Antipodal Triple Systems

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Abstract

An *antipodal triple system* of order v is a triple (V, B, f), where $\mid V \mid= v$, B is a set of cyclically oriented 3-subsets of V, and $f : V \longrightarrow V$ is an involution with one fixed point such that:

(i) $(V, B \cup f(B))$ is a Mendelsohn triple system.

(ii) $B \cap f(B) = \emptyset$.

(iii) f is an isomorphism between the Steiner triple system (STS) (V, B') and the STS $(V, f(B'))$, where B' is the same as B without orientation.

(iv) f preserves orientation.

An STS (V, B) is hemispheric if there exists a cyclic orientation B^* of its block set B and an involution f such that (V, B^*, f) is an antipodal system. We prove that for all admissible $v > 3$, there exists an antipodal system. This is the first step in establishing the conjecture that every $STS(V, B)$ of order $v > 3$ is hemispheric. It is known that this conjecture is true for $3 < v \leq 15$.

1. Introduction

The orientability of triple systems is a field of much interest. For a review of work in this area the reader is referred to the survey article of Colbourn and Rosa [5], where sections 2.2 and 3.2 are devoted to this problem, as are open problems 6, 7, and 8. Teirlinck [14] has shown the existence of a twofold triple system of order v whose block set decomposes into two isomorphic copies of a given Steiner triple system of order v for all $v > 3$. In this paper we will investigate the orientability of such twofold triple systems.

For completeness we include some basic definitions, as well as some new ones needed for this paper.

A Steiner triple system of order v, denoted STS(v), is a pair (V, B), where $\mid V \mid = v$, and B is a collection of 3-subsets (called blocks) of V such that every 2-subset of V is contained in exactly one block of B. A twofold triple system of order v, denoted TTS(v), is a pair (V, B), where $\mid V \mid = v$, and B is a collection of 3-subsets (blocks) of V such that every 2-subset of V is contained in exactly two blocks of B. A cyclically oriented 3-subset is one with an imposed cyclic order (a, b, c), representing the fact that $a < b, b < c$, and $c < a$. It is said to contain the ordered pairs (a, b), (b, c), and (c, a).

A Mendelsohn triple system of order v, denoted MTS(v), is a pair (V, B), where $\mid V \mid = v$, and B is a collection of cyclically ordered 3-subsets (blocks) of V such that every ordered 2-subset of V is contained in exactly one block of B.

A TTS is orientable if its blocks can be given the additional structure of cyclic ordering and made into an MTS. The question of orientability is a difficult one - Colbourn and Rosa [5] state that "the study of Mendelsohn Triple Systems derives much of its interest from the observation that orientability is apparently a subtle property". We begin by looking at a more modest question: Is every STS a subdesign of an orientable design? The answer here is trivially affirmative - simply take two copies of an $STS(v)$ and orient the blocks $\{a, b, c\}$ as (a, b, c) in one copy, and as (b, a, c) in the other copy.

Teirlinck [14] showed that given an $STS(v) = (V, B)$, there exists an $STS(v) = (V, B')$ such that $(V, B) \cong (V, B')$ and furthermore $B \cap B' = \emptyset$. However his construction is almost guaranteed to destroy any orientability properties. We attempt to integrate Teirlinck-like systems with orientability.

We define a *tier* of designs as an n-tuple $(V, B_0, B_1, ..., B_{n-2})$ such that $(V, B_0) \cong (V, B_j)$ and $B_i \cap B_j = \emptyset$, $0 \leq i < j \leq n-2$. When we wish to emphasize the isomorphisms and how the designs are actually linked we include them in the definition. That is, we define a *linked tier* of designs $LT(V, B_0, f_1, f_2, ..., f_{n-2})$ as the n-tuple in which (V, B_0) is an $STS(v)$, $f_i : V \to V$, $i = 1, 2, ..., n-2$, are one-to-one and onto maps, $f_0 = 1_v$, and $f_i(B_0) \cap f_j(B_0) = \emptyset$, $0 \leq i < j \leq n-2$. For example, a large set (cf. Teirlinck [15]) of $STS(9)$ is a linked tier of designs.

An antipodal triple system is a simple example of an orientable linked tier of designs. More formally, we define an *antipodal triple system* as a triple (V, B, f), where B is a set of cyclically ordered 3-subsets of V, and $f : V \to V$ is an involution with one fixed point such that:

(i) $(V, B \cup f(B))$ is a Mendelsohn triple system.

(ii) $B \cap f(B) = \emptyset$.

(iii) f is an isomorphism between the STS (V, B') and the STS $(V, f(B'))$, where B' is the same as B without orientation.

(iv) f preserves orientation.

An STS (V, B) is hemispheric if there exists a cyclic orientation B^* of its block set B and an involution f with one fixed point such that (V, B^*, f) is an antipodal system.

We use the involution $f(x) = -x \pmod{v}$ to establish that any cyclic $STS(v)$ with $v \equiv 1 \pmod{6}$ is hemispheric. This motivates the definition of hemispheric and antipodal systems (and in particular the requirement that f be an involution with exactly one fixed point), and the study of their existence. The generalisation to $v \equiv 3 \pmod{6}$ is much more difficult. For this case the mapping $f(x) = -x \pmod{v}$ cannot work, so we relaxed the conditions on f minimally to allow involutions which, like the mapping $f(x) = -x \pmod{v}$, have exactly one fixed point. Thus, we use involutions with one fixed point in the definition of antipodal systems.

