# Some conditions for $n$-extendable graphs

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Abstract

A Tutte style necessary and sufficient condition for n-extendable graphs is proposed. Let G be a connected graph and let u and v be two vertices of G such that $d_G(u,v) = 2$. We define divergence $\alpha^*(u,v)$ as follows: $I_{u,v}(w) = \max\{ |S| \mid w \in N_G(u) \cap N_G(v)$, S is an independent set containing u and v in $G[\{w\} \cup N_G(w)] \}$ and $\alpha^*(u,v) = \max_w\{ I_{u,v}(w) \mid w \in N_G(u) \cap N_G(v)\}$. It is proved that if for any two vertices u and v of G such that $d_G(u,v) = 2$, $|N(u) \cap N(v)| \geq \alpha^*(u,v) + 2n - 1$ and if G has even order, then G is n-extendable. It is also proved that if for each v in V(G), $G' = G[\{v\} \cup N_G(v)]$ and $\varkappa(G') \geq \alpha(G') + 2n - 1$ and if G has even order, then G is n-extendable.

## 1. Introduction and terminology

All graphs in this paper are finite, undirected, connected and simple.

Plummer [6] introduced the concept of n-extendable graphs. Let G be a graph on $\nu$ vertices with a perfect matching. Let n be a positive integer such that $n \le (\nu-2)/2$. G is said to be n-extendable if there are n independent edges in G and any n independent edges of G are contained in a perfect matching of G.

Let G be a connected graph. For a pair of vertices u and v of G, we use $N(u,v)$ to denote $N_G(u) \cap N_G(v)$ and $n(u,v)$ to denote $|N(u,v)|$. Let u and v be two vertices of G such that $d_G(u,v) = 2$. We define divergence $\alpha^*(u,v)$ as following:

$$I_{u,v}(w) = \max\{ |S| \mid w \in N(u,v),\ S \text{ is an independent set containing } u \text{ and } v \text{ in } G[\{w\} \cup N_G(w)] \}$$

and $\alpha^*(u,v) = \max_w\{ I_{u,v}(w) \mid w \in N(u,v) \}$.

We use $\omega(G)$ to denote the number of components of G. And we use $o(G)$ to denote the number of odd components of G.

All terminology and notation not defined in this paper are from [1].

In [2], [3] and [4], Holton and Lou showed that when the connectivity is large, line graphs, power graphs and total graphs are n-extendable. This shows that a locally dense graph is n-extendable under some conditions. In [5], Lou showed that a locally dense graph is Hamiltonian (and furthermore, pancyclic). Ronghua Shi [7] introduced a local condition for Hamiltonian graphs by using the concept of divergence. However, n-extendable graphs and Hamiltonian graphs have many similar properties. In this paper, we give a sufficient condition for n-extendable graphs which is a Shi's type condition. This condition implies a result of Plummer that for a connected graph G with even order, if $\delta(G) \geq \nu/2 + n$, then G is n-extendable.

First, we introduce a Tutte style necessary and sufficient condition for n-extendable graphs which is an immediate corollary of a result of Plummer (see [6]).

## 2. A necessary and sufficient condition for n-extendable graphs

In this section, we give a Tutte style necessary and sufficient condition for n-extendable graphs. It is an immediate corollary of the following lemma.

Lemma 1 Let $\nu$ and n be integers with $n \geq 2$, $\nu$ even and $\nu \geq 2n+2$. Let G be a graph with $\nu$ vertices. Then if G is n-extendable, it is also (n-1)-extendable.

Proof See [6]. □

**Theorem 2** A graph $G$ is $n$-extendable if and only if for any $S \subseteq V(G)$, $o(G-S) \leq |S| - 2d$, where $d = \min\{ind(S), n\}$ and $ind(S)$ is the maximum number of independent edges in $G[S]$.

