Bounds on the outer-independent Roman domination number of unicyclic and bicyclic graphs

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Abstract

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is called a Roman dominating function on a graph $G$ if for every vertex $v \in V(G)$ with $f(v) = 0$, there exists a vertex $u$ adjacent to $v$ with $f(u) = 2$. A Roman dominating function $f$ is an outer-independent Roman dominating function if the set $\{v \in V(G) : f(v) = 0\}$ is independent. The outer-independent Roman domination number of $G$ is the minimum weight of an outer-independent Roman dominating function on $G$. Chellali and Dehgardi [Commun. Comb. Optim. 6 (2021), 273–286] proved that the outer-independent Roman domination number of any tree $T$ of order $n \geq 3$ is bounded above by $5n/6$. In this paper, aiming to obtain best upper bounds for the outer-independent Roman domination number in cactus graphs, we prove an $8n/9$-upper bound for unicyclic graphs and a $9n/10$-upper bound for bicyclic graphs. We also characterize extremal unicyclic graphs as well as bicyclic graphs, achieving equality for the given bounds.
1 Introduction

For graph theory notation and terminology not given here we refer to [9]. We consider finite and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E(G)$. The number of vertices of $G$ is called the order of $G$ and is denoted by $n = n(G)$. The open neighborhood of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = N_G[v] = N(v) \cup \{v\}$. The degree of a vertex $v$, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to $G$), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of $G$, respectively. A vertex of degree one is referred as a leaf and a vertex adjacent to a leaf is referred as a support vertex. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to exactly one leaf. A component of a graph $G$ is a maximal connected subgraph of $G$. We denote by $P_n$, $C_n$ and $K_{m,n}$ the path of order $n$, the cycle of order $n$, and the complete bipartite graph such that one partite set has $m$ vertices and the other partite set has $n$ vertices, respectively. We refer to $K_{1,3}$, as a claw. For a subset $S$ of vertices of a graph $G$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. A graph $G$ is claw-free if $G[S] \not\cong K_{1,3}$ for any set $S$ of cardinality 4. A unicyclic graph is a graph obtained from a tree by adding precisely one edge. Equivalently, a unicyclic graph is a graph with precisely one cycle. A bicyclic graph is a graph with precisely two cycles. A cactus graph is a graph such that no pair of distinct cycles have a common edge. An independent set in a graph $G$ is a subset $S$ of vertices such that the subgraph induced by $S$ has no edges.

A function $f : V \rightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a Roman dominating function or just an RDF. The weight of an RDF $f$ is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on $G$ is called the Roman domination number of $G$ and is denoted by $\gamma_R(G)$. The mathematical concept of Roman domination was developed by Cockayne et al. [8]. Many variations, generalizations and applications of Roman domination parameters have been studied, and to see the latest progress until 2020 see [5, 6, 7].

Ahangar et al. [1] introduced the concept of outer-independent Roman domination in graphs. An RDF $f$ in a graph $G$ is an outer-independent Roman dominating function (OIRDF) on $G$ if the set $\{v \in V(G) : f(v) = 0\}$ is an independent set. The outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on $G$. The concept of outer-independent Roman domination in graphs was further studied in, for example, [2, 3, 10, 11, 12, 13]. Chellali and Dehgardi [4] proved that $\gamma_{oiR}(T) \leq 5n/6$ in any tree $T$ of order $n$, and they characterized trees achieving equality for this bound.

In this paper we present upper bounds for the outer-independent Roman domination number in unicyclic graphs as well as bicyclic graphs, and characterize unicyclic graphs and bicyclic graphs achieving equality for the given bounds. The organization of the paper is as follows. In Section 2, we present our main results, namely Propositions 2.1 and 2.2, and Theorems 2.3 and 2.4. In Section 3, we present a
proof for Propositions 2.1 and 2.2. In Section 4, we present a proof for Theorem 2.3. In Section 5, we present a proof for Theorem 2.4. In Section 6, we propose our suggested bounds and conjectures for cactus graphs. We make use of the following.

**Theorem 1.1 (Chellali et al. [4])** For any tree $T$ of order $n \geq 3$, $\gamma_{oIR}(T) \leq \frac{5n}{6}$.

For an RDF $f$ in a graph $G$, we denote by $V_i$ (or $V^f_i$ to refer to $f$) the set of all vertices of $G$ with label $i$ under $f$. Thus an RDF $f$ can be represented by a triple $(V_0, V_1, V_2)$, and we can use the notation $f = (V_0, V_1, V_2)$.

### 2 Main results

Let $H_1$ and $H_2$ be the graphs depicted in Figure 1.

![Graphs $H_1$ and $H_2$.](image)

Figure 1. Graphs $H_1$ and $H_2$.

We will prove the following.

**Proposition 2.1** For any integer $n \geq 3$, $\gamma_{oIR}(C_n) = \frac{3n+j}{4}$ if $n \equiv j \pmod{4}$, $j = 0, 1, 2, 3$.

**Proposition 2.2** For any integer $n \geq 3$, $\gamma_{oIR}(P_n) = \frac{3n+j}{4}$ if $n \equiv j \pmod{4}$, $j = 0, 1, 2$, and $\gamma_{oIR}(P_3) = \frac{3n-1}{4}$ if $n \equiv 3 \pmod{4}$.

**Theorem 2.3** For a unicyclic graph $G \neq K_3$, $\gamma_{oIR}(G) \leq \frac{8n}{9}$, with equality if and only if $G = H_1$, where $H_1$ is depicted in Figure 1.

**Theorem 2.4** For a bicyclic graph $G$ of order $n$, $\gamma_{oIR}(G) \leq \frac{9n}{10}$, with equality if and only if $G = H_2$, where $H_2$ is depicted in Figure 1.

### 3 Proof of Propositions 2.1 and 2.2

We only present the proof of Proposition 2.1; the proof of Proposition 2.2 is similar and is thus omitted.