The main result of the paper is that for all admissible $v > 3$ there exists an antipodal system. In fact computational evidence motivates the conjecture that *every* STS is a subdesign of an antipodal system. In particular, Gibbons and Mendelsohn [10] have shown that all $STS(v)$, $3 < v \leq 15$, are hemispheric. In addition they have shown that 1000 randomly generated $STS(19)$, as well as a smaller number of randomly generated $STS(21)$ and $STS(25)$, are hemispheric. This evidence was gathered using the search technique of simulated annealing. The authors note that for this application they were unable to formulate a search strategy based on hill-climbing or restricted backtrack, alternate techniques which are often used for problems of this kind.

2. Existence

The existence question is fraught with surprises and difficulties. The usual construction techniques must be handled most carefully, often failing in the least expected places. Wilson's fundamental construction, for example, works well to produce "antipodal GDD's" which may even be orientable, but the restrictions of the involution f thereby imposed may make it impossible to fill the holes to get a design. PBD closure is next to impossible because of the difficulty of getting the involution on the large design and the involutions on the design built on the blocks to be compatible. Although we shall be using specific cases of the Wilson construction, we shall not prove a general lemma because of the limited applicability. However the methods of 1-factorizations of cyclic graphs do yield results.

We begin with the following result which provides one of the main motivations for studying antipodal systems.

Theorem 1 *If* $v \equiv 1 \pmod 6$ *there exists an antipodal system.*

Proof: It is well known [4] that for all $v \equiv 1 \pmod 6$ there exists a cyclic triple system (Z_v, B). It is easily checked that if (Z_v, B) is oriented by orbits (that is, if $\{x, y, z\}$ is oriented as (x, y, z), then $\{x+i, y+i, z+i\}$ must be oriented as $(x+i, y+i, z+i)$), then (Z_v, B, f), where $f(x) = -x \pmod v$, has been strongly oriented. In fact this is almost true for $v = 6t+3$, $t \neq 1$. Since the block $\{0, 2t, 4t\}$ is fixed under f, the disjointness (rather than orientability) property is violated. □

The following (fragment swapping) method can be used to construct large numbers of hemispheric systems. Suppose (V, B, f) is an antipodal system containing the set of blocks $F = \{(a,c,e),(a,d,f),(b,c,f),(b,d,e)\}$. Let $G = \{(b,c,e),(b,d,f),(a,c,f),(a,d,e)\}$. Then $(V, (B-F) \cup G, f)$ is a different antipodal system.

For example, both $STS(13)$ are hemispheric - one because it is cyclic, and the other because it can be obtained from the first by a fragment swap.

We now turn our attention to the case $v \equiv 3 \pmod 6$. We note that a hemispheric $STS(v)$ does not exist for $v = 3$, but does exist for $v = 9, 15$ and 21. Examples of such designs are listed in the Appendices.

Theorem 2 *If* $v = 3(6t+1), t > 0$, *then there exists a hemispheric* $STS(v)$.

Proof: Let $V = Z_{6t+1} \times Z_3$. For brevity we shall write $(x, i) \in V$ as x_i. Let $f(x_i) = -x_{-i} \pmod{6t+1, 3}$, i.e. where the first component is taken modulo $6t+1$, and the second modulo 3. Let (Z_{6t+1}, B^*, g), where $g(x) = -x$, be an antipodal system as given by Theorem 1. In the constructed system of order $v = 3(6t+1)$ the blocks are defined and oriented as follows:

(a) $A = \{(x_1, y_{-1}, (x+y+1)_0) \mid x, y \in Z_{6t+1}\}$

(a') $f(A) = \{(-x_{-1}, -y_1, -(x+y+1)_0) \mid x, y \in Z_{6t+1}\}$

(b) $B = \{(x_i, y_i, z_i) \mid (x,y,z) \in B^*, i = 0, 1, 2\}$

(b') $f(B) = \{(-x_{-i}, -y_{-i}, -z_{-i}) \mid (x,y,z) \in -B^*, i = 0, 1, 2\}$

We claim that $(V, A \cup B, f)$ is an antipodal system. Clearly the orientation property is satisfied. Considering disjointness, suppose that $(x_1, y_{-1}, (x+y+1)_0)$ and $(-u_{-1}, -w_1, -(u+w+1)_0)$ are the same unoriented block. This implies that $x \equiv -w$ and $y \equiv -u$, and hence $x+y+1 \equiv -u-w-1 \equiv x+y-1 \pmod{6t+1}$, which is impossible as $6t+1$ is odd. □

Theorem 3 *For* $v = 3(6t+3)$ *there exists an antipodal system on* v *points.*

Proof: If $v \neq 27$ by [4] there exists a cyclic $STS(6t+3)$ with a distinguished set of blocks $\{i, 2t+i, 4t+i\}, i = 0, 1, ..., 2t$, called the *short orbit.* We shall distinguish the block $\{0, 2t, 4t\}$ from this set. We call the blocks in this orbit with $1 \leq i \leq t$ *positive*, and those with $t+1 \leq i \leq 2t$ *negative.* Note that $f(x) = -x$ sends positive blocks to negative blocks and vice versa.

