Three-Valued k-Neighborhood Domination in Graphs

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Abstract

Let $k \geq 1$ be an integer, and let $G = (V, E)$ be a graph. The closed k-neighborhood $N_k[v]$ of a vertex $v \in V$ is the set of vertices within distance k from v. A 3-valued function f defined on V of the form $f : V \rightarrow \{-1, 0, 1\}$ is a three-valued k-neighborhood dominating function if the sum of its function values over any closed k-neighborhood is at least 1. The weight of a three-valued k-neighborhood dominating function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The three-valued k-neighborhood domination number of a graph G, denoted $\gamma_k^-(G)$, equals the minimum weight of a three-valued k-neighborhood dominating function of G. For $k \geq 2$, we establish the existence of trees with three-valued k-neighborhood domination number less than any given negative number. We show that the decision problem corresponding to the problem of computing γ_k^- is NP-complete even when restricted to bipartite graphs.

1 Introduction

Let $G = (V, E)$ be a graph, and let v be a vertex in V. For $k \geq 1$ an integer, the *closed k-neighborhood* $N_k[v]$ of v is defined as the set of vertices within distance k from v. The *open k-neighborhood* $N_k(v)$ of v is $N_k[v] - \{v\}$. For a set S of vertices, we define the closed k-neighborhood $N_k[S]$ of S as the set of all vertices within distance k from some vertex of S. The open k-neighborhood $N_k(S)$ of S is $N_k[S] - S$. A set S of vertices is a *k-dominating set* if $N_k[S] = V$. The *k-domination number* of a graph G, denoted $\gamma_k(G)$, is the minimum cardinality of a k-dominating set in G.

Results on the concept of k-domination in graphs have been presented by, among others, Bacsó and Tuza [1, 2], Beineke and Henning [3], Bondy and Fan [4], Chang [5], Chang and Nemhauser [6, 7, 8], Fraisse [10], Fricke, Hedetniemi, and Henning [11, 12], Henning, Oellermann, and Swart [13, 14, 15, 16, 17], Meir and Moon [18], Mo and Williams [19], Slater [20], and Topp and Volkmann [21]. In this paper we introduce a new variation on the k-domination theme which we call three-valued k-neighborhood domination. In so doing we will attempt to describe a larger tapestry of domination results which increases our general understanding of domination parameters.

2 Definition of Three-Valued k-Neighorhood Domination

Let $g : V \rightarrow \{0, 1\}$ be a function which assigns to each vertex of a graph an element of the set $\{0, 1\}$. To simplify notation we will write $g(S)$ for $\sum g(v)$ over all v in the set S of vertices, and we define the *weight* of g to be $g(V)$. Further, we will write $g_k(v)$ for $g(N_k[v])$. We say g is a *k-dominating function* if for every $v \in V$, $g_k(v) \geq 1$. We say g is a minimal k-dominating function if there does not exist a k-dominating function $h : V \rightarrow \{0, 1\}$, $h \neq g$, for which $h(v) \leq g(v)$ for every $v \in V$. This is equivalent to saying that a k-dominating function is minimal if for every vertex v such that $g(v) > 0$, there exists a vertex $u \in N_k[v]$ for which $g_k(u) = 1$. An equivalent definition of the k-domination number of a graph G is $\gamma_k(G) = \min \{g(V) \mid g$ is a minimal k-dominating function on $G\}$.

In a similar manner we now define a three-valued k-neighborhood dominating function. A *three-valued k-neighborhood dominating function* is a function of the form $g : V \rightarrow \{-1, 0, 1\}$ such that for every $v \in V$, $g_k(v) \geq 1$. The *three-valued k-neighborhood domination number* for a graph G is $\gamma_k^-(G) = \min \{g(V) \mid g$ is three-valued k-neighborhood dominating function on $G\}$. For $k = 1$, this definition coincides with the notion of three-valued domination introduced and studied by Dunbar, Hedetniemi, Henning, and McRae [9].

