spectrum of A for the digraphs from Figures 1 and 2.

The eigenvalues:
Figure 1: $2, -1, -1, 0, 0, 0$;
Figure 2: $2, 0, -0.8679 + 1.1770i, -0.8679 - 1.1770i, -0.1321 + 1.1770i, -0.1321 - 1.1770i$.

Theorem 1 *For the numbers m_l ($l \geq 2$) of permutation cycles of even length of a (d)-digraph, $\sum_{l\ even} m_l$ is even.*

Proof
The eigenvalues of A in the last row of Figure 6 are mutually conjugate complex numbers and thus have equal number of occurences. These eigenvalues are different from those in the other rows (cf. (8) and (7) for $c_{l_q} \neq -1$). Then, the assertion follows. □

Since the digraph has no loops, trace$(A) = 0$, i.e.,

$$\begin{aligned} d - u + \sum_{l\ odd} \sum_{q=1}^{(l-1)/2} & [v_l x_1(l,q) + (m_l - v_l) x_2(l,q) \\ & + v_l \overline{x}_1(l,q) + (m_l - v_l)\overline{x}_2(l,q)] \\ + \sum_{l\ even} \sum_{q=1}^{\frac{1}{2}l-1} & [v_l x_1(l,q) + (m_l - v_l) x_2(l,q) \\ & + v_l \overline{x}_1(l,q) + (m_l - v_l)\overline{x}_2(l,q)] \\ + \frac{1}{2} \sum_{l\ even} & m_l(-1) = 0 \end{aligned} \tag{9}$$

The digraph G has m_1 vertices lying in 2-cycles of G (exactly these vertices are self-repeats). Thus trace$(A^2) = m_1$, but instead of squares of roots, we can use (4) and (5) (i.e., $x_1^2 = \lambda - 1 - x_1$, $x_2^2 = \lambda - 1 - x_2$, ...) and write

$$\begin{aligned} d^2 + u + \sum_{l\ odd} \sum_{q=1}^{(l-1)/2} & [v_l(\lambda(l,q) - 1 - x_1(l,q)) + (m_l - v_l)(\lambda(l,q) - 1 - x_2(l,q)) \\ & + v_l(\overline{\lambda}(l,q) - 1 - \overline{x}_1(l,q)) + (m_l - v_l)(\overline{\lambda}(l,q) - 1 - \overline{x}_2(l,q))] \\ + \sum_{l\ even} \sum_{q=1}^{\frac{1}{2}l-1} & [v_l(\lambda(l,q) - 1 - x_1(l,q)) + (m_l - v_l)(\lambda(l,q) - 1 - x_2(l,q)) \\ & + v_l(\overline{\lambda}(l,q) - 1 - \overline{x}_1(l,q)) + (m_l - v_l)(\overline{\lambda}(l,q) - 1 - \overline{x}_2(l,q))] \\ + \frac{1}{2} \sum_{l\ even} & m_l(-3) = m_1 \end{aligned} \tag{10}$$

Now we are going to simplify (9) and (10). Since $x_2 = -1 - x_1$, we have the following facts for the real parts of complex numbers.

$$\begin{aligned} re\{vx_1 + (m-v)x_2 + v\overline{x}_1 + (m-v)\overline{x}_2\} &= \\ re\{2vx_1 + 2(m-v)(-1-x_1)\} &= \\ -2(m-v) + 2(2v-m)\, re\{x_1\} \end{aligned}$$

Owing to the reasons of symmetry of (4) and (5) and by (9), (10) and Figure 6, we can take for x_1 any of the four roots from (7) and denote it simply by x. So (9) and (10) can be rewritten as follows.

$$\begin{aligned} d - u + \sum_{l\ odd} & \left[-(m_l - v_l)(l-1) + 2(2v_l - m_l) \sum_{q=1}^{(l-1)/2} re\{x(l,q)\} \right] \\ + \sum_{l\ even} & \left[-(m_l - v_l)(l-2) + 2(2v_l - m_l) \sum_{q=1}^{\frac{1}{2}l-1} re\{x(l,q)\} \right] \\ - \frac{1}{2} & \sum_{l\ even} m_l = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} d^2 + u + \sum_{l\ odd} & \left[-v_l(l-1) + 2m_l \sum_{q=1}^{(l-1)/2} c_{l_q} + 2(m_l - 2v_l) \sum_{q=1}^{(l-1)/2} re\{x(l,q)\} \right] \\ + \sum_{l\ even} & \left[-v_l(l-2) + 2m_l \sum_{q=1}^{\frac{1}{2}l-1} c_{l_q} + 2(m_l - 2v_l) \sum_{q=1}^{\frac{1}{2}l-1} re\{x(l,q)\} \right] \\ - \frac{3}{2} & \sum_{l\ even} m_l = m_1 \end{aligned} \tag{12}$$

The preceding observations can be summarized as follows.