We are now ready to construct our antipodal system. In a manner similar to the proof for Theorem 2, take $V = Z_{6t+3} \times Z_3$ and $f(x_i) = -x_{-i} \pmod{6t+3, 3}$. Let (Z_{6t+3}, B) be a cyclic STS with the blocks of the short orbit removed. Orient the blocks of B orbitwise as in Theorem 1 to form B^*, and then take $(Z_{6t+3}, B^* \cup -B^*)$ oriented with $(x, y, z) \in B^* \Rightarrow (-x, -y, -z) \in -B^*$. The blocks of our system, together with their orientations, will be:

(a) $A = \{(a_i, b_i, c_{i+1}), (a_i, b_{i+1}, c_i), (a_{i+1}, b_i, c_i) \mid (a, b, c) \in B^*, i = 0, 1, 2\}$

(a') $f(A) = \{(-a_{-i}, -b_{-i}, -c_{-i-1}), (-a_{-i}, -b_{-i-1}, -c_{-i}), (-a_{-i-1}, -b_{-i}, -c_{-i}) \mid (a, b, c) \in B^*, i = 0, 1, 2\}$

Let (X, B^*, f^*) be the antipodal system on 9 points given in the Appendix A2. For every positive block (a, b, c) of the short orbit, let $g : X \rightarrow \{a_0, b_0, c_0, a_1, b_1, c_1, a_{-1}, b_{-1}, c_{-1}\}$ be any one-to-one and onto map. We define the following block sets:

(b) $B = \{(g(x), g(y), g(z)) \mid (x, y, z) \in B^*\}$

(b') $f(B) = \{(-g(x), -g(y), -g(z)) \mid (x, y, z) \in B^*\}$

(c) $B = \{(-g(x), -g(y), -g(z)) \mid (x, y, z) \in f^*(B^*)\}$

(c') $B = \{(g(x), g(y), g(z)) \mid (x, y, z) \in f^*(B^*)\}$

Finally, on the points 0_i, $(2t)_i$, $(4t)_i$, $i = -1, 0, 1$ define h to be the following mapping:

x	0	1	2	3	4	5	6	7	8
$h(x)$	0_0	$(2t)_0$	0_1	$(2t)_1$	$(4t)_1$	$(2t)_{-1}$	$(4t)_{-1}$	0_{-1}	$(4t)_0$

We define the additional block sets:

(d) $D = \{(h(x), h(y), h(z)) \mid (x, y, z) \in B^*\}$

(d') $f(D) = \{(h(x), h(y), h(z)) \mid (x, y, z) \in f^*(B^*)\}$

It is easily seen that $(V, A \cup B \cup C \cup D, f)$ is the desired antipodal system.

For the case $v = 27$, first treat the blocks $\{0, 3, 6\}$, $\{1, 4, 7\}$, and $\{2, 5, 8\}$ as if they were the "short orbit". The blocks of B^* and $f(B^*) = -B^* \pmod 9$ become:

(b) $B^* = \{(0, 4, 8), (1, 5, 6), (2, 3, 7), (0, 1, 2), (3, 4, 5), (6, 7, 8), (2, 6, 4), (0, 7, 5), (1, 8, 3)\}$

(b') $f(B^*) = \{(0, 5, 1), (8, 4, 3), (7, 6, 2), (0, 8, 7), (6, 5, 4), (3, 2, 1), (7, 3, 5), (0, 2, 4), (8, 1, 6)\}$

The rest of the construction remains the same. $\square$

For the remaining case, viz. $v = 3(6t+5)$, let (X_n, E) be the complete graph with vertex set X_n. Now define:

$$D(n) = \begin{cases} \{i \mid 1 \le i \le (n-1)/2\} & \text{if} \quad n \equiv 1 \pmod 2 \\ \{i \mid 1 \le i \le (n-1)/2\} \cup \{\frac{1}{2} \cdot n/2\} & \text{if} \quad n \equiv 0 \pmod 2 \ . \end{cases}$$

The elements of $D(n)$ are called differences $(\mathrm{mod}\, n)$. For $n \equiv 0 \pmod 2$, the notation $\frac{1}{2} \cdot n/2$ means that from which we have the following 1-factor:

$$\Big\{\{i, i+n/2\} \mid 0 \le i < n/2\Big\} \ .$$

We need the following well known result [4]:

Lemma 1 *If $v \equiv 1 \pmod 6$, then all the $(v-1)/2$ differences of $D(v)$ can be partitioned into $(v-1)/6$ difference triples.*

Now let F_1 and F_2 be two ordered 1-factors over Z_n. F_1 and F_2 are said to be strictly disjoint if they are disjoint as unordered 1-factors.

Lemma 2 *Let $n \equiv 2^k \pmod{2^{k+1}}$, $k \ge 1$. If $d \ne n$, $d \equiv 2^s \pmod{2^{s+1}}$, $0 \le s \le k-1$, then from the difference $d \in D(n)$, we may form two strictly disjoint ordered 1-factors F_1 and F_2 of the form $F = \{(a_i, b_i) \mid 0 \le i < n/2\}$ such that*

$$\begin{aligned} a_i &\equiv t_1 \pmod{2^{s+1}}, \quad t_1 \in \{1, 2, \ldots, 2^s\} \ , \\ b_i &\equiv t_2 \pmod{2^{s+1}}, \quad t_2 \in \{2^s+1, 2^s+2, \ldots, 2^{s+1}\} \ . \end{aligned}$$

and $(-b_i+1, -a_i+1) \in F$ for each $(a_i, b_i) \in F$.