There is a wide variety of possible applications for this variation of domination. By assigning the values $-1, 0$ or $+1$ to the vertices of a graph we can model such things as networks of positive, neutral and negative electrical charges, networks of positive and negative spins of electrons, and networks of people or organizations in which global decisions must be made (e.g. positive, negative or neutral responses or preferences). In such a context, for example, the three-valued k-neighborhood domination number represents the minimum number of people whose positive votes can assure that all local groups of voters (represented by closed k-neighborhoods) have more positive than negative voters, even though the entire network may have far more people who vote negative than positive. In this paper we study situations in which, in spite of the presence of negative vertices, the closed k-neighborhoods of all vertices are required to maintain a positive sum.

Proposition 1 *A three-valued k-neighborhood dominating function g on a graph G is minimal if and only if for every vertex v with $g(v) = 0$ or 1, there exists a vertex $u \in N_k[v]$ with $g_k(u) = 1$.*

Proof. Let g be a minimal three-valued k-neighborhood dominating function and assume that there is a vertex $v \in V$ with $g(v) \geq 0$ and $g_k(u) \geq 2$ for every $u \in N_k[v]$. Define a new function $f : V \to \{-1, 0, 1\}$ by $f(v) = g(v) - 1$ and $f(u) = g(u)$ for all $u \neq v$. Then for all $u \in N_k[v]$, $f_k(u) = g_k(u) - 1 \geq 1$. For $w \notin N_k[v]$, $f_k(w) = g_k(w) \geq 1$. Thus f is a three-valued k-neighborhood dominating function. Since $f < g$, the minimality of g is contradicted.

Conversely, let g be a three-valued k-neighborhood dominating function such that for all $v \in V$ with $g(v) \geq 0$, there exists a $u \in N_k[v]$ with $g_k(u) = 1$. Assume g is not minimal. Then $f(w) \leq g(w)$ for all $w \in V$ and there is a $v \in V$ with $-1 \leq f(v) < g(v)$. So $g(v) \geq 0$ and by assumption, there is a $u \in N_k[v]$ with $g_k(u) = 1$. Since $f(v) \leq g(v) - 1$, and $f(w) \leq g(w)$ for all $w \in V$, we know that $f_k(u) \leq g_k(u) - 1 = 0$. This contradicts the fact that f is a three-valued k-neighborhood dominating function. □

Proposition 2 *For every graph G, $\gamma_k^-(G) \leq \gamma_k(G)$.*

Proof. Let D be a minimum k-dominating set in G. Let $g : V \to \{0, 1\}$ be the characteristic function on D, i.e., $g(v) = 1$ if $v \in D$ and $g(v) = 0$ if $v \in V - D$. Note that $g(V) = \gamma_k(G)$. By definition g is a three-valued k-neighborhood dominating function. Hence $\gamma_k^-(G) \leq g(V) = \gamma(G)$. □

3 Three-Valued k-Neighorhood Domination for Trees

In [9] it is shown that every tree T satisfies $\gamma_1^-(T) \geq 1$ with equality if and only if T is a star $K_{1,n}$. In this section we show that for $k \geq 2$, there exist trees with three-valued k-neighborhood domination number less than any negative integer.

Proposition 3 *For any integers $k \geq 2$ and $t \geq 1$, there exists a tree T with $\gamma_k^-(T) \leq -t$.*

Proof. Consider the tree T constructed as follows. Let m be a large positive integer, and let $H_1, H_2, \ldots, H_m$ be (disjoint) paths of length $2k-1$. Let the ends of H_i be u_i and v_i and let w_i and x_i be the vertices at distance $k-1$ from u_i and v, respectively. Let H_{m+1} be isomorphic to a star $K_{1,m-1}$ and let w_{m+1} denote the central vertex of H_{m+1}, and let $y_1, y_2, \ldots, y_{m-1}$ denote the end-vertices of H_{m+1}. Finally, let T be the tree obtained from the (disjoint) union of $H_1, H_2, \ldots, H_{m+1}$ by joining w_{m+1} to w_i with an edge, $i = 1, 2, \ldots, m$, and then subdividing the edge $y_i w_{m+1}$ $k-2$ times for each $i = 1, 2, \ldots, m-1$. (The tree T is sketched in Figure 1.) Let $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_m\}$, $W = \{w_1, w_2, \ldots, w_{m+1}\}$, $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_{m-1}\}$. Now let f assign the value 1 to each vertex of $W \cup X$, the value -1 to each vertex in $U \cup V \cup Y$, and the value of 0 to all remaining vertices of T. Then it is not too difficult to see that f is a three-valued k-neighborhood dominating function with $f(V(T)) = -m+2$. Hence $\gamma_k^-(T) \leq -m+2$. Letting $m = t+2$ completes the proof. □