Theorem 2 *For the numbers m_l of permutation cycles of length l, $l = 1, 2, ..., n$, of a (d)-digraph there are nonnegative integers u and v_l fulfilling (11) and (12).*

4. Special cases of permutation cycles

Now we are going to apply the preceding results to special cases when all the permutation cycles of a (d)-digraph are of the same length l, i.e., $M = m_l = n/l$.

In case $M = m_1$ (i.e., all the permutation cycles are trivial, $M = n$) we have no contradictions with Theorems 1 and 2. In fact, for each $d \geq 2$ there is such a (d)-digraph (the line digraph of the complete digraph K_{d+1} [6]).

Proposition 1 *If $M = m_2$ then $d = 3$.*

Proof.
By Theorem 2 (Eq. (11)) we have

$$d - u - \frac{1}{2}\frac{n}{2} = 0 \Longrightarrow d \geq \frac{n}{4} \Longrightarrow d \geq \frac{d^2+d}{4} \Longrightarrow d \leq 3.$$

Moreover, by Theorem 1 n must be divisible by 4. Therefore, for $d = 2$ no such digraph can exist. □
Note that Eq. (12) is also fulfilled.

Proposition 2 *If* $M = m_3$ *then* $d = 2$.

Proof.
By Theorem 2 (Eq. (11)) we get

$$d - u + \left[-(\frac{n}{3} - v_3)(3-1) + 2(2v_3 - \frac{n}{3})\ re\{x(3,1)\}\right] = 0 \qquad (13)$$

where

$$\begin{aligned} re\{x(3,1)\} &= -\frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{\sqrt{25 - 24\ cos\ \frac{2\pi}{3}} + 4\ cos\ \frac{2\pi}{3} - 3} \\ &= -\frac{1}{2} + \frac{1}{2\sqrt{2}}\sqrt{\sqrt{37} - 5} \end{aligned}$$

However, the last number is irrational. Hence in (13) we must have $2v_3 - n/3 = 0$, i.e., $v_3 = n/6$. Then (13) yields that $d = u + n/3 \geq (d^2+d)/3 \Longrightarrow d \leq 2$. □
Note that for $d = 2$ such a digraph does exist (see Figure 2).

Proposition 3 *If* $M = m_4$ *then* $d = 7$.

Proof.
By Theorem 2 (Eq. (11)) we get

$$\begin{aligned} & d - u + 2v_4 - \frac{5}{8}n = 0, \text{ where } v_4 \leq \frac{n}{4} \\ \Longrightarrow\quad & d + 2\frac{n}{4} \geq \frac{5}{8}n \Longrightarrow d \geq \frac{n}{8} \Longrightarrow d \leq 7. \end{aligned}$$

Morever, due to Theorem 1 n must be divisible by 8. This is possible only for $d = 7$, in which case v_4 must be $n/4 = 14$ and $u = 0$. □
Note that Eq. (12) is also fulfilled:

$$49 + 0 - 2 \cdot 14 + 28\ cos\ \frac{2\pi}{4} + 2(14 - 28)\ re\{x(4,1)\} - \frac{3}{2} \cdot 14 = 0$$

The following proposition deals with another special case.

