Decompositions of $K_{m,n}$ and K_n into Cubes

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Abstract

We consider the decomposition of the complete bipartite graph $K_{m,n}$ into isomorphic copies of a d-cube. We present some general necessary conditions for such a decomposition and show that these conditions are sufficient for $d = 3$ and $d = 4$. We also explore the d-cube decomposition of the complete graph K_n. Necessary and sufficient conditions for the existence of such a decomposition are known for d even and for d odd and n odd. We present a general strategy for constructing these decompositions for all values of d. We use this method to show that the necessary conditions are sufficient for $d = 3$.

1 Introduction

Let K be a simple graph and let G be a subgraph of K. A *G-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of edge-disjoint subgraphs of K each of which is isomorphic to G and such that the edge sets of the G_i's form a partition of the edge set of K. The graph K is said to be *G-decomposable.*

The decomposition of graphs has been and remains the focus of a great deal of research (see [1, 5, 11]). In particular, K_k-decompositions of K_n (balanced incomplete block designs, see M. Hall [4]) and C-decomposition of K_n, where C is a cycle of given length [7], have received much attention. The decomposition of K_n into complete bipartite graphs is explored in [3, 15] and into complete m-partite graphs in [6]. This problem has also been addressed for K_n in connection with trees and forests [10, 13]. The decomposition of $K_{m,n}$ into cycles of length $2k$ is explored in [14].

The *d-cube* is the graph Q_d whose vertex set is the set of all binary d-tuples, $V(Q_d) = (\mathbf{Z}_2)^d$, and whose edge set consists of all pairs of vertices which differ in exactly one coordinate. Q_d has 2^d vertices, $d2^{d-1}$ edges and is bipartite.

In 1966, Rosa [12] introduced the notion of $\alpha-$valuation of a graph G as follows: Let $|E(G)| = n$ and let ψ be a one-to-one mapping of $V(G)$ into $N = \{0, 1, \ldots, n\}$. Then ψ is called an *$\alpha-$valuation of G* if (i) the set $\{|\psi(u) - \psi(v)| : uv \in E(G)\}$ equals $\{1, 2, \ldots, n\}$ and (ii) there exists a number λ such that, for every $(u, v) \in E(G)$, $\min\{\psi(u), \psi(v)\} \leq \lambda < \max\{\psi(u), \psi(v)\}$. A graph G admitting an $\alpha-$valuation is necessarily bipartite.

The following theorem establishing a connection between $\alpha-$valuations of a graph G and $G-$decompositions of complete graphs was proved by Rosa [12].

Theorem 1.1 *If a graph G with n edges has an $\alpha-$valuation, then there exists a G-decomposition of the complete graph K_{2cn+1}, for every positive integer c.*

A bipartite analogue of Theorem 1.1 was proved in [2].

Theorem 1.2 *If a graph G with n edges has an $\alpha-$valuation, then there exists a G-decomposition of the complete bipartite graph $K_{nx,ny}$, for all positive integers x and y.*

In 1981, Kotzig [8] showed that Q_d has an $\alpha-$valuation for all positive integers d, thus establishing necessary and sufficient conditions for the existence of Q_d-decompositions of K_n when d is even and in the case when d is odd and n is odd. The case d odd and n even remains open. The purpose of this paper is to explore Q_d-decompositions of $K_{m,n}$ and of K_n.

2 d-Cube Decompositions of $K_{m,n}$

In this section, we consider decompositions of the complete bipartite graph $K_{m,n}$ (where we assume $m \leq n$) into copies of a d-cube. In such a decomposition, the degree of each vertex of the d-cube (which is d-regular) must divide the degree of each vertex of $K_{m,n}$ and the number of edges of the d-cube must divide the number of edges of $K_{m,n}$. This implies:

(2.1) $d \mid m$ and $d \mid n$, and

(2.2) $d2^{d-1} \mid mn$.

Also, since the d-cube is a bipartite graph with each "part" of size 2^{d-1}, we need

(2.3) $2^{d-1} \leq m \leq n$.

The 2-cube is simply a 4-cycle and it is shown in [14] that the necessary conditions (2.1)-(2.3) are sufficient in this case. To establish sufficiency for $d = 3$ and $d = 4$, we will present decompositions of $K_{m,n}$ where the vertex set of $K_{m,n}$ is $(\mathbf{Z}_m \times \{0\}) \bigcup (\mathbf{Z}_n \times \{1\})$ (with the obvious bipartition) and the ordered pair (x, y) of this vertex set is represented by x_y. For $d = 3$ we have:

Theorem 2.1 *For $m \leq n$, a 3-cube decomposition of $K_{m,n}$ exists if and only if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$.*

Proof. We get the necessary conditions from (2.1)-(2.3). Under these conditions, either

(2.4) $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 6$,
(2.5) $m \equiv 3 \pmod 6$ and $n \equiv 0 \pmod{12}$, or
(2.6) $m \equiv 0 \pmod{12}$ and $n \equiv 3 \pmod 6$.