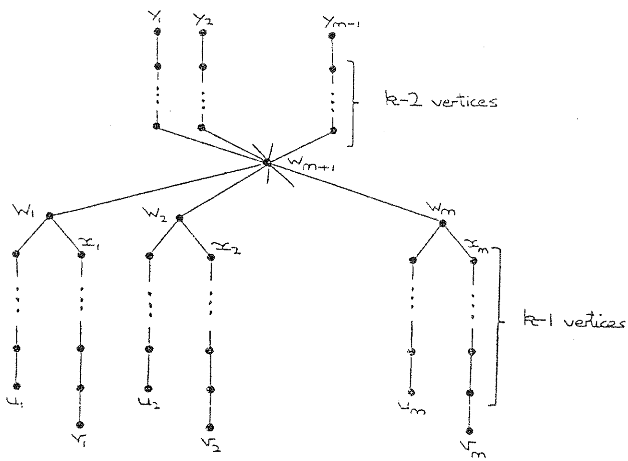

Figure 1. A tree T with $\gamma_k^-(T) \leq -m+2$.

4 Complexity Issues

In this section we consider the following decision problem corresponding to the problem of computing $\gamma_k^-(G)$ for any fixed integer $k \geq 1$.

THREE-VALUED k-NEIGHBORHOOD DOMINATION ($TkND$)
INSTANCE: A graph $G = (V, E)$ and a positive integer $r \leq |V|$.
QUESTION: Is there a three-valued k-neighborhood dominating function of weight r or less for G?

The purpose of this section is to establish the following result.

Theorem 1 *$TkND$ is NP-complete when restricted to bipartite graphs.*

Proof. It is obvious that $TkND$ is a member of NP since we can, in polynomial time, guess at a partition of the vertex set of G into three subsets V_1, V_2, and V_3, where the vertices of V_1, V_2, and V_3 are assigned the values $-1, 0$ and 1, respectively, and then verify that the funcion $f : V \to \{-1, 0, 1\}$ corresponding to this partition has weight at most r and is a three-valued k-neighborhood dominating function.

To show that $TkND$ is an NP-complete problem, we will establish a polynomial transformation from the well-known NP-complete problem $3SAT$. Let I be an instance of $3SAT$ consisting of the (finite) set $C = \{c_1, \ldots, c_m\}$ of three literal clauses in the n variables $x_1, \ldots, x_n$. We transform I to the instance (G_I, r) of $TkND$ in which $r = 3(n + m)$ and G_I is the bipartite graph constructed as follows.

Let H be the bipartite graph obtained from a $(4k+2)$-cycle C by attaching a path P of length $2k+1$ at some vertex e. Let x and $\overline{x}$ be the vertices on C at distance k from e and let f, g and h be the vertices on P at distance 1, $k+1$ and $2k+1$, respectively, from e. Further, let a, b and d be the three vertices on the x-$\overline{x}$ path of H that does not contain e at distance 1, $k+1$ and $2k+1$ from x, respectively. (So d is adjacent to $\overline{x}$.) Let $H_1, H_2, \ldots, H_n$ be n disjoint copies of H. Corresponding to each variable x_i we associate the graph H_i. Let $x_i, \overline{x}_i, a_i, b_i, d_i, e_i, f_i, g_i$ and h_i be the names of the vertices of H_i that are named $x, \overline{x}, a, b, d, e, f, g$ and h, respectively, in H.

Next we construct the bipartite graph F as follows. Attach three (disjoint) paths of length $k-1$ to a vertex c of a $(2k+2)$-cycle. Let y_1 and y_2 (w_1 and w_2) be the two vertices on the cycle at distance 1 (k, respectively) from c. Let u_1, u_2, u_3, u_4, u_5 be a path of length 4 and identify the vertex u_2 with the vertex at distance $k+1$ from c. Finally let $\mathcal{I} = \{1, 2, 4, 5\}$ and attach a u_i-v_i path of length k to each vertex u_i for $i \in \mathcal{I}$. Let F denote the resulting (bipartite) graph. Let $F_1, F_2, \ldots, F_m$ be m disjoint copies of F. Corresponding to each clause c_j we associate the graph F_j. Let $c_j, y_{j,1}, y_{j,2}, w_{j,1}, w_{j,2}, u_{j,i}$ ($i = 1, 2, \ldots, 5$) and $v_{j,i}$ ($i \in \mathcal{I}$) be the names of the vertices of F_j that are named $c, y_1, y_2, w_1, w_2, u_i$ ($i = 1, 2, \ldots, 5$) and v_i ($i \in \mathcal{I}$), respectively, in F. If $k = 1$, then we let $y_{j,i} = w_{j,i}$.