Proof: Let

$$\begin{aligned} F_1 &= \Big\{(i, d+i) \mid i \in Z_n \ , \quad i \equiv 1, 2, \ldots, 2^s \pmod{2^{s+1}}\Big\} \\ F_2 &= \Big\{(i, -d+i) \mid i \in Z_n \ , \quad i \equiv 2^s+1, 2^s+2, \ldots, 2^{s+1} \pmod{2^{s+1}}\Big\} \ . \end{aligned}$$

It can be easily seen that F_1 and F_2 are the desired ordered 1-factors. $\square$

Lemma 3 *Let $n \equiv 2^k \pmod{2^{k+1}}$, $k \ge 1$, $n \equiv 2 \pmod 6$. Suppose $6t \le n-8$ and we can form t difference triples from $D(n)$. Let R_s denote the set of differences $d \ne n/2$ not contained in the t difference triples with $d \equiv 2^s \pmod{2^{s+1}}$ when $s < k$ or $d \equiv 0 \pmod{2^k}$ when $s = k$. Let $r_s = |R_s|$. If $r_k = 0$, and $r_{k-1} \ge 1$. Then there exists an antipodal triple system of order $2n-(6t+1)$ containing a subsystem of order $n-(6t+1)$.*

Proof: Since $n \equiv 2 \pmod 6$ and $6t \leq n-8$, then $n-(6t+1) \equiv 1 \pmod 6$ and $n-(6t+1) \geq 7$. By Theorem 1, there is an antipodal triple system of order $v = n-(6t+1)$. Let (X, A, g) be such a system, where $X = \{a_i \mid i \in Z_v\}$ and $g(a_i) = a_{-i}$. Now let $Y = X \cup Z_n$ and

$$f(y) = \begin{cases} g(y) & \text{if } y \in X \\ -y+1 & \text{if } y \in Z_n \,. \end{cases}$$

Then $f: Y \to Y$ is an involution with fixed point a_0. For each of the t difference triples, say (a, b, c), where $a+b \equiv c$ or $a+b+c \equiv 0 \pmod n$, form n cyclically oriented triples $(i, a+1, a+b+i)$, $i \in Z_n$. Let B be the set of all nt such triples.

Let $\{a_1, a_2, \ldots, a_{(v-1)/2}\}$ be partitioned into k subsets X_i, $0 \leq i \leq k-1$, such that $|X_i| = r_i$, $0 \leq i \leq k-1$. Then

$$X = \{a_0\} \cup \Big\{ \bigcup_{i=0}^{k-1} (X_i) \cup f(X_i) \Big\} \,.$$

For $0 \leq i \leq k-1$, if $r_i \neq 0$, let $X_i = \{a_{i1}, \ldots, a_{ir_i}\}$, $R_i = \{d_{i1}, d_{i2}, \ldots, d_{ir_i}\}$. If $0 \leq i < k-1$, then for each $d_{ij} \in R_i$, form two strictly disjoint ordered 1-factors F^1_{ij} and F^2_{ij} satisfying the conditions in Lemma 1 and let B_i be the following set of cyclically oriented triples:

$$B_i = \bigcup_{j=1}^{r_i} \Big\{ \{(a_{ij}, x, y) \mid (x,y) \in F^1_{ij} \Big\} \cup \{(f(a_{ij}), x, y) \mid (x,y) \in F^2_{ij}\} \,, \quad 0 \leq i < k-1 \,.$$

If $i = k-1$, then for each $d_{k-1,j} \in R_{k-1}$, $2 \leq j \leq r_{k-1}$, form two strictly disjoint ordered 1-factors $F^1_{k-1,j}$ and $F^2_{k-1,j}$ satisfying the conditions in Lemma 2. For $d_{k-1,1} \in R_{k-1}$, since $k \geq 2$, then we can form the following two strictly disjoint ordered 1-factors:

$$\begin{aligned} F^1_{k-1,1} &= \Big\{(i, d_{k-1,1}+i) \mid i \in Z_n, i \equiv 1, 2, \ldots, 2^{k-2} \pmod{2^k}\Big\} \cup \\ &\qquad \Big\{(i, -d_{k-1,1}+i) \mid i \in Z_n, i \equiv 2^{k-2}+1, \ldots, 2^{k-1} \pmod{2^k}\Big\} \,, \\ F^2_{k-1,1} &= \Big\{(i, -d_{k-1,1}+i) \mid i \in Z_n, i \equiv 1, 2, \ldots, 2^{k-2} \pmod{2^k}\Big\} \cup \\ &\qquad \Big\{(i, d_{k-1,1}+i) \mid i \in Z_n, i \equiv 2^{k-2}+1, \ldots, 2^{k-1} \pmod{2^k}\Big\} \,. \end{aligned}$$

For $\frac{1}{2} \cdot n/2$, let

$$F_0 = \{(i, i+n/2) \mid i \in Z_n \,, \quad i \equiv 1, 2, \ldots, 2^{k-1} \pmod{2^k}\Big\} \,.$$