Let $V(C_n) = \{v_1, \ldots, v_n\}$, where $v_i$ is adjacent to $v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $v_1$ is adjacent to $v_n$. It is evident that $\gamma_{oIR}(C_n) < n$ for $n \geq 4$, since assigning 2 to $v_2$, 0 to
we deduce that $v$ is as small as possible. If $n = 3$, then $V_0 = \emptyset$, and so we change both $f(v_i)$ and $f(v_{i-1})$ to 1 to obtain a $g_{oiR}(C_n)$-function with fewer vertices assigned 2 than under $f$, contradicting the choice of $f$. Thus, $|\{v_{i-1}, v_i\} \cap V_0| = 2$, for each vertex $v_i \in V_2$. Let $v_i \in V_2$. As noted, $\{v_{i-1}, v_i\} \subseteq V_0$. Observe that $v_{i+2} \notin V_0$ and $v_{i-2} \notin V_0$. Suppose that $v_{i+2} \in V_2$. Then we change both $f(v_i)$ and $f(v_{i-1})$ to 1, leading as before to a contradiction on the choice of $f$. Thus, $v_{i+2} \in V_1$, and likewise, $v_{i-2} \in V_1$. Let $G'$ be obtained from $C_n$ by removing $v_{i-1}$ and $v_i$ and then joining $v_{i-2}$ to $v_{i+3}$. Then $G' = C_{n-4}$ and $f|_{G'}$ is an OIRDF for $G'$, implying that $\gamma_{oiR}(G') = \gamma_{oiR}(C_{n-4}) \leq \gamma_{oiR}(C_n) - 3$, since $f(v_i) + f(v_{i+2}) = 3$.

On the other hand, let $g = (V_0, V_1, V_2)$ be a $g_{oiR}(C_{n-4})$-function such that $|V_2|$ is as small as possible. If $V_1 = \emptyset$, then $n - 4$ is even (since by the choice of $g$, both neighbors of a vertex of $V_0$ belong to $V_2$ and both neighbors of a vertex of $V_2$ belong to $V_0$) and we may assume that $V_2 = \{v_{2i+1} : 0 \leq i < \frac{n-4}{2}\}$ and $V_0 = \{v_2i : 1 \leq i \leq \frac{n-4}{2}\}$, and so $\gamma_{oiR}(C_{n-4}) = n - 4$, a contradiction, since $n \geq 8$. Thus, there is a vertex $v_j$ such that $v_j \in V_1$. Then we form a graph $G''$ by replacing $v_i$ with a path $abcede$, and join $a$ to $v_{i-1}$ and $e$ to $v_{i+1}$. Observe that $G'' = C_n$. Then $h$ defined on $G''$ by $h(x) = g(x)$ if $x \notin \{a, b, c, d, e\}$, $h(a) = h(e) = 1$, $h(b) = h(d) = 0$ and $h(c) = 2$, is an OIRDF on $G''$, and so $\gamma_{oiR}(G'') \leq w(g) + 3 = \gamma_{oiR}(C_{n-4}) + 3$. We deduce that $\gamma_{oiR}(C_n) \leq \gamma_{oiR}(C_{n-4}) + 3$. Hence $\gamma_{oiR}(C_n) = \gamma_{oiR}(C_{n-4}) + 3$. Now it is straightforward to prove that $\gamma_{oiR}(C_n) = \frac{3n+2}{4}$ using induction on $n$.

4 Proof of Theorem 2.3

We prove this by induction on the order $n$. For the base step of the induction it is easy to see that the result holds if $n \leq 5$. Thus let $n \geq 6$ and assume the result holds for all unicyclic graph $G'$ of order $5 \leq n' < n$, that is, $\gamma_{oiR}(G') \leq \frac{3n'}{4}$, with equality if and only if $G' = H_1$. Now consider the unicyclic graph $G$ of order $n$. Assume $G$ has no leaves. Then $G = C_n$. By Proposition 2.1, $\gamma_{oiR}(C_n) = \frac{3n+2}{4}$ if $n \equiv j \pmod{4}$, $j = 0, 1, 2, 3$. Since $n \geq 6$, we find that $\frac{3n+2}{4} < 8n/9$. We thus assume that $G$ has at least one leaf. Let $C$ be the unique cycle of $G$, and $x_0$ be a vertex of $C$ such that $\deg(x_0)$ is as maximum as possible and $x_0 \ldots x_d$, where $d \geq 1$, be a path from $x_0$ to a farthest leaf $x_d$ of $G$, where $x_i$ is outside $C$ for $i = 1, \ldots, d$. Let $|V(C)| = k$, and $V(C) = \{x_0, y_1, y_2, \ldots, y_{k-1}\}$, where $k \geq 3$, $x_0$ is adjacent to $y_1$ and $y_{k-1}$, and $y_i$ is adjacent to $y_{i+1}$ for $i = 1, 2, \ldots, k-2$.

Lemma 4.1 If $\deg(y_i) = 2$ for some $i \in \{1, 2, \ldots, k-1\}$, then $\gamma_{oiR}(G) < 8n/9$.

Proof. Suppose that $\deg(y_i) = 2$ for some $i \in \{1, 2, \ldots, k-1\}$, and let $T = G - y_i$. 


By Theorem 1.1, \( \gamma_{oIR}(T) \leq 5(n - 1)/6 \). Then we extend any \( \gamma_{oIR}(T) \)-function to an OIRDF for \( G \) by assigning 1 to \( y_i \). Thus, \( \gamma_{oIR}(G) \leq 5(n - 1)/6 + 1 < 8n/9 \), since \( n \geq 6 \).

By Lemma 4.1, we have \( \deg(y_i) \geq 3 \) for \( i = 1, 2, \ldots, k - 1 \).