The construction of our instance of $TkND$ is completed by joining the vertex c_j to the three special vertices that name the three literals in clause c_j if $k = 1$, or by joining the three end-vertices of F_j at distance $k-1$ from c_j to the three special vertices that name the three literals in clause c_j, each end-vertex being joined to exactly one literal, and vice-versa, for $k \geq 2$. Let G_I denote the resulting graph. Observe that G_I is a bipartite graph. The graph G_I associated with $(x_1 \vee \overline{x}_2 \vee x_n) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_n)$ is depicted in Figure 2.

It is easy to see that G_I has order $(6k+3)n+(9k+3)m$ and size $(6k+3)n+(9k+6)m$ and the construction can be accomplished in polynomial time. All that remains to be shown is that I has a satisfying truth assignment if and only if $\gamma_k^-(G_I) \leq r$, where $r = 3(n+m)$.

First suppose I has a satisfying truth assignment $t : \{x_1, x_2, \ldots, x_n\} \rightarrow \{T, F\}$. We construct a three-valued k-neighborhood dominating function f of G_I of weight $f(V(G_I)) \leq 3(n+m)$. This will show that $\gamma_k^-(G_I) \leq 3(n+m)$. For each $i = 1, 2, \ldots, n$, do the following. If $t(x_i) = T$, then let $f(x_i) = f(d_i) = f(g_i) = 1$ and let $f(v) = 0$ for the remaining vertices of H_i. On the other hand, if $t(x_i) = F$, then let $f(\overline{x}_i) = f(a_i) = f(g_i) = 1$ and let $f(v) = 0$ for the remaining vertices of H_i. For each $j = 1, 2, \ldots, m$, do the following. Let $f(u_{j,i}) = 1$ for $i \in \mathcal{I}$, let $f(u_{j,3}) = -1$ and let $f(v) = 0$ for the remaining vertices of F_j. Then it is straightforward to verify that f is a minimal three-valued k-neighborhood dominating function of weight $3(n+m)$. The only vertices whose closed k-neighborhoods under f give any doubt are the vertices c_j. But the closed k-neighborhoods of these vertices under f maintain a positive sum because I has a satisfying truth assignment. This shows that $\gamma_k^-(G_I) \leq 3(n+m)$.

Conversely assume that $\gamma_k^-(G_I) \leq 3(n+m) = r$. Let g be a three-valued k-neighborhood dominating function of weight $g(V(G_I)) = \gamma_k^-(G_I)$.

Claim 1 *$g(V(H_i)) \geq 3$ for all $i = 1, 2, \ldots, n$.*

Proof. Since g is a three-valued k-neighborhood dominating function, $g_k(v) \geq 1$ for every $v \in V(G_I)$. Hence, since $N_k[b_i]$, $N_k[e_i]$ and $N_k[h_i]$ are pairwise disjoint and their union is $V(H_i)$, we have $g(V(H_i)) \geq 3$. □

Next we show that $g(V(F_j)) \geq 3$ for all j. For each $j = 1, 2, \ldots, m$, let $N_j = N_k[c_j] \cap V(F_j)$. We proceed further by proving four claims.