**Proof** Suppose $G$ is $n$-extendable. Let $S$ be any subset of $V(G)$. Let $d = \min\{ind(S), n\}$. By Lemma 1, $G$ is $d$-extendable. Let $e_i = u_i v_i$ $(i = 1,2,\ldots,d)$ be $d$ independent edges in $G[S]$. Let $G' = G - \{u_i, v_i \mid i = 1,2,\ldots,d\}$. By the $d$-extendability of $G$, $G'$ has a perfect matching. Let $S' = S - \{u_i, v_i \mid i = 1,2,\ldots,d\}$. Then $o(G'-S') = o(G-S) \leq |S'| = |S| - 2d$, as required.

Conversely, suppose that for any $S \subseteq V(G)$, $o(G-S) \leq |S| - 2d$, where $d = \min\{ind(S), n\}$. Let $e_i = u_i v_i$ $(i = 1,2,\ldots,n)$ be any $n$ independent edges of $G$. Let $G' = G - \{u_i, v_i \mid i = 1,2,\ldots,n\}$. For any $S' \subseteq V(G')$, let $S = S' \cup \{u_i, v_i \mid i = 1,2,\ldots,n\}$, then $o(G'-S') = o(G-S) \leq |S|-2n = |S'|$. So $G'$ has a perfect matching. Hence $G$ is $n$-extendable. □

## 3. Local conditions for n-extendable graphs

In this section, we introduce some sufficient local conditions for n-extendable graphs. This type of condition was first introduced by Ronghua Shi for Hamiltonian graphs. We first give a connectivity result.

Lemma 3 Let $G$ be a connected graph. If $n(u,v) \geq \alpha^*(u,v) + k - 1$ for any two vertices $u$ and $v$ of $G$ such that $d_G(u,v) = 2$, then $G$ is $(k+1)$-connected $(k \geq 1)$.

Proof Suppose that $G$ is not $(k+1)$-connected. Let $S \subseteq V(G)$ be a minimum vertex cutset such that $|S| \leq k$. Assume $C_1$ and $C_2$ are two components of $G-S$. Let $w \in S$. By the minimality of $|S|$, $w$ is adjacent to a vertex $u$ in $C_1$ and a vertex $v$ in $C_2$. Then $d_G(u,v) = 2$. So $n(u,v) \geq \alpha^*(u,v) + k - 1$. But $N(u,v) \subseteq S$ and $\alpha^*(u,v) \geq 2$. Hence $|S| \geq \alpha^*(u,v) + k - 1 \geq k+1$, a contradiction. □

Theorem 4 Let $G$ be a connected graph and let $k \geq 0$. If $n(u,v) \geq \alpha^*(u,v) + k$ for any two vertices $u$ and $v$ of $G$ such that $d_G(u,v) = 2$, then $\omega(G-S) \leq |S| - k$ for all cutsets $S \subseteq V(G)$.

Proof Let $S \subseteq V(G)$ be a vertex cutset of $G$. By Lemma 3, $|S| \geq k+2$. Let $|S| = s$, $\omega(G-S) = t$ and $C_1, C_2, \ldots, C_t$ be the components of $G-S$. Let $S = \{v_1, v_2, \ldots, v_s\}$ and $k_i$ be the number of components in $G-S$ which are adjacent to $v_i$. Without loss of generality, assume that $k_1 \leq k_2 \leq \ldots \leq k_s$.

Let $k_{m_j} = \max\{k_i \mid v_i$ is adjacent to $C_j$ and $1 \leq i \leq s\}$ $(j = 1,2,\ldots,t)$. Without loss of generality, assume $k_{m_1} \leq k_{m_2} \leq \ldots \leq k_{m_t}$.

We choose $S \subseteq V(G)$ such that $|S|-k-\omega(G-S)$ is as small as possible.

Claim 1 There is no $k_i$ such that $k_i \leq 1$ $(1 \leq i \leq s)$.

Suppose this is not the case and $k_i \leq 1$ for some $i$. We use $S' = S - \{v_i\}$ to replace $S$. Then $\omega(G-S') \geq \omega(G-S)$. But $|S'| = |S| - 1$. So $|S'|-k-\omega(G-S') < |S|-k-\omega(G-S)$, a contradiction to the choice of $S$.

