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On round-robin tournaments with a unique maximum score

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Abstract

Richard Arnold Epstein (1927–2016) published the first edition of "The Theory of Gambling and Statistical Logic" in 1967. He introduced some material on round-robin tournaments (complete oriented graphs) with n labeled vertices in Chapter 9; in particular, he stated, without proof, that the probability that there is a unique vertex with the maximum score tends to 1 as n tends to infinity. Our goal here is to give a proof of this result along with some historical remarks and comments.

1 Introduction

In a classical round-robin tournament, each of n players wins or loses a game against each of the other $n-1$ players (see Moon [18]). Let X_{ij} equal 1 or 0 according to whether player i wins or loses the game played against player j, for $1 \leq i, j \leq n$, $i \neq j$, where $X_{ij}+X_{ji}=1$. We assume that all $\binom{n}{2}$ pairs (X_{ij}, X_{ji}) are independently distributed with $P(X_{ij}=1)=P(X_{ji}=0)=1/2$. Let

$$s_i := s_i(n) = \sum_{j=1, j\neq i}^{n} X_{ij}$$

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denote the score of player $i, 1 \leq i \leq n$, after playing against all the other $n-1$ players. We refer to $(s_1, s_2, \ldots, s_n)$ as the score sequence of the tournament. When the identity of the players with a given score in a tournament is of no particular significance, we often rearrange the elements of the score sequence in nondecreasing order and refer to this rearranged sequence as the nondecreasing score sequence of the tournament. These tournaments can be represented by complete oriented graphs in which the vertices represent the players and each pair of distinct vertices i and j is joined by an edge oriented from i to j or from j to i according to whether $X_{ij} = 1$ or $X_{ji} = 1$. The frequency of a given n-vertex nondecreasing score sequence is the sum of the frequencies (or the total number) of all labelled n-vertex tournaments that contain the same number of vertices of score k as the number of elements of value k in the given nondecreasing sequence, for $0 \leq k \leq n-1$.

Round-robin tournaments can be considered as a model of paired comparison experiments used in an attempt to rank a number of objects with respect to some criterion–or at least to determine if there is any significant difference between the objects—when it is impracticable to compare all the objects simultaneously: see, e.g., Zermelo [22], David [6], David and Edwards [7], and Aldous and Kolesnik [1]. In particular, David [5] generated the score sequences of tournaments with n players for $3 \leq n \leq 8$ and their frequencies by expanding products of the form

$$F(n) = \prod_{1 \leq i < j \leq n} \left(\frac{1}{2} w_i + \frac{1}{2} w_j \right).$$

For example,

$$F(3) = \frac{1}{2^3} \left(w_1^2 w_2 + w_1 w_2^2 + w_1^2 w_3 + w_1 w_3^2 + w_2^2 w_3 + w_2 w_3^2 + 2 w_1 w_2 w_3 \right);$$

so the joint probability mass function of the scores are

$$P(s_1 = 2, s_2 = 1, s_3 = 0) = \cdots = P(s_1 = 0, s_2 = 1, s_3 = 2) = \frac{1}{8}, \quad \text{and}$$
$$P(s_1 = 1, s_2 = 1, s_3 = 1) = \frac{2}{8};$$

and the frequencies of the nondecreasing score sequences $(0,1,2)$ and $(1,1,1)$ are six and two, respectively. David [5] used this information to develop, among other things, tests for deciding whether the maximum in a given outcome was significantly larger than the expected value $(n-1)/2$ of a given score.

Let r_n denote the probability that an ordinary tournament with n labeled vertices has a unique vertex with maximum score. Epstein [8, p. 353] gave the values $r_4 = .5, r_5 = .586, r_6 = .627, r_7 = .581$, and $r_8 = .634$ without further explanation. However, a reference to David [5] is given a few pages later, so presumably he deduced these values for r_i from Table 1 in [5], except for one error: the value for r_8 should have been $160,241,152/2^{28} = .5969\cdots$. Epstein also stated, without a proof or reference, that as n increases indefinitely, r_n approaches unity. Some later editions of his book contain more material on tournaments but the material on r_n remains unchanged. A survey paper by Guy [12], on various unsolved problems, mentions Epstein's problem on r_n as being still unsolved. Alon [2] referred us to a paper by Erdős and Wilson [9] that considered the analogous problem for the vertices of maximum degree in a random labelled graph in which pairs of distinct vertices are joined by an edge with probability 1/2.