**Lemma 4.2** If \( v \in C \) is a strong support vertex of degree at least four such that all its neighbors but two are leaves, or \( v \in V(G) - C \) is a strong support vertex such that all its neighbors but one are leaves, then \( \gamma_{oIR}(G) < 8n/9 \).

**Proof.** Let \( v \in C \) be a strong support vertex of degree at least four such that all its neighbors but two are leaves and let \( v' \) and \( v'' \) be the neighbors of \( v \) in \( C \). Let \( T' \) be the tree obtained from \( G \) by removal of edges \( v'v \) and \( v''v \), such that \( T' \) contains \( v' \). If \( |V(T')| = 2 \), then \( n = 1 + \deg(v) \) and by assigning 2 to \( v \), 1 to \( v' \) and 0 to each other vertex we obtain that \( \gamma_{oIR}(G) = 3 < 8n/9 \), since \( \deg(v) \geq 4 \). Thus assume that \( |V(T')| = 3 \). By Theorem 1.1, \( \gamma_{oIR}(T') \leq 5(n - (\deg(v) - 1))/6 \). Then we extend any \( \gamma_{oIR}(T') \)-function to an OIRDF for \( G \) by assigning 2 to \( v \) and 0 to its leaf neighbors. Thus, \( \gamma_{oIR}(G) \leq 5(n - (\deg(v) - 1))/6 + 2 < 8n/9 \), since \( \deg(v) \geq 4 \). The proof for the case \( v \in V(G) - C \) is similar and is omitted.

We proceed with Lemma 4.3.

**Lemma 4.3** If \( d = 1 \), then \( \gamma_{oIR}(G) < 8n/9 \).

**Proof.** Assume that \( d = 1 \). Then \( n = 2k \), since \( \deg(y_i) \geq 3 \), for \( i = 1, 2, \ldots, k - 1 \). It is straightforward to see that if \( k = 3 \) then \( \gamma_{oIR}(G) = 5 < 8n/9 \), if \( k = 4 \), then \( \gamma_{oIR}(G) = 6 < 8n/9 \), if \( k = 5 \) then \( \gamma_{oIR}(G) = 8 < 8n/9 \) and if \( k = 6 \) then \( \gamma_{oIR}(G) = 9 < 8n/9 \). Thus assume that \( k \geq 7 \). Let \( T \) be the tree obtained by removing the edges \( y_{k-1}x_0 \) and \( y_2y_3 \) such that \( T \) contains \( y_3 \). By Theorem 1.1, \( \gamma_{oIR}(T) \leq 5(n - 6)/6 \). We assign 2 to \( x_0 \), 1 to \( y_2 \) and the leaf-neighbors of \( y_1 \) and \( y_2 \), and 0 to \( x_1 \) and \( y_1 \) to extend any \( \gamma_{oIR}(T) \)-function to an OIRDF for \( G \). Then \( \gamma_{oIR}(G) \leq 5(n - 6)/6 + 5 < 8n/9 \). We thus assume that \( d \geq 2 \). By Lemma 4.2, \( \deg(x_{d-1}) = 2 \), and any neighbor of \( x_{d-2} \), with the exception of \( x_{d-3} \) (if \( d \geq 3 \)) and with the exception of its neighbor on \( C \) (if \( d = 2 \)), is a leaf or a support vertex of degree two. We proceed with Lemma 4.4.

**Lemma 4.4** If \( d \geq 3 \), then \( \gamma_{oIR}(G) < 8n/9 \).

**Proof.** Assume that \( d \geq 3 \). We consider the following two cases.

**Case 1.** \( \deg(x_{d-2}) \geq 3 \). Assume that \( x_{d-2} \) has \( a \) neighbors as support vertices of degree two and \( b \) neighbors as leaves. Clearly \( a \geq 1 \) and \( a + b \geq 2 \). Let \( G' \) be the component of \( G - x_{d-2} \) that contains \( x_{d-3} \). If \( G' = K_3 \), then \( d = 3 \) and \( n = 2a + b + 4 \). Then we assign 2 to \( x_1 \), 0 to each neighbor of \( x_1 \), and 1 to any other vertex of \( G \) to obtain that \( \gamma_{oIR}(G) \leq a + 4 < 8n/9 \), since \( a + b \geq 2 \). Thus, assume that \( G' \neq K_3 \). By the inductive hypothesis, \( \gamma_{oIR}(G') \leq 8n'/9 \). Let \( f' \) be a \( \gamma_{oIR}(T') \)-function. Then
we extend \( f' \) to an OIRDF for \( G \) by assigning 2 to \( x_{d-2} \), 0 to its neighbors which are leaves or support vertices of degree two, and 1 to any other vertex. Since \( a \geq 1 \) and \( a+b \geq 2 \), we obtain that

\[
\gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 2 + a \\
\leq \frac{8}{9}(n-2a-b-1) + 2 + a \\
= \frac{8n-7a-8b+10}{9} < \frac{8}{9}n.
\]

\textbf{Case 2.} \( \deg(x_{d-2}) = 2 \). If \( x_{d-3} \) has a neighbor \( w_1 \) outside \( C \) such that there is a path \( x_{d-3}w_1w_2w_3 \) from \( x_{d-3} \) to a leaf \( w_3 \), then \( w_3 \) plays the role of \( x_d \), and thus, by Lemma 4.2 and Case 1 of the proof of Lemma 4.4, we may assume that \( \deg(w_1) = \deg(w_2) = 2 \). Thus, any neighbor of \( x_{d-3} \) outside \( C \) is a leaf, a support vertex of degree two, or a vertex of degree two which is adjacent to a support vertex of degree two.