If $v_j$ is adjacent to a vertex $u$ in component $C$ and a vertex $v$ in component $C'$, by the hypothesis of the theorem, $u$ and $v$ have at least $k_j + k$ common neighbours in $S$. Therefore, if $v_j$ is adjacent to at least two components of $G-S$, then all components adjacent to $v_j$ each have at least $k_j + k$ neighbours in $S$. Consider all vertices in $S$ adjacent to a component $C_i$, then $C_i$ has at least $k_{m_i} + k$ neighbours in $S$ by the definition of $k_{m_i}$. For the convenience of explanation, if a vertex in $S$ is adjacent to $k$ components of $G-S$, then we say that it sends $k$ edges to the components. If a component $C$ of $G-S$ has $k$ neighbours in $S$, then we say that $C$ sends $k$ edges to $S$. Now the vertices in $S$ send $k_1 + k_2 + \ldots + k_s$ edges to the components of $G-S$. And the components of $G-S$ send at least $(k_{m_1} + k) + (k_{m_2} + k) + \ldots + (k_{m_t} + k)$ edges to $S$. So we have

$$k_1 + k_2 + \ldots + k_s$$
$$\geq (k_{m_1} + k) + (k_{m_2} + k) + \ldots + (k_{m_t} + k) \qquad (1)$$

and so $$\sum_{i=1}^{t} k_i + \sum_{j=t+1}^{s} k_j \geq \sum_{i=1}^{t} k_{m_i} + tk \qquad (2)$$

<u>Claim 2</u> $\sum_{i=1}^{t} k_i \le \sum_{i=1}^{t} k_{m_i}$

By induction, we shall prove that $k_{m_i} \ge k_i$ $(i = 1,2, \ldots,t)$. Then the claim holds. By the definition of $k_{m_i}$, $k_{m_1} \ge k_1$.

Assume that $k_{m_i} \ge k_i$ for all $i < j$. Now $i = j$. If there is a component $C_p \in \{C_1,C_2,\ldots,C_j\}$ such that $C_p$ is adjacent to $v_q$ for some $q \ge j$, then $k_{m_j} \ge k_{m_p} \ge k_q \ge k_j$. Otherwise, $C_1,C_2,\ldots,C_j$ are only adjacent to $v_1,v_2,\ldots, v_{j-1}$. Then $k_1 + k_2 + \ldots + k_{j-1} \ge (k_{m_1} + k) + (k_{m_2} + k) + \ldots + (k_{m_j} + k)$. By the induction hypothesis, $k_{m_i} \ge k_i$ $(i = 1,2,\ldots,j-1)$ and $k_{m_j} \ge 1$. So $k_{m_1} + k_{m_2} + \ldots + k_{m_{j-1}} + k_{m_j} > k_1 + k_2 + \ldots + k_{j-1}$, a contradiction.

Hence we have that $k_{m_j} \ge k_j$.

By (2) and Claim 2, $\sum_{j=t+1}^{s} k_j \ge tk$ (3)

But there are at most t components adjacent to $v_1$, $v_2,\ldots,v_s$ respectively. Hence

$$k_i \le t \quad (i = 1,2,\ldots,s) \tag{4}$$

By (3) and (4), $(s-t)t \ge \sum_{j=t+1}^{s} k_j \ge tk$ (5)

By (5), we have $s-t \ge k$. Therefore $t \le s-k$. So

$$\omega(G-S) \le |S|-k. \qquad \square$$

<u>Corollary 5</u> Let G be a connected graph with even order. If $n(u,v) \ge \alpha^*(u,v) + 2n - 1$ for any two vertices u and v of G such that $d_G(u,v) = 2$, then G is n-extendable.