If (2.4) is satisfied, then $K_{m,n}$ can clearly be decomposed into isomorphic copies of $K_{6,6}$. Since $K_{m,n}$ and $K_{n,m}$ are isomorphic, (2.5) and (2.6) are equivalent. In either case, $K_{m,n}$ can be decomposed into a collection of graphs each of which is isomorphic to either $K_{6,6}$ or $K_{9,12}$. So for sufficiency, we only need to give 3-cube decompositions of $K_{6,6}$ and $K_{9,12}$.

We give the desired decomposition in an $m \times n$ array whose rows are labelled 0_0 through $(m-1)_0$ and whose columns are labelled 0_1 through $(n-1)_1$. The (i_0, j_1) entry of the array is q_k if edge (i_0, j_1) of $K_{m,n}$ is an edge of cube q_k in the decomposition. For a decomposition of $K_{6,6}$ into 3-cubes, consider:

	0_1	1_1	2_1	3_1	4_1	5_1
0_0	q_1	q_3	q_1	q_1	q_3	q_3
1_0	q_3	q_1	q_1	q_1	q_3	q_3
2_0	q_1	q_1	q_2	q_1	q_2	q_2
3_0	q_1	q_1	q_1	q_2	q_2	q_2
4_0	q_3	q_3	q_2	q_2	q_3	q_2
5_0	q_3	q_3	q_2	q_2	q_2	q_3

The set of 3-cubes $\{q_1, q_2, q_3\}$ forms a Q_3-decomposition of $K_{6,6}$.

For a decomposition of $K_{9,12}$ into 3-cubes, consider:

	0_1	1_1	2_1	3_1	4_1	5_1	6_1	7_1	8_1	9_1	10_1	11_1
0_0	q_1	q_1	q_1	q_4	q_2	q_2	q_2	q_4	q_3	q_3	q_3	q_4
1_0	q_1	q_1	q_5	q_1	q_2	q_2	q_5	q_2	q_3	q_3	q_5	q_3
2_0	q_1	q_6	q_1	q_1	q_2	q_6	q_2	q_2	q_3	q_6	q_3	q_3
3_0	q_7	q_1	q_1	q_1	q_7	q_2	q_2	q_2	q_7	q_3	q_3	q_3
4_0	q_4	q_5	q_5	q_4	q_8	q_9	q_5	q_4	q_8	q_9	q_8	q_9
5_0	q_4	q_5	q_5	q_4	q_8	q_9	q_8	q_9	q_8	q_9	q_5	q_4
6_0	q_4	q_5	q_6	q_7	q_7	q_6	q_5	q_4	q_7	q_6	q_5	q_4
7_0	q_7	q_6	q_6	q_7	q_7	q_6	q_8	q_9	q_8	q_9	q_8	q_9
8_0	q_7	q_6	q_6	q_7	q_8	q_9	q_8	q_9	q_7	q_6	q_8	q_9

The set of 3-cubes $\{q_1, q_2, \ldots, q_9\}$ forms a Q_3-decomposition of $K_{9,12}$. ∎

For $d = 4$ we have:

Theorem 2.2 *A 4-cube decomposition of $K_{m,n}$ (where $m \leq n$) exists if and only if $m \equiv n \equiv 0 \pmod 4$, $mn \equiv 0 \pmod{32}$ and $n \geq m \geq 8$.*

Proof. Again, (2.1)-(2.3) give the necessary conditions. As in Theorem 2.1, we find under these necessary conditions, that $K_{m,n}$ can be decomposed into a collection of graphs each of which is isomorphic to either $K_{8,8}$ of $K_{8,12}$.

For a decomposition of $K_{8,8}$ into 4-cubes, consider:

	0_1	1_1	2_1	3_1	4_1	5_1	6_1	7_1
0_0	q_1	q_1	q_1	q_2	q_2	q_1	q_2	q_2
1_0	q_1	q_1	q_2	q_1	q_1	q_2	q_2	q_2
2_0	q_2	q_2	q_2	q_1	q_1	q_2	q_1	q_1
3_0	q_1	q_2	q_2	q_2	q_1	q_1	q_2	q_1
4_0	q_1	q_2	q_1	q_1	q_2	q_2	q_2	q_1
5_0	q_2	q_1	q_1	q_1	q_2	q_2	q_1	q_2
6_0	q_2	q_1	q_2	q_2	q_1	q_1	q_1	q_2
7_0	q_2	q_2	q_1	q_2	q_2	q_1	q_1	q_1

The set of 4-cubes $\{q_1, q_2\}$ forms a Q_4-decomposition of $K_{8,8}$.