Proposition 4 *If $M = m_n = 1$ (i.e., there is only one permutation cycle (of length n)) then no (d)-digraph can exist.*

Proof.
Since $n = d(d+1)$ is even, by Theorem 1 m_n must be even, which is a contradiction. □

5. The special case $d = 3$

In this section we will study the special case of $d = 3$ ($n = 12$). According to [1, Lemma 4] if a (3)-digraph G contains a 2-cycle then each vertex of G is contained in a 2-cycle. In other words, if there is a permutation cycle of repeats of length 1, then each permutation cycle is of length 1. Thus considering all other partitions of 12 into feasible lengths of permutation cycles, it is sufficient to deal with lengths at least 2. There are exactly 21 such partitions (12, 10+2, 9+3, ... , 2+2+2+2+2+2). Ten of them are excluded by Theorem 1 (9 cases) and Proposition 2. Further 8 cases are excluded by Theorem 2 (using it as above). This was checked independently by computing possible spectra of adjacency matrices on a computer. Unfortunately this spectral theory leaves 3 partitions (2+2+2+2+2+2, 4+4+2+2, 4+3+3+2) as possible. Consequently, we will next consider the following 4 cases:

- $M = m_1 = 12$ (as mentioned earlier, at least one such digraph exists)
- $M = m_2 = 6$
- $M = 4, m_2 = 2$ and $m_4 = 2$
- $M = 4, m_2 = 1, m_3 = 2$ and $m_4 = 1$.

First we are going to show that a (3)-digraph with $M = m_1$ is a line digraph. Recall a characterization from [7]: A digraph G is a line digraph iff for any three arcs (u_1, v_1), (u_1, v_2), (u_2, v_1) of G there is also the arc (u_2, v_2) in G.

Lemma 1 *Let G be a (3)-digraph with all the permutation cycles of repeats of length 1. If arcs (x_0, x_1), (x_0, x_2), (x_0, x_3) are in G then arcs (y, x_1), (y, x_2) and (y, x_3) are also in G whenever one of them is in G.*

Proof.
Since x_0 is a selfrepeat it lies in a unique 2-cycle (x_0, z, x_0) and we can assume that $z = x_3$. Moreover we can suppose that the subdigraph of Figure 8 is in G. As each vertex is a selfrepeat, we have

Property 1 *G contains no three arcs (u, v), (v, w) and (u, w) and no four arcs (u, v), (v, w), (u, v'), and (v', w).*

Denote $S_0 := \{x_1, x_2, x_3\}$, $S_1 := \{x_4, x_5, x_6\}$, $S_2 := \{x_7, x_8, x_9\}$, $S_3 := \{x_0, x_{10}, x_{11}\}$.

There are no arcs going from $S_3 - \{x_0\}$ to $S_0 \cup S_3$ (otherwise x_3 has two repeats). Hence the vertex y from the statement of above lemma belongs to $S_1 \cup S_2$. We can assume that $y \in S_1$. First observe that there are exactly 3 arcs from S_1 to S_0 (because we have to reach x_2, x_3 from x_1 and x_1 is a selfrepeat). Let G contain arc (x_6, x_2) and denote the remaining two outneighbours of x_6 by p and q. By Property 1, neither of p or q can be in $S_2 \cup \{x_0\}$. Also, none of them can be in S_1 (otherwise x_1 has two repeats). Therefore, we have $p, q \in S_0 \cup \{x_{10}, x_{11}\}$. Since there are no arcs (x_{10}, x_3) and (x_{11}, x_3) in G and x_6 must reach x_3, then either p or q, say p, is x_3 itself. This implies that $q = x_1$ (to reach all from x_6). Then $y = x_6$ and the lemma is proved. □

Thus we have the following assertion.

Corollary 1 *If a (3)-digraph G has a selfrepeat then G is the line digraph of K_4 (the complete digraph on 4 vertices).*

Proof.
By [1, Lemma 4] each vertex of G is a selfrepeat. Then by Lemma 1 for any two vertices u, v we have $N^+(u) = N^+(v)$ whenever $N^+(u) \cap N^+(v) \neq \emptyset$. Now according to the above mentioned result from [7] we see that G is a line digraph of a digraph H. Since H must be diregular of degree 3 with every arc lying in a 2-cycle, it must be K_4. □

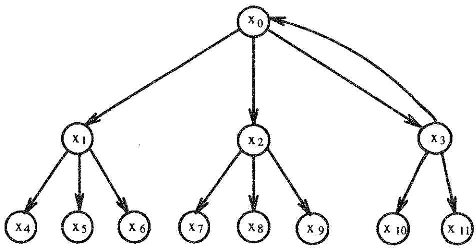


Figure 8:

Our next aim is to show that there is no other (3)-digraph. Consequently, from now on, we can assume $r(x) \neq x$, $\forall x \in G$. However, not all the following results will require this assumption in full.