Proof Suppose G is not n-extendable. There are n independent edges $e_i = u_iv_i$ $(i = 1,2,\ldots,n)$ such that $G - \{u_i, v_i \mid i = 1,2,\ldots,n\}$ has no perfect matching. Let $G' = G - \{u_i, v_i \mid i = 1,2,\ldots,n\}$. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that $o(G'-S') > |S'|$. In fact, $o(G'-S') \ge |S'| + 2$. Let $S = S' \cup \{u_i, v_i \mid i = 1,2,\ldots,n\}$. Then $\omega(G-S) = \omega(G'-S') \ge o(G'-S') \ge |S'| + 2 = |S| - 2n + 2$. But by Theorem 4, $\omega(G-\bar{S}) \le |S| - (2n-1) = |S| - 2n + 1$, a contradiction. $\square$

Next we shall give another kind of local density condition for n-extendable graphs which concerns the induced subgraph on each vertex of G and its neighbours.

<u>Theorem 6</u> Let G be a connected graph. For any $v \in V(G)$, let $G' = G[\{v\} \cup N_G(v)]$ and suppose that for any

two nonadjacent vertices $u,w \in V(G')$, $d_{G'}(u) + d_{G'}(w) \geq d_G(v) + 1 + k$. Then for any two vertices u and v at distance 2 in G apart, $n(u,v) \geq \alpha^*(u,v) + k$ $(k \geq 0)$.

Proof Let u and v be two vertices in $V(G)$ such that $d_G(u,v) = 2$, and $w \in N(u,v)$. Then $u,v \in N(w)$. So u and v are two nonadjacent vertices in $G' = G[\{w\} \cup N_G(w)]$. By the hypothesis of this theorem, $d_{G'}(u) + d_{G'}(v) \geq d_G(w) + 1 + k$.

Let $n'(u,v) = |N_{G'}(u) \cap N_{G'}(v)|$. Let $\alpha'(u,v)$ be the order of the maximum independent vertex set in G' which contains u and v.

There are at most $(d_G(w) + 1 - 2) - (\alpha'(u,v) - 2)$ vertices adjacent to either u or v in G'. So $n(u,v) \geq n'(u,v) \geq [d_G(w) + 1 + k] - [(d_G(w) + 1 - 2) - (\alpha'(u,v) - 2)] = \alpha'(u,v) + k$.

But w can be any vertex in $N(u,v)$. Hence $n(u,v) \geq \alpha^*(u,v) + k$. □

Corollary 7 Let G be a connected graph with even order. For any vertex v in $V(G)$, let $G' = G[\{v\} \cup N_G(v)]$ and suppose that for any two nonadjacent vertices u and w in $V(G')$,

$$d_{G'}(u) + d_{G'}(w) \geq d_G(v) + 2n.$$

Then G is n-extendable.

Proof By Theorem 6, for any two vertices u and v in $V(G)$ of distance 2, $n(u,v) \geq \alpha^*(u,v) + 2n - 1$. By Corollary 5, G is n-extendable. □

Corollary 5 also implies a result of Plummer. The following corollary is due to Plummer [6].

Corollary 8  Let G be a connected graph of even order. If $\delta(G) \geq \nu/2 + n$, then G is n-extendable.

Proof  Let $u, v \in V(G)$ such that $d_G(u,v) = 2$. Let w be a vertex in $N(u,v)$. By the hypothesis, $d_G(u) + d_G(v) \geq \nu + 2n$. Let $\alpha(u,v)$ be the order of the maximum independent vertex set in G which contains u and v. Then there are at most $(\nu-2) - (\alpha(u,v) - 2)$ vertices adjacent to u or v in G. So

$$\begin{aligned} n(u,v) &\geq [\nu+2n] - [(\nu-2) - (\alpha(u,v) - 2)] \\ &= \alpha(u,v) + 2n \\ &\geq \alpha^{*}(u,v) + 2n \\ &> \alpha^{*}(u,v) + 2n - 1. \end{aligned}$$

By Corollary 5, G is n-extendable. □

Now we introduce another kind of local condition for n-extendable graphs concerning the local connectivity and local independence number. The proof of the following theorem is the same as in that of Theorem 4. We use $\varkappa(G)$ to denote the connectivity of G and $\alpha(G)$ to denote the independence number of G.

Theorem 9 Let G be a connected graph and let $k \geq 0$. For any $v \in V(G)$, let $G' = G[\{v\} \cup N_G(v)]$ and suppose that $\varkappa(G') \geq \alpha(G') + k$. Then $\omega(G-S) \leq |S| - k$ for any cutset $S \subseteq V(G)$.