Assume that \( d \geq 4 \). Let \( G' \) be the component of \( G - x_{d-3}x_{d-4} \) that contains \( x_{d-4} \). It is evident that \( G' \neq K_3 \). By the inductive hypothesis, \( \gamma_{oIR}(G') \leq 8n'/9 \). Let \( f' \) be a \( \gamma_{oIR}(G') \)-function. If \( \deg(x_{d-3}) = 2 \), then \( n' = n - 4 \), and we extend \( f' \) to an OIRDF for \( G \) by assigning 2 to \( x_{d-1} \), 0 to \( x_d \) and \( x_{d-2} \) and 1 to \( x_{d-3} \) to obtain

\[
\gamma_{oIR}(G) \leq \gamma_{oIR}(G')+3 \leq \frac{8}{9}(n-4) + 3 < 8n/9.
\]

Thus, assume that \( \deg(x_{d-3}) \geq 3 \). Assume that \( x_{d-3} \) is adjacent to \( c \) leaves, \( b \) support vertices of degree two and \( a \) vertices of degree two each of which is adjacent to a support vertex of degree two (all outside \( C \)). Then \( a \geq 1 \) and \( a+b+c \geq 2 \). Then we extend \( f' \) to an OIRDF for \( G \) by assigning 2 to \( x_{d-3} \) and each support vertex at distance 2 from \( x_{d-3} \), 1 to any leaf at distance two from \( x_{d-3} \) and 0 to any other vertex in \( V(G) - V(G') \). Then

\[
\gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 2 + 2a+b \\
\leq \frac{8}{9}(n-3a-2b-c-1) + 2 + 2a+b \\
= \frac{8n-6a-7b-8c+10}{9} < \frac{8}{9}n,
\]

since \( a+b+c \geq 2 \).

Next assume that \( d = 3 \). Let \( G' \) be the component of \( G - \{x_0y_1, x_0y_k \} \) that contains \( y_1 \). If \( G' \neq K_2 \), then by Theorem 1.1, \( \gamma_{oIR}(G') \leq 8n'/9 \), and as before, we can extend a \( \gamma_{oIR}(G') \)-function to an OIRDF for \( G \) to obtain that \( \gamma_{oIR}(G) < 8n/9 \). Thus, assume that \( G' = K_2 \). Assume that \( x_0 \) is adjacent to \( c \) leaves, \( b \) support vertices of degree two and \( a \) vertices of degree two each of which is adjacent to a support vertex of degree two. Then \( n = 3a + 2b + c + 3, a \geq 1 \) and \( a+b+c \geq 1 \). Then we define a function \( f \) on \( V(G) \) by assigning 2 to \( x_{d-3} \) and each support vertex at distance 2 from \( x_{d-3} \), 1 to \( y_1 \) and to any leaf at distance two from \( x_{d-3} \) and 0 to any other vertex in \( V(G) \). Then \( \gamma_{oIR}(G) \leq 3 + 2a + b < \frac{8}{9}(3a + 2b + c + 3) = \frac{8}{9}n \), as desired.

By Lemmas 4.3 and 4.4, we may assume for the next that \( d = 2 \). Furthermore, any neighbor of \( x_0 \) outside \( C \) is a leaf or a support vertex of degree two. Let \( a \) be the
number of support neighbors of \( x_0 \) (outside \( C \)) and \( b \) be the number of leaf neighbors of \( x_0 \) (outside \( C \)).

Assume that \( \deg(x_0) \geq 4 \). Then \( a + b \geq 2 \). Let \( G' \) be the component of \( G - x_0 \) that contains \( y_i \). Then \( G' \) is a tree of order at least four. By Theorem 1.1, \( \gamma_{oilR}(G') \leq 5n'/6 \). Let \( f' \) be a \( \gamma_{oilR}(T') \)-function. Then we extend \( f' \) to an OIRDF for \( T \) by assigning 2 to \( x_0 \), 0 to its neighbors outside \( C \), and 1 to any other vertex, and thus we can obtain that \( \gamma_{oilR}(G) \leq \frac{5}{6}(n - 2a - b - 1) + a + 2 < \frac{5}{6}n \). We thus assume that \( \deg(x_0) = 3 \). By the choice of \( x_0 \), any vertex on \( C \) is of degree three which either is a support vertex or is adjacent to a support vertex of degree two.

Assume that \( y_i \) is a support vertex for some \( i \in \{1, 2, \ldots, k - 1\} \) and \( z_i \) is the leaf adjacent to \( y_i \). Let \( T = G - \{y_i, z_i\} \). Then \( |V(T)| \geq 5 \). By Theorem 1.1, \( \gamma_{oilR}(T) \leq 5(n - 2)/6 \). Then we extend any \( \gamma_{oilR}(T) \)-function to an OIRDF for \( T \) by assigning 1 to both \( y_i \) and \( z_i \), and so \( \gamma_{oilR}(G) \leq \frac{5}{6}(n - 2) + 2 < \frac{5}{6}n \). We thus assume that \( y_i \) is a vertex of degree three adjacent to a support vertex of degree two, for \( i = 1, 2, \ldots, k - 1 \).

Assume that \( k \geq 5 \). Let \( T \) be the tree containing \( y_2 \) that is obtained by removing the edges \( y_1y_2 \) and \( y_{k-1}x_0 \). Note that \( |V(T)| \geq 6 \). Let \( G' \) be the unicyclic graph obtained from \( T \) by joining \( y_2 \) to \( y_{k-1} \). By the inductive hypothesis, \( \gamma_{oilR}(G') \leq \frac{5}{6}n' \). Let \( f' \) be a \( \gamma_{oilR}(G') \)-function. Clearly \( f'(y_2) \neq 0 \) or \( f'(y_{k-1}) \neq 0 \), since they are adjacent in \( G' \). Without loss of generality, assume that \( f'(y_2) \neq 0 \). Let \( y_i' \) be the support vertex adjacent to \( y_2 \), and let \( y_i'' \) be the leaf adjacent to \( y_i' \). Clearly we may assume that \( f(y_i') + f(y_i'') = 2 \). Let \( f \) be a function defined on \( V(G) \) by \( f(x_0) = 2 \), \( f(x) = f'(x) \) if \( x \in V(G') \setminus \{y_i', y_i''\} \), \( f(x) = 0 \) if \( x \in \{x_1, y_1, y_i'', y_i', y_i\} \), \( f(y_i') = 2 \), and \( f(x) = 1 \) otherwise. Then \( f \) is an OIRDF for \( G \), and so \( \gamma_{oilR}(G) \leq \frac{5}{6}(n - 6) + 5 < \frac{5}{6}n \).