Stockmeyer [20] has recently pointed out that MacMahon [14] generated the score sequences and their frequencies for tournaments with up to 9 vertices and his results agree with David's for $n = 8$. It follows from MacMahon's data that $r_9 = 42,129,744,768/2^{36} = .6130\cdots$. As a partial check, we obtained the same value for r_9, as given earlier, by determining the number of ways of constructing 9-vertex tournaments with a unique vertex v of maximum score by adjoining v to an 8-vertex tournament with any given 8-vertex nondecreasing score sequence. We then tried to determine the value of r_{10}, which was unknown to us at the time, in the same way from information about the 9-vertex case. In doing this we discovered that MacMahon's values of $361,297,520$ for the nondecreasing score sequence $(2,2,3,3,4,4,6,6,6)$ and its complement were incorrect; these frequencies should have been divisible by 9, since there are 9 choices for the label of the winner of the match between the two vertices of score 4 (in both cases). These two frequency values should each be increased by $10,000$; and, using these corrected values we found that $r_{10} = 21,293,228,876,800/2^{45} = .6051\cdots$. Later, we also discovered that Doron Zeilberger [21] had extended MacMahon's work and had generated the nondecreasing score sequences and their frequencies for tournaments with up to 15 vertices using the Maple program. (We remark that Zeilberger's frequencies for the two sequences mentioned earlier agree with the corrections we gave.)

This additional data as well as Monte-Carlo simulations up to $n = 100,000$ (see Malinovsky and Moon [16]) strongly suggest that Epstein's Conjecture is correct. In the next section, we show that this is indeed the case.

2 The Uniqueness of the Maximum Score

2.1 Useful Facts and Notation

Let (p_{ij}) denote a probability matrix such that $p_{ij} + p_{ji} = 1$, and $p_{ij} = P(X_{ij} = 1)$ for $1 \leq i \leq j \leq n$, $p_{ii} = 0$ for $1 \leq i \leq n$; and where the variables X_{ij} and X_{ji} are as defined in Section 1. Huber [13] used a coupling argument to establish the following inequality for the joint distribution function of the scores $s_1, \ldots, s_n$ in a round-robin tournament:

$$P\left(s_1 < k_1, \ldots, s_m < k_m\right) \leq P\left(s_1 < k_1\right) \cdots P\left(s_m < k_m\right), \tag{2.1}$$

where $m \leq n$, for any such probability matrix (p_{ij}), and any numbers $(k_1, \ldots, k_m)$; the inequality also holds if the $<$ sign is replaced by the $\leq$ sign throughout. We assume here that $p_{ij} = 1/2$ for all $i \leq j$. As we shall see presently, Huber's inequality has implications for the maximum scores in tournaments.

For expository convenience we introduce some notation and relations that we shall need later. Let

$$b(n-1,j) = P(s_i = j) = \binom{n-1}{j}\frac{1}{2^{n-1}}$$

and

$$B(n-1,j) = P(s_i > j) = \sum_{k>j} b(n-1,k)$$

for $0 \leq j \leq n-1$ and $1 \leq i \leq n$.

Next, let

$$t_{n-1} = \lceil (n-1)/2 + x_{n-1}((n-1)/4)^{1/2} \rceil \tag{2.2}$$

where

$$x_{n-1} = (2\log(n-1) - (1+\epsilon)\log(\log(n-1)))^{1/2} \tag{2.3}$$

for an arbitrary constant ϵ between 0 and 1, say. It is not difficult to see that

$$x_{n-1} \leq ((t_{n-1} - (n-1)/2)\,((n-1)/4)^{-1/2} \leq x_{n-1} + ((n-1)/4)^{-1/2}. \tag{2.4}$$

It follows from (2.4) and definition (2.3) that $x_{n-1} \to \infty$ and $x_{n-1} = o(n^{1/6})$ as $n \to \infty$, and the same conclusion holds when x_{n-1} is replaced by

$$((t_{n-1} - (n-1)/2)\,((n-1)/4)^{-1/2}.$$

Consequently, we may appeal to relation (4.5.1) in Rényi [19, p. 204] and relations (2.7) and (6.7) in Feller [10, pp. 180 & 193] to conclude that

$$b(n-1,t_{n-1}) \sim \left(\frac{2}{\pi(n-1)}\right)^{1/2} e^{-\frac{x_{n-1}^2}{2}} \sim \frac{\sqrt{2}(\log(n-1))^{(1+\epsilon)/2}}{\sqrt{\pi(n-1)^3}} \tag{2.5}$$

and

$$B(n-1,t_{n-1}) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{x_{n-1}}e^{-\frac{x_{n-1}^2}{2}} \sim \frac{(\log(n-1))^{\epsilon/2}}{\sqrt{4\pi}(n-1)}. \tag{2.6}$$

2.2 Main Result

Theorem. *The probability that a random n-vertex tournament T_n has a unique vertex of maximum score tends to 1 as n tends to infinity. In particular, if t_{n-1} is defined as in (2.2) and (2.3) and $s^\star$ denotes the maximum value of the scores $s_1, \ldots, s_n$ in T_n, then the following statements hold:*

(i) [Huber [13]] $P(s^\star > t_{n-1}) \to 1$ as $n \to \infty$.