Corollary 10 Let G be a connected graph. For any $v \in V(G)$, let $G' = G[\{v\} \cup N_G(v)]$ and suppose that $\varkappa(G') \geq \alpha(G') + 2n - 1$. Then G is n-extendable.

Proof By Theorem 9 and the same argument as in Corollary 5, the result follows. □

## 4. Concluding remarks

Now we give some remarks to show the sharpness of the results in this paper.

Remark 1 The bound of $n(u,v)$ in Lemma 3 is sharp. Let $H = K_k$ and let u and v be two vertices not in $V(H)$. We construct G by joining each of u and v to all vertices of H. Then $n(u,v) \geq \alpha^*(u,v) + k - 2$. But G is only k-connected.

Remark 2 The bound of $n(u,v)$ in Corollary 5 is sharp.

Let $H = K_{2n}$ and let u and v be two vertices not in $V(H)$. We construct G by joining each of u and v to all vertices of H. Then we have only two vertices of distance 2 and $n(u,v) \geq \alpha^*(u,v) + 2n - 2$. Let $V(H) = \{u_1, u_2, \ldots, u_{2n}\}$ Then $e_i = u_{2i-1}u_{2i}$ $(i = 1,2,\ldots,n)$ do not lie in any perfect matching of G. The above graph also shows the sharpness of the bound of the degree sum in Corollary 7. And it also shows the sharpness of the bound on connectivity in Corollary 10.

Remark 3 Corollary 5 and Corollary 10 include many n-extendable graphs which Corollary 8 cannot describe. Let $H_i = K_{2n+1}$ $(i = 1,2,\ldots,m$ and m is a positive even integer not less than 2). Let $G = H_1 \cup H_2 \cup \ldots \cup H_m \cup \{uv \mid u \in V(H_i)$ and $v \in V(H_{i+1})$, $i = 1,2,\ldots,m-1\}$. Then G satisfies the hypotheses of Corollary 5 and Corollary 10 and G is n-extendable. But no vertex in G has degree at least $\nu/2 + n$ when m is sufficiently large.

Remark 4 The local conditions in Corollary 5 and Corollary 10 cannot imply each other.

Let $H = (U,V)$ be a $K_{m,m}$, where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_m\}$. Let $H_1$ be a $K_{2n}$ with vertex set $\{w_1, w_2, \ldots, w_{2n}\}$ and $H_2$ be a $K_{2n}$ with vertex set $\{x_1, x_2,$

$\ldots, x_{2n}\}$. Let $G = H \cup H_1 \cup H_2 \cup \{ u_i w_j \mid 1 \le i \le m, 1 \le j \le 2n \} \cup \{ v_i x_j \mid 1 \le i \le m, 1 \le j \le 2n \} \cup \{ w_i x_j \mid 1 \le i \le 2n, 1 \le j \le 2n \}$. Then G satisfies that $n(u,v) \ge \alpha^*(u,v) + 2n - 1$ for any two vertices u and v of G such that $d_G(u,v) = 2$. But consider $G' = G[\{v_1\} \cup N_G(v_1)]$. $\varkappa(G') \ge \alpha(G') + 2n - 1$ doesn't hold.

Let H be a $K_{4n+1}$ with vertex set $\{v_1, v_2, \ldots, v_{4n+1}\}$. Let $u_1$ and $u_2$ be two vertices not in $V(H)$. Let $G = H \cup \{u_1, u_2\} \cup \{u_1 v_1, u_2 v_1\} \cup \{u_1 v_i \mid i = 2, 3, \ldots, 2n+1\} \cup \{ u_2 v_j \mid j = 2n+2, 2n+3, \ldots, 4n+1\}$. Then G satisfies that for any $v \in V(G)$, $\varkappa(G[\{v\} \cup N_G(v)]) \ge \alpha(G[\{v\} \cup N_G(v)]) + 2n - 1$. But $d_G(u_1, u_2) = 2$ and $u_1$ and $u_2$ have only one common neighbour.

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