If \( k = 4 \), then \( n = 12 \). Let \( f \) be a function defined on \( V(G) \) by \( f(x_0) = 2 \), \( f(x) = 0 \) if \( x \) is adjacent to \( x_0 \), and \( f(x) = 1 \) otherwise. Then \( f \) is an OIRDF for \( G \) with weight \( 2 + (n - 4) = n - 2 \), and clearly, \( \gamma_{oilR}(G) \leq n - 2 < \frac{5}{6}n \). Thus, we assume that \( k = 3 \). Then \( n = 9 \) and \( G = H_1 \). Let \( f \) be a function defined on \( V(G) \) by \( f(x_1) = 2 \), \( f(x_0) = f(x_2) = 0 \), and \( f(x) = 1 \) otherwise. Then \( f \) is an OIRDF for \( G \) with weight \( 8 = \frac{8}{9}n \).

We thus have proved the upper bound. If the equality holds, then following the above proof, we deduce that \( G = H_1 \).

5 Proof of Theorem 2.4

We prove this theorem by induction on the order \( n \). Clearly, \( n \geq 5 \). For the base step of the induction, if \( n \leq 9 \) then we choose a vertex \( x \) with two non-adjacent neighbors \( y \) and \( z \), and assign 2 to \( x \), 0 to both \( y \) and \( z \), and 1 to other vertices of \( G \), and so \( \gamma_{oilR}(G) \leq n - 1 < 9n/10 \). Thus let \( n \geq 10 \) and assume that the result holds for all bicyclic graphs of order \( n' < n \). Now consider the bicyclic graph \( G \) of order \( n \). Let \( C_1 \) and \( C_2 \) be the cycles of \( G \), and let \( V(C_1) = \{v_1, \ldots, v_k\} \) and \( V(C_2) = \{u_1, \ldots, u_l\} \). Without loss of generality, assume that \( d(C_1, C_2) = d(v_1, u_1) \). Let \( d(C_1, C_2) = s \) and \( P \) the shortest path between \( v_1 \) and \( u_1 \). The following lemma can be proved similarly
to the proof of Lemma 4.2, and thus we omit its proof.

Lemma 5.1 If \( v \in P \cup C_1 \cup C_2 \) is a strong support vertex such that all of its neighbors outside \( P \cup C_1 \cup C_2 \) are leaves, or \( v \in V(G) - P \cup C_1 \cup C_2 \) is a strong support vertex such that all of its neighbors but one are leaves, then \( \gamma_{oR}(G) < \frac{9n}{10} \).

We proceed with Lemma 5.2.

Lemma 5.2 If \( x \) is a vertex on \( P \cup C_1 \cup C_2 \) and there is a path of length \( d \geq 3 \) from \( x \) to a farthest leaf \( x_d \) from \( x \) that intersects \( P \cup C_1 \cup C_2 \) only in \( x \), then \( \gamma_{oR}(G) < \frac{9n}{10} \).

Proof. Assume that there is a path \( xx_1x_2 \ldots x_d \) to a leaf \( x_d \) and \( \{x, x_1, \ldots, x_d\} \cap (V(P) \cup V(C_1) \cup V(C_2)) = \{x\} \). By Lemma 5.1, \( \deg(x_{d-1}) = 2 \). Assume that \( \deg(x_{d-2}) \geq 3 \). Then we may assume that each neighbor of \( x_{d-2} \) with the exception of \( x_{d-3} \) is a leaf or a support vertex of degree two by Lemma 5.1. Assume that \( x_{d-2} \) has \( a \) neighbors as support vertices of degree two and \( b \) neighbors as leaves. Clearly \( a + b \geq 2 \). Let \( G' \) be the component of \( G - x_{d-2} \) that contains no leaf-neighbor or support neighbor of \( x_{d-2} \). By the inductive hypothesis, \( \gamma_{oR}(G') \leq \frac{9n'}{10} \). Let \( f' \) be a \( \gamma_{oR}(G') \)-function. Then we extend \( f' \) to an OIRDF for \( G \) by assigning 2 to \( x_{d-2} \), 0 to its neighbors which are leaves or support vertices of degree two, and 1 to any other vertex. Since \( a + b \geq 2 \), we obtain that

\[
\gamma_{oR}(G) \leq \gamma_{oR}(G') + 2 + a \leq \frac{9}{10}(n - 2a - b - 1) + 2 + a \\
= \frac{9n - 8a - 9b + 11}{10} < \frac{9n}{10}.
\]

We thus assume that \( \deg(x_{d-2}) = 2 \).

Assume that \( d \geq 4 \). Assume \( \deg(x_{d-3}) \geq 3 \). Then we may assume that each neighbor of \( x_{d-3} \) with the exception of \( x_{d-4} \) is a leaf, a support vertex of degree two, or a vertex of degree two that is adjacent to a support vertex of degree two. Assume that \( x_{d-3} \) has \( a \) neighbors as vertices of degree two that are adjacent to support vertices of degree two, \( b \) neighbors as support vertices of degree two, and \( c \) neighbors as leaves. Clearly \( a + b + c \geq 2 \). Let \( A \) be the set of all such neighbors of \( x_{d-3} \). Let \( G' \) be the component of \( G - x_{d-3} \) that contains no vertex of \( A \). By the inductive hypothesis, \( \gamma_{oR}(G') \leq \frac{9n'}{10} \). Let \( f' \) be a \( \gamma_{oR}(G') \)-function. We extend \( f' \) to an OIRDF for \( G \) by assigning 2 to \( x_{d-3} \) and each support vertex at distance 2 from \( x_{d-3} \), 1 to any leaf at distance two from \( x_{d-3} \) and 0 to any other vertex in \( V(G) - V(G') \). Then