For a decomposition of $K_{8,12}$ into 4-cubes, consider:

	0_1	1_1	2_1	3_1	4_1	5_1	6_1	7_1	8_1	9_1	10_1	11_1
0_0	q_1	q_1	q_1	q_1	q_2	q_2	q_2	q_2	q_3	q_3	q_3	q_3
1_0	q_3	q_3	q_3	q_3	q_1	q_1	q_1	q_1	q_2	q_2	q_2	q_2
2_0	q_1	q_1	q_3	q_3	q_1	q_1	q_2	q_2	q_3	q_3	q_2	q_2
3_0	q_3	q_3	q_1	q_1	q_2	q_2	q_1	q_1	q_2	q_2	q_3	q_3
4_0	q_1	q_3	q_1	q_3	q_1	q_2	q_1	q_2	q_3	q_2	q_3	q_2
5_0	q_3	q_1	q_3	q_1	q_2	q_1	q_2	q_1	q_2	q_3	q_2	q_3
6_0	q_3	q_1	q_1	q_3	q_1	q_2	q_2	q_1	q_2	q_3	q_3	q_2
7_0	q_1	q_3	q_3	q_1	q_2	q_1	q_1	q_2	q_3	q_2	q_2	q_3

The set of 4-cubes $\{q_1, q_2, q_3\}$ forms a Q_4-decomposition of $K_{8,12}$. ∎

3 d-Cube Decompositions of K_n

In this section we examine Q_d-decompositions of the complete graph K_n. Kotzig [8] proved the following results concerning Q_d-decompositions of K_n:

(3.1) If d is even and there is a Q_d-decomposition of K_n, then $n \equiv 1 \pmod{d2^d}$.

(3.2) If d is odd and there is a Q_d-decomposition of K_n then either
(a) $n \equiv 1 \pmod{d2^d}$ or
(b) $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod d$.

(3.3) There is a Q_d-decomposition of K_n if $n \equiv 1 \pmod{d2^d}$.

The consequence of result (3.3) is that the necessary conditions described in (3.1) and (3.2) are sufficient whenever n is odd (and hence, whenever d is even). However, if d is odd and n is even, then (3.2) tells us that

(3.4) $n \equiv 0 \pmod{2^d}$ and

(3.5) $n \equiv 1 \pmod{d}$.

But Kotzig's results do not tell us if these conditions are sufficient. For a given d (odd), equations (3.4) and (3.5) have solution

$$n \equiv a_d \pmod{d2^d} \qquad (*)$$

for a unique a_d with $0 \leq a_d < d2^d$.

We now consider the question of whether or not (*) is sufficient for the existence of a Q_d-decomposition of K_n. This problem had been mentioned previously in the literature, see [5, 9]. It is known that for sufficiently large n, K_n can be decomposed into d-cubes whenever (*) is satisfied (see [16]) and it is known that there is a Q_3-decomposition of K_{16} (see [9]).

Theorem 3.1 *Let r, a and d be any positive integers with $d \geq 2$ and let $n = rd2^d + a$. Then there is a Q_d-decomposition of K_n if there are Q_d-decompositions of K_a and $K_{d2^d,a-1}$.*

Proof. Let $A = \{x_1, x_2, \ldots, x_a\}$ and let $V_1, V_2, \ldots, V_r$ be mutually disjoint sets of size $d2^d$ such that $V_i \cap A = \emptyset$ for all i. For each $i, j \in \{1, 2, \ldots, r\}$ with $i < j$, let $C_{i,j}$ be a Q_d-decomposition of $K_{d2^d,d2^d}$ where the bipartition is (V_i, V_j) (which exists for all positive integers d by Theorem 1.2). For each $i \in \{1, 2, \ldots, r\}$, let C_i be a Q_d-decomposition of K_{d2^d+1} (which exists for all $d \geq 2$, see [8]) where the vertex set of K_{d2^d+1} is $V_i \cup \{x_1\}$ and let D_i be a Q_d-decomposition of $K_{d2^d,a-1}$ where the bipartition is $(V_i, \{x_2, x_3, \ldots, x_a\})$. Let B be a Q_d-decomposition of K_a where the vertex set of K_a is A.

Then, C is a Q_d-decomposition of K_n where

$$C = \left(\bigcup_{i,j \in \{1,2,\ldots,r\}, i<j} C_{i,j} \right) \cup \left(\bigcup_{i=1}^{r} C_i \right) \cup \left(\bigcup_{i=1}^{r} D_i \right) \cup B$$

and the vertex set of K_n is V with $V = (\bigcup_{i=1}^{r} V_i) \cup A$. ∎

For $d = 3$, the conditions of the theorem are satisfied since there is a Q_3-decomposition of K_{16} (see [9]) and of $K_{15,24}$ by Theorem 2.1. Therefore:

Corollary 3.1 *There is a Q_3-decomposition of K_n if and only if $n \equiv 1$ or $16 \pmod{24}$.*

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