We remark that $(f(b), f(a)) \in F^2_{k-1,1}$ for each $(a,b) \in F^1_{k-1,1}$ and $(f(b), f(a)) \in F^1_{k-1,1}$ for each $(a,b) \in F^2_{k-1,1}$. Now let

$$\begin{aligned} B_{k-1} &= \{(a_{k-1,1}, x, y) \mid (x,y) \in F_0\} \cup \\ &\qquad \Big\{(f(a_{k-1,1}), xy) \mid (x,y) \in F^1_{k-1,1}\Big\} \cup \{a_0, x, y) \mid (x,y) \in F^2_{k-1,1}\} \cup \\ &\qquad \Big\{ \bigcup_{j=2}^{r_{k-1}} \{(a_{k-1,j}, x, y) \mid (x,y) \in F^1_{k-1,j}\} \cup \{(f(a_{k-1,j}), x, y) \mid (x,y) \in F^2_{k-1,j}\Big\} \Big\}. \end{aligned}$$

Let $B = A \cup B' \cup f(B') \cup \Big\{ \bigcup_{i=0}^{k-1} (B_i \cup f(B_i)) \Big\}$. Then (Y, B, f) is an antipodal triple system of order $2n-(6t+1)$ containing (X, A, g) as a system. $\square$

Lemma 4 *For each $t \geq 1$, there exists an antipodal triple system of order $90t + 15 - 6m$ containing a subsystem of order $42t + 7 - 6m$, $0 \leq m \leq 6t + 1$.*

Proof: Let $n = 46t + 8 = 8\ (6t + 1)$ in Lemma 3. Since all the $3t$ differences of $D(6t + 1)$ can be partitioned into t difference triples by Lemma 1, then all the $3t$ differences $8i$, $1 \leq i \leq 3t$, of $D(48t + 8 = 8)$ can be partitioned into t difference triples. Further, we partition all the differences $d \equiv 1, 2,$ or $3 (\bmod 4)$ into $6t + 1$ difference triples:

$$(2, 24t + 1, 24t + 3)\ ,$$
$$(8i + 1, 8i + 5, 16i + 6),\ (8i + 3, 8i + 7, 16i + 10)\ , \qquad 0 \leq i \leq 3t - 1\ .$$

For $0 \leq m \leq 6t + 1$, decompose $6t + 1 - m$ of the above $6t + 1$ difference triples into differences so that the total number of the remaining difference triples is $t + m$. It can be checked that $r_3 = 0$, $r_2 = 3t$, $r_1 = 6t + 1 - m$, $r_0 = 12t + 2 - 2m$. Thus, by Lemma 3, there exists an antipodal triple system of order $90t + 15 - 6m$ containing a subsystem of order $42t + 7 - 6m$. □

Theorem 4 *There exists an antipodal triple system of order v for each $v \equiv 15 \ (\bmod 18)$.*

Proof: By Lemma 4, for each $t \geq 1$, if $v \equiv 3 \ (\bmod 6)$, $54t + 9 \leq v \leq 90t + 15$, then there is an antipodal triple system of order v. Thus, it can be proved by repeatedly using Lemma 4 that, for every $v \equiv 3 \ (\bmod 6)$, $v \geq 63$, $v \neq 111$, there is an antipodal triple system of order v. As a consequence, we have proved that there is an antipodal triple system of order v for every $v \equiv 15 \ (\bmod 18)$, $v \geq 69$. Antipodal triple systems of order 15, 33 and 51 are constructed in the Appendices. This, of course, covers the case of $v \equiv 3(6t + 3)$ but the construction is more complicated and more starting cases are needed. □

3. Computational construction method

3.1. Simulated annealing

Simulated annealing is a variant of the state space search technique for solving combinatorial optimization problems. Such a problem can be specified as a set Σ of feasible solutions (or states) together with a cost $c(S)$ associated with each feasible solution S. An optimal solution corresponds to a feasible solution with overall (i.e. global) minimum cost.

In simulated annealing we define, for each feasible solution $S \in \Sigma$, a set T_S of transformations (or transitions), each of which can be used to change S into another feasible solution S'. The set of solutions that can be reached from S by applying a transformation from T_S is called the neighborhood $N(S)$ of S.

The general simulated annealing algorithm works by randomly choosing an initial feasible solution and then generating a set of sequences (or Markov chains) of trials. In each trial, we examine a randomly chosen transition of the current feasible solution S. If the transition results in a feasible solution S' of equal or lower cost, then S' is accepted as the new current feasible solution. If the transition results in a feasible solution S' of higher cost, then S' is accepted with probability $e^{-(c(S')-c(S))/T}$, where T is the controlling temperature of the simulation. The temperature is lowered in small steps with the system being allowed to approach "equilibrium" at each temperature through a sequence of trials at this temperature. Usually this is done by setting $T := \alpha T$, where α (the *control decrement*) is a real number slightly less than 1. After an appropriate stopping condition is met, the current feasible solution is taken as the solution of the problem at hand. With a general optimization problem the hope is that this is close to an optimal solution. With an existence problem, where we cannot be satisfied just with an approximation to an optimal solution, we must repeat the experiment until an optimal solution is found.