\[
\gamma_{oR}(G) \leq \gamma_{oR}(G') + 2 + 2a + b \leq \frac{9}{10}(n - 3a - 2b - c - 1) + 2 + 2a + b \\
= \frac{9n - 7a - 8b - 9c + 11}{10} < \frac{9}{10}n,
\]
since \(a + b + c \geq 2\). Thus assume that \(\deg(x_{d-3}) = 2\). Let \(G' = G - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}\}\). By the inductive hypothesis, \(\gamma_{oIR}(G') \leq 9n'/10\). Let \(f'\) be a \(\gamma_{oIR}(G')\)-function. Then we extend \(f'\) to an OIRDF for \(G\) by assigning 2 to \(x_{d-1}\), 1 from \(x_{d-3}\), and 0 to \(x_d\) and \(x_{d-2}\). Then \(\gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 3 \leq \frac{9}{10}(n - 4) + 3 < \frac{9}{10}n\).

Thus assume that \(d = 3\). Let \(G' = G - \{x, x_1, x_2, x_3\}\). If \(G'\) is connected, then \(G' \neq K_3\), since \(G\) is a bicyclic graph. By Theorem 2.3, \(\gamma_{oIR}(G') \leq 8n'/9\). Let \(f'\) be a \(\gamma_{oIR}(G')\)-function. Then we extend \(f'\) to an OIRDF for \(G\) by assigning 2 to \(x_2\), 1 to \(x\), and 0 to \(x_1\) and \(x_3\). Then \(\gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 3 \leq \frac{8}{9}(n - 4) + 3 < \frac{9}{10}n\). Next assume that \(G'\) is disconnected. If \(G'\) has no \(K_3\)-components, then as before we find that \(\gamma_{oIR}(G) < \frac{9}{10}n\). Thus assume that \(G'\) has some \(K_3\)-components. If \(G'\) has two \(K_3\)-components, then \(|C_1| = |C_2| = 3\), \(s = 2\) and \(P = v_1xu_1\). Then \(n = 10\) and \(\gamma_{oIR}(G) = 8 < \frac{9}{10}n\). Thus assume that \(G'\) has precisely one \(K_3\)-component. Without loss of generality, assume that \(C_1\) is such component. Let \(G' = G - \{x, x_1, x_2, x_3, v_1, v_2, v_3\}\). By Theorem 2.3, \(\gamma_{oIR}(G') \leq 8n'/9\). Let \(f'\) be a \(\gamma_{oIR}(G')\)-function. Then we extend \(f'\) to an OIRDF for \(G\) by assigning 2 to \(x\), 1 to \(x_2, x_3, v_2, v_3\), and 0 to \(x_1\) and \(v_1\). Then \(\gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 6 \leq \frac{8}{9}(n - 7) + 6 < \frac{9}{10}n\). \(\square\)

**Lemma 5.3** If \(x\) is a vertex in \(P \cup C_1 \cup C_2\) with \(x \notin \{v_1, u_1\}\) and \(\deg(x) \geq 4\), then \(\gamma_{oIR}(G) < \frac{9}{10}n\).

**Proof.** Assume that \(x\) is a vertex in \(P \cup C_1 \cup C_2\) and \(x \notin \{v_1, u_1\}\) and \(\deg(x) \geq 4\). By Lemmas 5.1 and 5.2, each neighbor of \(x\) outside \(P \cup C_1 \cup C_2\) is a leaf or a support vertex of degree two.

Assume that \(x\) has \(a\) neighbors as support vertices of degree two and \(b\) neighbors as leaves. Clearly \(a + b \geq 2\). Let \(G'\) be the component of \(G - x\) that contains no leaf-neighbor or support neighbor of \(x\). If \(G'\) has no \(K_3\)-components, then by Theorem 2.3, \(\gamma_{oIR}(G') \leq 8n'/9\). Let \(f'\) be a \(\gamma_{oIR}(G')\)-function. Then we extend \(f'\) to an OIRDF for \(G\) by assigning 2 to \(x\), 0 to its neighbors which are leaves or support vertices of degree two, and 1 to any other vertex. Since \(a + b \geq 2\), we obtain that \(\gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 2 + a \leq \frac{8}{9}(n - 2a - b - 1) + 2 + a < \frac{9}{10}n\). Thus assume that \(G'\) has some \(K_3\)-components. If \(G'\) has two \(K_3\)-components, then \(|C_1| = |C_2| = 3\), \(s = 2\) and \(P = v_1xu_1\). Then \(n = 2a + b + 7\) and \(\gamma_{oIR}(G) = 6 + a < \frac{9}{10}n\). Thus assume that \(G'\) has precisely one \(K_3\)-component. Without loss of generality, assume that \(C_1\) is such component. Let \(G'' = G' - \{v_1, v_2, v_3\}\). By Theorem 2.3, \(\gamma_{oIR}(G'') \leq 8n'/9\). Let \(f'\) be a \(\gamma_{oIR}(G'')\)-function. Then we extend \(f'\) to an OIRDF for \(G\) by assigning 2 to \(x\), 1 to \(v_2, v_3\) and the leaves at distance two from \(x\), and 0 to each other vertex. Then \(\gamma_{oIR}(G) \leq \gamma_{oIR}(G'') + 4 + a \leq \frac{8}{9}(n - 2a - b - 4) + 4 + a < \frac{9}{10}n\). \(\square\)

By Lemma 5.3 if \(x\) is a vertex in \(P \cup C_1 \cup C_2\) and \(x \notin \{v_1, u_1\}\), then we may assume that \(\deg(x) \leq 3\). Similarly, we may assume that the following hold.