Claim 2 *$g(N_j) \geq -1$.*

Proof. Suppose, to the contrary, that $g(N_j) < -1$. Then $1 \leq g_k(y_{j,1}) = g(N_j) - g(w_{j,2}) + g(u_{j,2}) < -1 - g(w_{j,2}) + g(u_{j,2}) \leq -1 - (-1) + 1 = 1$, which is impossible. □

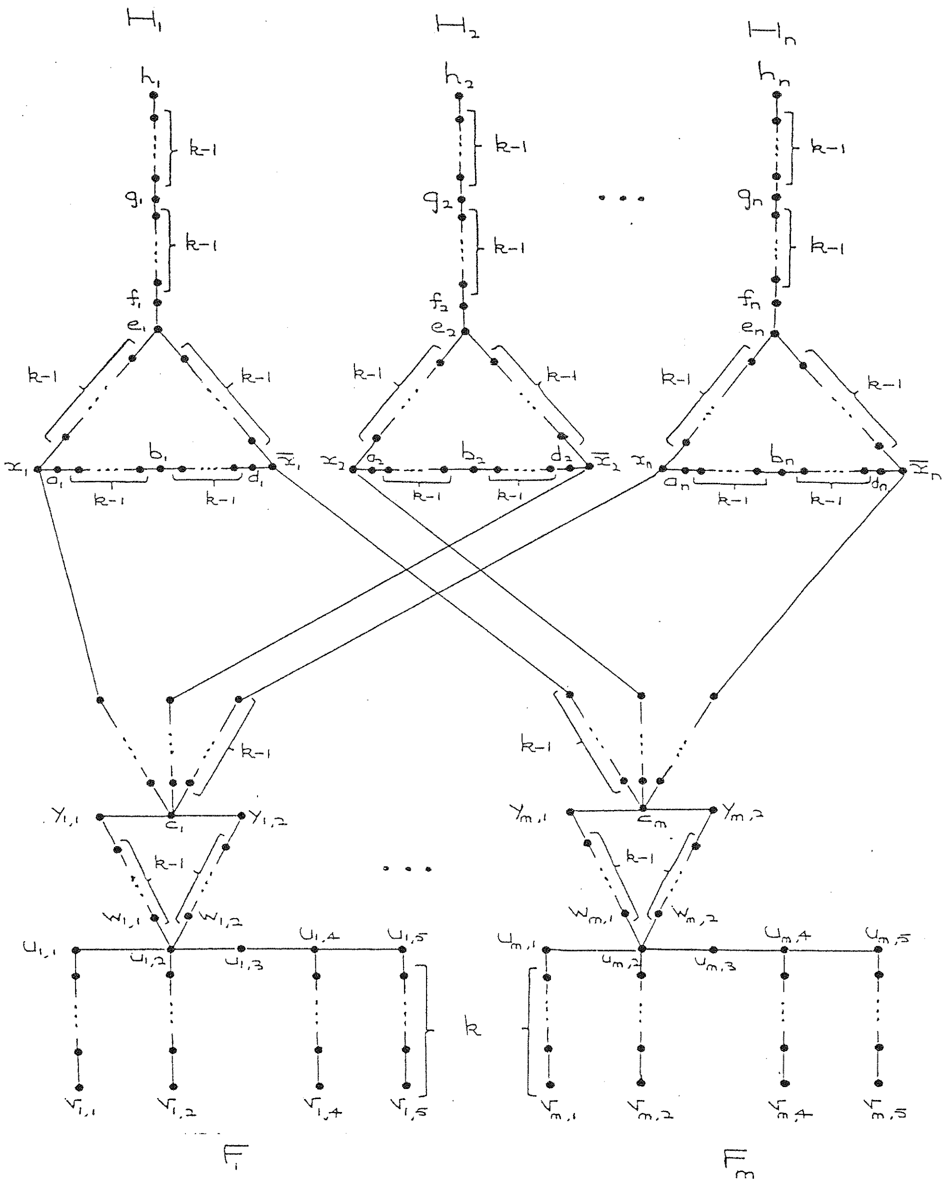

Figure 2. The graph G_I for $(x_1 \vee \overline{x}_2 \vee x_n) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_n)$.

Claim 3 *If* $g(N_j) = -1$, *then* $g(V(F_j)) \geq 4$.