3.2. Results

The algorithm described in Section 3.1 was applied to the construction of antipodal systems of orders $v = 7, 9, 13, 15, 19, 21$ and 25. Using a Sun Sparcstation 2, the average times (in seconds) to build antipodal systems based on randomly chosen STS of these orders are as follows:

v	*Time*
7	0.01
9	0.01
13	0.04
15	0.13
19	7.5
21	124.3
25	7978.6

However, the main purpose of this was to gather evidence to support the following conjecture:

Conjecture 1 *Every STS is a subdesign of an antipodal triple system.*

Indeed the algorithm constructed antipodal triple systems for the unique $STS(7)$, the unique $STS(9)$, each of the two $STS(13)$, and each of the 80 $STS(15)$, thus proving the conjecture for $v \leq 15$. For the case $v = 19$ we generated 1000 random $STS(19)$ and in each case were able to build an antipodal system containing the generated $STS(19)$. We also generated 10 random $STS(21)$, and 1 random $STS(25)$ and found an antipodal system containing each of them. We believe this provides strong evidence in support of the conjecture.

Some examples of the generated systems are listed in the Appendices. The full set of generated systems are contained in Gibbons and Mendelsohn [10].

4. Concluding remarks

The obvious open question is whether every $STS(v)$ is hemispheric.

Some of our early computational attempts (otherwise known as bugs) gave rise to the following conjecture: If (V, B) is an STS and $f : V \to V$ is any one-to-one map such that $B \cap f(B) = \emptyset$, then there exists a conjugate $h = g^{-1}fg$ of f in S_n, such that $h(B) \cap B = \emptyset$ and $(V, B \cup h(B))$ is orientable with h preserving orientation.

A further open question is whether there exists for $k > 1$ a linked tier of designs $(V, B, f_1, f_2, ..., f_k)$ such that (V, B, f_i) $i = 1, ..., k$ is antipodal. Could there even be a large set of such designs, i.e. such a linked tier with $k = v - 2$?

We would also comment that in the field of design theory, simulated annealing does not normally compete well with other probabilistic techniques such as hill-climbing. However in this case, not only did simulated annealing successfully construct antipodal systems for $v \leq 25$, but also there was no obvious way of modelling the problem as either a backtrack search or a hill-climb.

As a final comment we note that the question of *halving* triple systems is discussed by Das and Rosa [6]. They examine the orders v for which there exists an $STS(v)$ (V, B) admitting a partition of its block set $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, such that (V, B_1) and (V, B_2) are isomorphic hypergraphs. Such a Steiner triple system is said to be *halvable.* They extend this concept to a *twofold triple system* (TTS), where every 2-subset of elements is contained in exactly two triples. They show that there exists a $TTS(v)$ with the halving property for all admissible orders $v \equiv 0, 1 \pmod 3$. The question of whether an $MTS(v)$ with $\lambda = 2$ can be halved into two isomorphic directed hypergaphs is not dealt with here. However, we can formulate a $\lambda = 1$ directed version of this concept as follows. An $MTS(V, B)$ can be halved if there is a partition of its block set $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$ such that (V, B_1) is isomorphic to (V, B_2) as undirected hypergraphs. The results of this paper show that for all $v \equiv 1, 3 \pmod 6$ there is an $STS(v)$ (V, B) which can be doubled and directed to form a halvable MTS with both of its halves isomorphic as undirected hypergraphs to (V, B).

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APPENDICES

In cases A1 - A7, we list the oriented blocks of D_1 and set $f(x) \equiv -x \pmod{v}$.

A1. $v = 7$

```
 1  2  3    5  1  0    1  6  4    2  5  6    2  0  4    5  3  4    3  0  6
```

A2. $v = 9$

```
 1  2  5    3  1  8    6  1  0    1  4  7    2  3  0    6  2  4    8  2  7
 3  6  7    4  3  5    8  6  5    8  4  0    5  7  0
```

A3. $v = 13$

```
STS(13) #1:
   1  0  2    1  3  4    1  5  9    1  8  6    1 12 11    1  7 10    0  9  3
   0  5  4    0  6 11    0 12  7    0 10  8    2  3  6    2  5 12    2 11  9
   2 10  4    2  7  8    3  8  5    3 11  7    3 10 12    4  6 12    4  8 11
   4  7  9    9  8 12    9 10  6    5  6  7    5 10 11
STS(13) #2:
   1  0  2    1  4  3    1 12  5    1  9  6    1  7  8    1 11 10    0 12  3
   0  5  4    0  6  8    0  9 11    0 10  7    2  4 11    2 12 10    2  5  7
   2  6  3    2  9  8    3  9  5    3  7 11    3 10  8    4 12  8    4  6  7
   4 10  9   12  7  9   12 11  6    5  6 10    5  8 11
```

A4. $v = 15$

```
STS(15) #1:
   1  2  3    1  5  4    1  0  6    1 13  9    1 12  7    1 10 14    1  8 11
   2  4  6    2  5  0    2  9 12    2 13  7    2 14  8    2 11 10    3  4  0
   3  6  5    3  9  7    3 12 13    3 10  8    3 14 11    4 13 14    4  7  8
   4 10  9    4 11 12    5 12  8    5 10 13    5 14  9    5 11  7    6  9 11
   6 13  8    6 12 10    6  7 14    0 10  7    0 14 12    0 11 13    0  8  9
STS(15) #2:
   1  2  3    1  4  5    1  6  7    1 14 12    1  0 10    1 13 11    1  8  9
   2  6  4    2  7  5    2 14 10    2  0 12    2 11  9    2  8 13    3  4  7
   3  5  6    3 10 12    3  0 14    3  8 11    3  9 13    4 10  8    4  0  9
   4 13 14    4 11 12    5 12 13    5  0  8    5 11 14    5  9 10    6 12  8
   6 10 11    6  0 13    6  9 14    7 14  8    7 12  9    7  0 11    7 13 10
```