**Lemma 5.4** \(3 \leq \deg(v_1) \leq 4\) and \(3 \leq \deg(u_1) \leq 4\).

From Lemmas 5.1 and 5.2, we may assume that for any vertex \(x \in P \cup C_1 \cup C_2 - \{v_1, u_1\}\) with \(\deg(x) = 3\), \(x\) is a support vertex or adjacent to a support vertex of
degree two, and if \( \deg(x) = 4 \) for \( x \in \{u_1, v_1\} \), then \( x \) is a support vertex or adjacent to a support vertex of degree two.

**Lemma 5.5** If \( n > 10 \), then \( \gamma_{oIR}(G) < \frac{n}{10}n \).

**Proof.** Assume that \( n > 10 \). Assume that there is a vertex \( x \in C_1 \cup C_2 \) such that \( \deg(x) = 2 \). Clearly, \( x \notin \{v_1, u_1\} \). Let \( G' = G - x \). By Theorem 2.3, \( \gamma_{oIR}(G') \leq 8n' / 9 \).

Let \( f' \) be a \( \gamma_{oIR}(G') \)-function. Then we extend \( f' \) to an OIRDF for \( G \) by assigning 1 to each vertex. Then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 1 \leq \frac{8}{9}(n - 1) + 1 < \frac{n}{10}n \). Thus assume that each vertex of \( C_1 \cup C_2 \) with the exception of \( v_1 \) and \( u_1 \) are of degree 3.

Assume that \( |C_1| = 3 \). Let \( G' \) be the component of \( G - v_1 \) containing \( u_2 \). Then \( G' \) is a tree or a unicyclic graph. If \( G' \) is a tree then by Theorem 1.1, \( \gamma_{oIR}(G') < \frac{2}{5}n' \) and if \( G' \) is a unicyclic graph, then by Theorem 2.3, \( \gamma_{oIR}(G') < \frac{8}{9}n' \). We extend any \( \gamma_{oIR}(G') \)-function to an OIRDF for \( G \) by assigning 1 to \( v_1 \) and the remaining vertices are assigned values as follows. Assume that each vertex in \( \{v_1, v_2, v_3\} \) is a support vertex. Then assign 2 to \( v_3 \), 0 to \( v_2 \) and the leaf-neighbor of \( v_3 \), 1 to each other vertex. If \( G' \) is a tree, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 6 \leq \frac{2}{5}(n - 7) + 6 < \frac{n}{10}n \), and if \( G' \) is a unicyclic graph, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 6 \leq \frac{8}{9}(n - 7) + 6 < \frac{n}{10}n \). Thus, assume without loss of generality, that \( v_3 \) is not a support vertex, (note that the other possibilities are similar). Assume that both \( v_1 \) and \( v_2 \) are support vertices. Then we assign 2 to \( v_3 \), 0 to \( v_2 \) and the support-neighbor of \( v_3 \), and 1 to each other vertex. If \( G' \) is a tree, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 6 \leq \frac{2}{5}(n - 7) + 6 < \frac{n}{10}n \), and if \( G' \) is a unicyclic graph, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 6 \leq \frac{8}{9}(n - 7) + 6 < \frac{n}{10}n \). Thus assume that \( v_2 \) is not a support vertex. If \( \deg(v_1) = 3 \), then we extend any \( \gamma_{oIR}(G') \)-function to an OIRDF for \( G \) by assigning 2 to \( v_3 \), 0 to \( v_2 \) and the support-neighbor of \( v_3 \), and 1 to each other vertex, and as before the result is valid no matter \( G' \) is a tree or a unicyclic graph. Thus, assume that \( \deg(v_1) = 4 \). If \( v_1 \) is a support vertex, then assign 2 to \( v_3 \), 0 to \( v_2 \) and the support-neighbor of \( v_3 \), and 1 to each other vertex. If \( G' \) is a tree, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 7 \leq \frac{2}{5}(n - 8) + 7 < \frac{n}{10}n \), and if \( G' \) is a unicyclic graph, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 7 \leq \frac{8}{9}(n - 8) + 7 < \frac{n}{10}n \). Thus assume that \( v_1 \) is not a support vertex. Then similarly, we find that if \( G' \) is a tree then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 8 \leq \frac{2}{5}(n - 9) + 8 < \frac{n}{10}n \), and if \( G' \) is a unicyclic graph, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 8 \leq \frac{8}{9}(n - 9) + 8 < \frac{n}{10}n \).

Next assume that \( |C_1| > 3 \). Let \( G' \) be the component of \( G - \{v_2, v_3, v_4\} \) containing \( v_1 \). Then \( G' \) is a unicyclic graph, and by Theorem 2.3, \( \gamma_{oIR}(G') < \frac{8}{9}n' \). Observe that \( \deg(v_2) = \deg(v_3) = \deg(v_4) = 3 \). We assign 2 to \( v_2 \), 0 to \( v_3 \) and the neighbor of \( v_2 \) outside \( C_1 \), and 1 to any other vertex. If none of \( v_2, v_3, v_4 \) is a support vertex, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 8 \leq \frac{2}{5}(n - 9) + 8 < \frac{n}{10}n \). If precisely, one vertex in \( \{v_2, v_3, v_4\} \) is a support vertex, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 7 \leq \frac{2}{5}(n - 8) + 7 < \frac{n}{10}n \). If precisely, two vertices in \( \{v_2, v_3, v_4\} \) are support vertices, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 6 \leq \frac{8}{9}(n - 7) + 6 < \frac{n}{10}n \). Finally, if each vertex in \( \{v_2, v_3, v_4\} \) is a support vertex, then \( \gamma_{oIR}(G) \leq \gamma_{oIR}(G') + 5 \leq \frac{8}{9}(n - 6) + 5 < \frac{n}{10}n \). □

Thus assume by Lemma 5.5 that \( n = 10 \). If \( |C_1| \geq 4 \), then assigning 2 to \( v_1 \), 0 to \( v_2, v_k \) and a neighbor of \( v_1 \) outside \( C_1 \), and 1 to each other vertex of \( G \) yields an
OIRDF for $G$, and thus $\gamma_{\text{oiR}}(G) \leq n - 2 < \frac{9}{10}n$. Thus assume that $|C_1| = 3$ and likewise, $|C_2| = 3$.