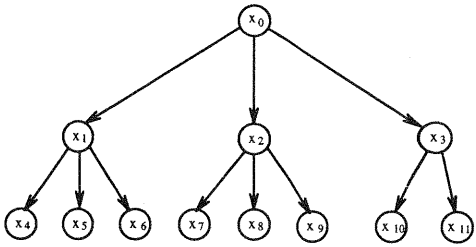
Lemma 2 *If G is a (3)-digraph and $x_0 \in V(G)$, $r(x_0) \neq x_0$, $r^2(x_0) = x_0$ then $\delta(x_0, r(x_0)) = \delta(r(x_0), x_0) = 2$.*

Proof

We can suppose that we have in G the following subdigraph (Figure 9). If $\delta(x_0, r(x_0)) = 1$ then $(x_0, r(x_0)) \in G$ and also $(r(x_0), r^2(x_0)) \in G$. Since all x_i are distinct and $r(x_0) \neq x_0$, this is not possible. Hence $\delta(x_0, r(x_0)) = \delta(r(x_0), x_0) = 2$. □

Corollary 2 *Refer to Figure 9. If $r(x_0) \neq x_0$ and $r^2(x_0) = x_0$ then $(r(x_0), x_0) \notin G$.*

If $r(x_0) \neq x_0$ and $r^2(x_0) = x_0$, then in view of Lemma 2 we can assume without loss of generality that G contains the subdigraph shown in Figure 10 excepting the dotted arcs. Since we have to reach x_0 from x_1, x_2 and x_3 and $(x_9, x_0) \notin G$ (by Corollary 2), without loss of generality we can suppose that (x_4, x_0), (x_7, x_0), $(x_{11}, x_0) \in G$ (the dotted arcs in Figure 10).



and $(x_3, r(x_0))$ in G

Figure 9:

Lemma 3 *Refer to Figure 10. If $r(x_0) \neq x_0$ and $r^2(x_0) = x_0$ then $r(x_2) \in \{x_4, x_7\}$ and $r(x_3) \in \{x_4, x_{11}\}$.*

Proof

By assumption (see Figure 10) $r(x_0) = x_9$ and $r(x_9) = x_0$.
$(x_2, x_9) \in G$ so $(r(x_2), r(x_9)) \in G$ i.e., $(r(x_2), x_0) \in G$, i.e., $r(x_2) \in \{x_4, x_7, x_{11}\}$. Similarly, $r(x_3) \in \{x_4, x_7, x_{11}\}$. Now $r(x_2) \neq x_{11}$. To see this assume $r(x_2) = x_{11}$. But $(x_7, x_{11}) \notin G$ since otherwise $r(x_7) = x_0$. Also $(x_9, x_{11}) \notin G$ since otherwise $r(x_3) = x_{11}$. Hence $r(x_2) \neq x_{11}$. Similarly, $r(x_3) \neq x_7$. Hence $r(x_2) \in \{x_4, x_7\}$ and $r(x_3) \in \{x_4, x_{11}\}$. □

Lemma 4 *Refer to Figure 10. If $r(x_0) \neq x_0$, $r^2(x_0) = x_0$ and $(x_0, x_i) \in G$ then $r^2(x_i) \neq x_i$, $i = 1, 2, 3$.*

Proof

Suppose $r^2(x_2) = x_2$. Then by Lemma 2, $\delta(x_2, r(x_2)) = 2$. Using Lemma 3, it follows

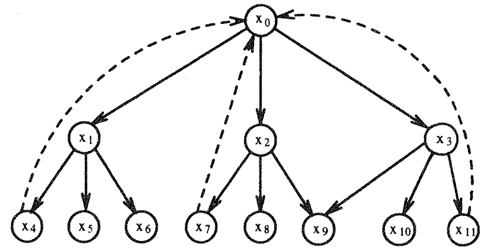


Figure 10:

that $r(x_2) = x_4$, Then $r(x_3) = x_{11}$. By Lemma 2, since $\delta(x_3, r(x_3)) = 1$ then $r^2(x_3) \neq x_3$. Since $r^2(x_0) = x_0$ and $r^2(x_2) = x_2$ and $(x_0, x_3) \in G$ so also $(r^2(x_0), r^2(x_3)) \in G$, i.e., $(x_0, r^2(x_3)) \in G$. It follows that $r^2(x_3) = x_1$ and $r^2(x_1) = x_3$. Since $(x_3, x_9) \in G$ then $(r^2(x_3), r^2(x_9)) \in G$, that is, $(x_1, x_9) \in G$ which is a contradiction and so $r^2(x_2) \neq x_2$. The vertex x_3 being isomorphic to x_2 (according to Figure 10) we have $r^2(x_3) \neq x_3$.