⋮

⋮

STS(15) #79:

1 2 3	1 4 5	1 14 6	1 13 7	1 0 11	1 12 9	1 8 10
2 14 4	2 6 5	2 7 0	2 12 13	2 9 8	2 10 11	3 14 8
3 7 4	3 13 11	3 0 5	3 12 10	3 9 6	4 0 12	4 11 6
4 10 9	4 8 13	5 7 10	5 13 14	5 11 9	5 8 12	14 12 11
14 9 7	14 10 0	6 13 10	6 0 8	6 12 7	7 11 8	13 0 9

STS(15) #80:

1 0 2	1 14 3	1 4 13	1 5 6	1 9 10	1 7 11	1 8 12
0 14 4	0 5 3	0 6 13	0 7 10	0 11 8	0 12 9	2 3 13
2 5 8	2 10 14	2 9 4	2 11 6	2 12 7	14 13 7	14 5 11
14 9 8	14 12 6	3 4 12	3 10 6	3 7 8	3 11 9	4 5 7
4 10 11	4 8 6	13 9 5	13 11 12	13 8 10	5 12 10	6 9 7

A5. $v = 19$

A random hemispheric STS(19):

1 2 13	1 3 0	1 17 4	5 1 15	1 6 7	1 16 18	1 8 11
12 1 14	1 9 10	2 3 16	17 2 10	2 4 5	0 2 12	6 2 18
2 7 15	8 2 9	11 2 14	17 3 5	4 3 14	3 6 15	13 3 9
7 3 18	8 3 10	12 3 11	17 0 9	6 17 11	17 13 7	16 17 14
17 8 15	12 17 18	0 4 10	6 4 13	16 4 8	7 4 12	9 4 18
4 11 15	5 0 7	5 6 10	5 13 14	5 16 12	8 5 18	9 5 11
6 0 16	0 13 8	0 11 18	0 14 15	6 8 12	9 6 14	13 16 11
12 13 10	13 18 15	16 7 9	10 16 15	7 8 14	10 7 11	12 9 15
10 18 14						

A6. $v = 21$

A random hemispheric STS(21):

1 2 12	3 1 16	4 1 18	1 5 20	6 1 11	1 7 19	8 1 15
1 0 14	9 1 10	17 1 13	2 3 20	2 4 10	5 2 7	6 2 18
8 2 14	0 2 11	2 9 13	2 15 17	2 19 16	4 3 13	3 5 15
6 3 7	3 8 9	0 3 19	10 3 14	3 18 17	3 12 11	4 5 9
4 6 0	4 7 12	4 8 16	4 15 14	19 4 17	20 4 11	5 6 8
0 5 13	5 10 11	18 5 14	19 5 12	5 16 17	6 9 14	10 6 16
15 6 20	6 19 13	12 6 17	7 8 11	7 0 9	7 10 20	15 7 18
7 16 13	17 7 14	0 8 18	8 10 19	8 12 13	17 8 20	0 10 17
0 15 16	12 0 20	15 9 19	9 18 12	9 16 20	17 9 11	15 10 12
18 10 13	13 15 11	18 19 20	16 18 11	14 19 11	12 16 14	14 20 13

A7. $v = 25$

A random hemispheric STS(25):

0 1 4	0 2 8	3 0 13	5 0 21	6 0 22	0 19 10	0 7 24
9 0 20	0 15 18	0 17 12	0 11 23	14 0 16	1 2 14	3 1 8
5 1 24	6 1 23	1 19 13	1 7 20	9 1 22	10 1 18	1 15 12
17 1 21	1 11 16	3 2 12	4 2 6	2 5 13	19 2 17	2 7 9
2 24 15	2 20 21	2 22 11	10 2 16	2 18 23	4 3 9	3 5 15
3 6 14	19 3 21	3 7 10	3 24 20	22 3 23	3 17 16	3 11 18
4 5 14	19 4 7	8 4 13	4 24 11	20 4 18	4 22 21	10 4 12
4 15 16	4 17 23	5 6 9	19 5 18	5 7 12	5 8 16	5 20 22
10 5 23	17 5 11	6 19 20	6 7 17	6 8 15	6 24 16	10 6 11
21 6 13	12 6 18	19 8 14	19 9 16	24 19 23	19 22 12	19 15 11
7 8 22	15 7 21	7 11 13	7 14 18	16 7 23	9 8 18	24 8 21
20 8 11	8 10 17	12 8 23	24 9 17	9 10 15	11 9 14	21 9 23
9 12 13	24 22 10	24 12 14	18 24 13	10 20 13	20 15 17	12 20 16
14 20 23	22 15 14	22 17 18	16 22 13	10 21 14	15 13 23	14 17 13
21 11 12	21 16 18					