**Lemma 5.6** If there is a vertex $x$ with three neighbors $x_1, x_2, x_3$ such that $G[\{x, x_1, x_2, x_3\}] = K_{1,3}$, then $\gamma_{\text{oiR}}(G) < \frac{9}{10}n$.

**Proof.** Assume that $x$ is a vertex with three neighbors $x_1, x_2, x_3$ such that $G[\{x, x_1, x_2, x_3\}] = K_{1,3}$. Then assigning 2 to $x$, 0 to $x_1, x_2, x_3$, and 1 to each other vertex of $G$ yields an OIRDF for $G$, and thus $\gamma_{\text{oiR}}(G) \leq n - 2 < \frac{9}{10}n$. \[\square\]

By Lemma 5.6, we assume that $G$ is claw-free. We next continue according to values of $s = d(v_1, u_1)$.

Assume that $s = 0$. By Lemma 5.6, $\deg(v_1) = 4$. If $\deg(v_2) = \deg(v_3) = \deg(u_2) = \deg(u_3) = 3$, then we may assume, without loss of generality, that $u_3$ is not a support vertex, while $v_2, v_3, u_2$ are support vertices. Then $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. Thus, assume, without loss of generality, that $\deg(v_2) = 2$. Then $\deg(v_3) = \deg(u_2) = \deg(u_3) = 3$, since $n = 10$. If $v_3$ is a support vertex, then neither of $u_2$ and $u_3$ is a support vertex, and $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. Thus assume that $v_3$ is not a support vertex. Then one of $u_2$ and $u_3$ is a support vertex and the other one is not a support vertex, and we can see that $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$.

Thus $1 \leq s \leq 5$. Let $P : v_1w_1\ldots w_{s-1}u_1$. If $s = 5$, then by assigning 2 to $w_1, u_2, 1$ to $w_3, v_2, v_3, u_2$, and 0 to each other vertex we obtain that $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. If $s = 4$, then we may assume by Lemma 5.6 that precisely one vertex in $\{v_2, v_3, u_2, u_3\}$ is a support vertex. Assume that $v_2$ is such a vertex. Then by assigning 2 to $w_2, v_2, 0$ to $v_3, w_1, u_3$ and the leaf-neighbor of $v_2$, and 1 to each other vertex we obtain that $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. If $s = 3$, then by Lemma 5.6, either there is precisely one vertex of $\{v_2, v_3, u_2, u_3\}$ that is adjacent to a support vertex of degree two or there are two vertices in $\{v_2, v_3, u_2, u_3\}$ that are support vertices. In each of these possibilities, we can see that $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. If $s = 2$, then by Lemma 5.6, either there are precisely three vertices of $\{v_2, v_3, u_2, u_3\}$ that are support vertices or there are two vertices $x, y$ in $\{v_2, v_3, u_2, u_3\}$ such that $x$ is a support vertex and $y$ is adjacent to a support vertex of degree two. In each of these possibilities, we can see that $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$.

Finally, assume that $s = 1$. If $\deg(v_2) = \deg(v_3) = \deg(u_2) = \deg(u_3) = 3$, then $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. Thus assume, without loss of generality, that $\deg(u_2) = 2$. Assume that $\deg(u_3) = 3$. If $u_3$ is a support vertex, then we may assume that $v_2$ is a support vertex and $v_3$ is a vertex adjacent to a support vertex of degree two. Then $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. Thus, assume that $u_3$ is not a support vertex. Then $u_3$ is adjacent to a support vertex of degree two. Then either both $v_2$ and $v_3$ are support vertices or precisely one of them is adjacent to a support vertex of degree two. Then we observe that $\gamma_{\text{oiR}}(G) = 8 < \frac{9}{10}n$. We thus assume that $\deg(u_3) = 2$. Then each of $v_2$ and $v_3$ is adjacent to a support vertex of degree two. Consequently, $G = H_2$. 

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6 Concluding remarks

By Theorem 2.3, \( \gamma_{oiR}(G) \leq \frac{8n}{9} \) for a unicyclic graph \( G \neq K_3 \), and by Theorem 2.4, \( \gamma_{oiR}(G) \leq \frac{9n}{10} \) if \( G \) has two cycles. It is a good problem to investigate such a bound for cactus graphs. It seems that if a cactus graph \( G \) has three cycles then \( \gamma_{oiR}(G) \leq \frac{10n}{11} \), and if it has four cycles then \( \gamma_{oiR}(G) \leq \frac{11n}{12} \). Figure 2 illustrates two graphs achieving equality of the above proposed bounds. Furthermore, if the above bounds are correct then perhaps \( \gamma_{oiR}(G) \leq \frac{8+4k}{9+4k}n \) if \( G \) is a cactus graph with \( k \) cycles, but this earlier bound does not seem to be sharp. We propose these problems for researchers.

**Conjecture 6.1** If \( G \neq K_3 \) is a cactus graph of order \( n \), then \( \gamma_{oiR}(G) \leq \frac{11n}{12} \), with equality if and only if \( G = H_4 \), where \( H_4 \) is the graph depicted in Figure 2.

![Figure 2](image)

**Figure 2.** Cactus graphs with outer-independent Roman domination number \( \frac{10n}{11} \) and \( \frac{11n}{12} \). The graph \( H_3 \) has outer-independent Roman domination number 10 and the graph \( H_4 \) has outer-independent Roman domination number 11.

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References


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