(ii) *If* $W_n = W_n(T_n)$ *denotes the number of ordered pairs of distinct vertices* u *and* v *in* T_n *such that* $s_u = s_v = h$ *for some integer* h *such that* $t_{n-1} < h \leq n-1$, *then* $P(W_n > 0) \to 0$ *as* $n \to \infty$.

PROOF:

Proof of (i) [Huber [13]]

Huber [13] observed that the required conclusion follows from the facts that

$$P\left(s^\star < t_{n-1}\right) \leq \left(1 - B(n-1, t_{n-1})\right)^n \leq e^{-nB(n-1,t_{n-1})} \leq (1+o(1))e^{-\frac{(\log(n-1))^{\epsilon/2}}{\sqrt{4\pi}}} \to 0, \tag{2.7}$$

as $n \to \infty$, appealing to the definition of $B\left(n-1, t_{n-1}\right)$, inequality (2.1), the inequality $1 - c \leq e^{-c}$, and relation (2.6).

Proof of (ii)

We now turn to conclusion (ii). In view of conclusion (i), we may restrict our attention to tournaments T_n in which the maximum value $s^\star$ of the scores realized by the vertices is at least as large as $t = t_{n-1}$. Recall that $W_n = W_n(T_n)$ denotes the number of ordered pairs of distinct vertices u and v of T_n such that $t < s_u = s_v$ where $t \leq n-1$, i.e.

$$W_n = \sum_{1 \leq v < u \leq n} I(t < s_u = s_v).$$

Let s'_u and s'_v denote the scores of two such vertices u and v in their matches with the remaining $n-2$ players and note that s'_u and s'_v are independent variables. Then it follows that

$$\begin{aligned} P\left(s_u = h, s_v = h\right) &= 1/2P(s'_u = h-1)P(s'_v = h) + 1/2P(s'_u = h)P(s'_v = h-1) \\ &= \binom{n-2}{h-1}(1/2)^{n-2}\binom{n-2}{h}(1/2)^{n-2} \\ &= 4\frac{h}{n-1}\left(1 - \frac{h}{n-1}\right)\binom{n-1}{h}(1/2)^{n-1}\binom{n-1}{h}(1/2)^{n-1} \\ &\leq \left(b(n-1,h)\right)^2. \end{aligned} \tag{2.8}$$

Hence,

$$\begin{aligned} E(W_n) &= E\left(\sum_{1 \leq v < u \leq n} I(t < s_u = s_v)\right) = n(n-1)E\left(I(t < s_1 = s_2)\right) \\ &= n(n-1)P\left(t < s_1 = s_2\right) = n(n-1)\sum_{h=t+1}^{n-1} P(s_1 = h, s_2 = h) \\ &\leq n(n-1)\sum_{h=t+1}^{n-1} b(n-1,h)^2 \leq n(n-1)b(n-1,t+1)B(n-1,t) \\ &\leq n(n-1)b(n-1,t)B(n-1,t) \sim \frac{(\log(n-1))^{1/2+\epsilon}}{\pi\sqrt{2(n-1)}} \to 0, \end{aligned}$$

as $n \to \infty$. Consequently, appealing to (2.5), (2.6), and to the fact that $W_n = W_n I(W_n > 0) \geq I(W_n > 0)$, we find that

$$1 - P(W_n = 0) = P(W_n > 0) \leq E(W_n) \to 0,$$

as required. $\square$

3 Remarks

Remark 3.1 For the sake of completeness, we mention an upper bound, found by Huber [13], for the maximum score of all tournaments T_n, except for a fraction that tends to zero as $n \to \infty$. Let $t' = t'_{n-1}$ be defined as $t = t_{n-1}$ was defined earlier except that the ϵ in relation (2.3) is replaced by $-\epsilon$ and without the ceiling function, it turns out that a relation corresponding to (2.6) is

$$B(n-1, t'_{n-1}) \sim \frac{(\log(n-1))^{-\epsilon/2}}{\sqrt{4\pi}(n-1)}.$$

Hence, it follows from Boole's inequality that

$$P\left(s^{\star} > t'\right) \leq nB(n-1, t') = O\left((\log(n-1))^{-\epsilon/2}\right), \tag{3.1}$$

as $n \to \infty$. From (2.7) and (3.1) Huber [13] concluded that

$$s^{\star} - \frac{n-1}{2} - \sqrt{\frac{n-1}{4}}\sqrt{2\log(n-1)} \to 0$$

in probability as $n \to \infty$.

Remark 3.2 Malinovsky and Moon [15] and Malinovsky and Rinott [17] have extended Huber's inequality to a more general round-robin tournament model and to other tournaments and games models, respectively.

Remark 3.3 Malinovsky and Moon [16] gave another proof of (i) by the 2nd Moment Method frequently applied to probabilistic problems in Graph Theory (see Alon and Spencer [3, Chapter 4]). Bollobás [4] and Frieze and Karoński [11] (see also references therein) have, among other things, derived numerous results on the distribution of the degrees of vertices in ordinary graphs in which edges are present with probability p.

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