A8. A Construction for $v = 33$

Let (X, A, g) be an antipodal triple system of order 13 where $X = \{a_i \mid a_i \in Z_{13}\}$ and $g(a_i) = a_{-i}$. Let $Y = X \cup Z_{20}$ and

$$f(y) = \begin{cases} g(y) & \text{if } \ y \in X \\ -y+1 & \text{if } \ y \in Z_{20} \,. \end{cases}$$

Difference triple: $(3, 5, 8)$. From the differences 6 and $\frac{1}{2} \cdot 10$, form the following 3 ordered 1-factors:

$$\begin{aligned} F_0 &= \{(4i+1, 4i+11),\ (4i+2, 4i+12) \mid 0 \leq i \leq 4\}\ ; \\ F_1^1 &= \{(4i+1, 4i+7),\ (4i+2, 4i-4) \mid 0 \leq i \leq 4\}\ ; \\ F_1^2 &= \{(4i+1, 4i-5),\ (4i+2, 4i+8) \mid 0 \leq i \leq 4\}\ . \end{aligned}$$

From the differences 2 and 4, form the following 4 ordered 1- factors:

$$\begin{aligned} F_2^1 &= \{(0,2),(18,16),(12,14),(10,8),(6,4),(19,1),(5,3),(7,9),(13,11),(17,15)\}\ ; \\ F_2^2 &= \{(0,4),(18,2),(12,16),(10,14),(6,8),(17,1),(13,15),(7,11),(5,9),(19,3)\}\ ; \\ F_3^1 &= \{(0,16),(14,18),(10,12),(2,6),(8,4),(5,1),(3,7),(9,11),(15,19),(17,13)\}\ ; \\ F_3^2 &= \{(0,18),(14,16),(10,6),(2,4),(8,12),(3,1),(5,7),(15,11),(17,19),(9,13)\}\ . \end{aligned}$$

From the remaining differences 1,7 and 9, form 6 ordered 1- factors $F_4^1, F_4^2, F_5^1, F_5^2, F_6^1$ and F_6^2 satisfying the conditions in Lemma 2. Now let

$$\begin{aligned} B_0 &= \{(i, 3+i, 8+i) \mid i \in Z_{20}\} \\ B_1 &= \{(a_0, x, y) \mid (x,y) \in F_1^1\} \cup \{(a_1, x, y) \mid (x,y) \in F_1^2\} \cup \{(f(a_1), x, y) \mid (x,y) \in F_0\} \\ B_2 &= \bigcup_{i=2}^{6} \{(a_i, x, y) \mid (x,y) \in F_i^1\} \cup \{ \bigcup_{i=2}^{6} \{(f(a_i), x, y) \mid (x,y) \in F_i^2\} \\ B &= A \cup B_0 \cup f(B_0) \cup B_1 \cup f(B_1) \cup B_2 \cup f(B_2)\ . \end{aligned}$$

Then (Y, B, f) is an antipodal triple system of order 33 containing (X, A, g) a subsystem.

A9. A Construction for $v = 51$

Let (X, A, g) an antipodal triple system of order 19 where $X = \{a_i \mid i \in Z_{19}\}$ and $g(a_i) = a_{-i}$. Let $Y = X \cup Z_{32}$ and

$$f(y) = \begin{cases} g(y) & \text{if } y \in X \\ -y+1 & \text{if } y \in Z_{32} . \end{cases}$$

Differences triples: $(1,3,4), (5,7,12)$. From the differences 2,6 and $\frac{1}{2} \cdot 16$, form the following 5 ordered 1- factors:

$$\begin{aligned}
F_0 &= \{(4i+1, 4i+3), (4i+2, 4i+4) \mid 0 \le i \le 7\} ;\\
F_1^1 &= \{(4i+1, 4i-5), (4i+2, 4i+8) \mid 0 \le i \le 7\} ;\\
F_1^2 &= \{(4i+1, 4i+7), (4i+2, 4i-4) \mid 0 \le i \le 7\} ;\\
F_2^1 &= \{(4i, 4i+16), (4i+5, 4i+21)(4i+18, 4i+2), (4i+19, 4i+3) \mid 0 \le i \le 3\} ;\\
F_2^2 &= \{(4i, 4i+2), (4i+5, 4i+3)(4i+18, 4i+16), (4i+19, 4i+21) \mid 0 \le i \le 3\} .
\end{aligned}$$

From the remaining differences 8,9,10,11,13,14 and 15, from 14 ordered 1-factors F_i^1 and F_i^2, $3 \le i \le 9$, satisfying the conditions in Lemma 2. Let

$$\begin{aligned}
B_0 &= \{(i, 1+i, 4+i), (i, 5+i, 12+i) \mid i \in Z_{32}\}\\
B_1 &= \{(a_0, x, y) \mid (x, y) \in F_1^1\} \cup \{(a_1, x, y) \mid (x, y) \in F_1^2\} \cup \{(f(a_1), x, y) \mid (x, y) \in F_0\}\\
B_2 &= \bigcup_{i=2}^{9} \{a_i, x, y) \mid (x, y) \in F_i^1\} \cup \bigcup_{i=2}^{9} \{a_i, x, y) \mid (x, y) \in F_i^2\}\\
B &= \mathcal{A} \cup \Big\{ \bigcup_{i=0}^{2} \{B_i \cup f(B_i)\} \Big\} .
\end{aligned}$$

Then (Y, B, f) is an antipodal triple system of order 51 containing (X, A, g) as a subsystem.

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