Proof. Since $1 \leq g_k(y_{j,1}) = g(N_j) - g(w_{j,2}) + g(u_{j,2}) \leq -1-(-1)+1 = 1$, we must have equality throughout in the above inequalities. In particular, this means $g(u_{j,2}) = 1$ and $g(w_{j,2}) = -1$. Similarly, if we consider the vertex $y_{j,2}$, we may show that $g(w_{j,1}) = -1$. Now let z_j denote the vertex that immediately precedes $v_{j,2}$ on the $u_{j,2}$–$v_{j,2}$ path. (If $k = 1$, then $z_j = u_{j,2}$.) Then $g(V(F_j)) = g_k(z_j) + (g(N_j) - g(w_{j,1}) - g(w_{j,2})) + (g_k(v_{j,1}) - g(u_{j,1})) + g_k(v_{j,4}) + g_k(v_{j,5}) \geq 1 + (-1+2) + (1-1) + 1 + 1 = 4$, as asserted. □

Claim 4 *If* $g(N_j) = 0$, *then* $g(V(F_j)) \geq 3$.

Proof. Since $V(F_j)$ can be partitioned into the six subsets N_j, $\{u_{j,3}\}$, $N_k[v_{j,i}]$ $(i \in \mathcal{I})$, it follows that $g(V(F_j)) = g(N_j) + g(u_{j,3}) + \sum_{i \in \mathcal{I}} g_k(v_{j,i}) \geq 0 - 1 + 4 = 3$, with equality if $g(u_{j,3}) = -1$ and $g_k(v_{j,i}) = 1$ for $i \in \mathcal{I}$. □

Claim 5 *If* $g(N_j) \geq 1$, *then* $g(V(F_j)) \geq 4$.

Proof. Proceeding as in the proof of Claim 4, we have $g(V(F_j)) = g(N_j) + g(u_{j,3}) + \sum_{i \in \mathcal{I}} g_k(v_{j,i}) \geq 1 - 1 + 4 = 4$. □

As an immediate consequence of Claims 2, 3, 4 and 5, we have the following result:

Claim 6 $g(V(F_j)) \geq 3$ *for all* $j = 1, 2, \ldots, m$.

By Claims 1 and 6, we have

$$g(V(G_I)) = \sum_{i=1}^{n} g(V(H_i)) + \sum_{j-1}^{m} g(V(F_j)) \geq 3(n+m)$$

with equality if and only if $g(V(H_i)) = 3$ for all i and $g(V(F_j)) = 3$ for all j. However, by assumption, $g(V(G_I)) \leq 3(n+m)$. Consequently, $g(V(G_I)) = 3(n+m)$ and $g(V(H_i)) = 3$ for all i and $g(V(F_j)) = 3$ for all j. In particular, since $g(V(F_j)) = 3$, it follows from Claims 2, 3, 4 and 5, that $g(N_j) = 0$. Since $g_k(c_j) \geq 1$ for all j, this implies the existence of a set $S \subseteq \cup_{i=1}^{n} \{x_i, \overline{x}_i\}$ with $g(v) = 1$ for all $v \in S$ and such that $\{c_1, c_2, \ldots, c_m\} \subseteq N_k(S)$. We show next that S contains at most one of the vertices x_i and $\overline{x}_i$.

Claim 7 *If* $g(x_i) = 1 = g(\overline{x}_i)$, *then* $g(V(H_i)) \geq 4$.

Proof. Since $V(H_i)$ can be partitioned into the five subsets $N_k[f_i]$, $N_k[h_i] - \{g_i\}$, $N_k[b_i]$, $\{x_i\}$ and $\{\overline{x}_i\}$, it follows that $g(V(H_i)) = g_k(f_i) + (g_k(h_i) - g(g_i)) + g_k(b_i) + g(x_i) + g(\overline{x}_i) \geq 1 + (1-1) + 1 + 1 + 1 = 4$, as asserted. □

Since $g(V(H_i)) = 3$ for all i, it follows from Claim 7 that S contains at most one of the vertices x_i and $\overline{x}_i$. Thus we can use S to obtain a truth assignment $t : \{x_1, x_2, \ldots, x_n\} \to \{T, F\}$. We merely set $t(x_i) = T$ if $x_i \in S$ and $t(x_i) = F$ if $x_i \notin S$. By our construstion of the graph G_I, and from the definition of S, it follows that each clause c_j of I contains some variable $x_i \in S$ or $\overline{x}_i \in S$. Hence this truth assignment satisfies each of the clauses c_j of C. Hence I has a satisfying truth assignment.

Therefore, I has a satisfying truth assignment if and only if $\gamma_k^-(T) \leq r$, where $r = 3(n+m)$, completing the proof. □

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