Next suppose $r^2(x_1) = x_1$, $r^2(x_2) \neq x_2$, $r^2(x_3) \neq x_3$. Then $r^2(x_2) = x_3$ and $r^2(x_3) = x_2$ since the arcs $(r^2(x_0), r^2(x_2))$ and $(r^2(x_0), r^2(x_3))$ must be in G.

Consider the repeat of x_1, $r(x_1)$. By Lemma 2, $\delta(x_1, r(x_1)) = \delta(r(x_1), x_1) = 2$ and so $r(x_1) \notin \{x_0, x_1, x_4, x_5, x_6\}$. Also $r(x_1) \neq x_9$ since $x_9 = r(x_0)$. $r(x_1) \neq x_2$ since $(x_2, x_9) \in G$ while $(x_1, x_0) \notin G$. Similarly $r(x_1) \neq x_3$. Furthermore, $r(x_1) \neq x_7$ since $(x_7, x_0) \in G$ and $(x_1, x_9) \notin G$. Similarly, $r(x_1) \neq x_{11}$. Also $r(x_1) \neq x_8$ since we can have at most one (x_i, x_8) for $i = 4, 5, 6$. Similarly $r(x_1) \neq x_{10}$. Thus $r^2(x_1) \neq x_1$. □

Corollary 3 *If G is a (3)-digraph then the permutation cycles of repeats are not all of length 2.*

Lemma 5 *If G is a (3)-digraph and G contains a permutation cycle of repeats of length 2 then G also contains a permutation cycle of repeats of length ≥ 6.*

Proof

Assume the subdigraph of Figure 10 is in G. Assume $r^2(x_0) = x_0, r(x_0) \neq x_0$. By Lemma 4, $r^2(x_1) \neq x_1, r^2(x_2) \neq x_2, r^2(x_3) \neq x_3$. Since (x_0, x_i), $i = 1, 2, 3$ is in G we must also have $(r^2(x_0), r^2(x_i))$, $i = 1, 2, 3$ in G, that is, $(x_0, r^2(x_i)) \in G$ for $i = 1, 2, 3$. Assume without loss of generality $r^2(x_2) = x_1$. Then $r^2(x_1) = x_3$ (since $r^2(x_3) \neq x_3$) and $r^2(x_3) = x_2$. Now consider $x_2, r(x_2), r^2(x_2), r^3(x_2), r^4(x_2), r^5(x_2)$, i.e., $x_2, r(x_2), x_1, r(x_1), x_3, r(x_3)$. These are six different vertices in G. To see this assume $r(x_1) = x_2$. Then $x_3 = r^2(x_1) = r(x_2)$ and $x_1 = r^2(x_2) = r(x_3)$. Hence $r(N^+(x_0)) = N^+(x_0)$. By Lemma A, $r(N^+(x_0)) = N^+(r(x_0))$ which implies that $N^+(x_9) = N^+(r(x_0)) = \{x_1, x_2, x_3\}$. Then x_9 has two repeats, namely x_0 and x_2 which is a contradiction. The other nontrivial cases can be excluded similarly. Thus $x_2, r(x_2), r^2(x_2), r^3(x_2), r^4(x_2), r^5(x_2)$ are all distinct vertices and G contains a permutation cycle of repeats of length ≥ 6. □

Corollary 4 *If G is a (3)-digraph then the permutation cycles of repeats are not of lengths 2,2,4,4.*

Corollary 5 *The permutation cycles of repeats of a* (3)*-digraph are not of lengths 2,3,3,4.*

Summarizing the results of this section we get:

Theorem 3 *There is exactly one (3)-digraph, namely the line digraph of K_4.*

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