AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **88(1)** (2024), Pages 52–76

Quest for graphs of Frank number 3

János Barát*

Alfréd Rényi Institute of Mathematics
University of Pannonia, Department of Mathematics
8200 Veszprém, Egyetem utca 10., Hungary
barat@renyi.hu

Zoltán L. Blázsik†

Alfréd Rényi Institute of Mathematics
MTA–ELTE Geometric and Algebraic Combinatorics Research Group
SZTE Bolyai Institute
6720 Szeged, Aradi vértanúk tere 1, Hungary
blazsik@renyi.hu

Abstract

In an orientation O of a graph G, an edge e is deletable if $O - e$ is strongly connected. For a 3-edge-connected graph G, Hörsch and Szigeti defined the Frank number as the minimum k for which G admits k orientations such that every edge e of G is deletable in at least one of the k orientations. They conjectured the Frank number is at most 3 for every 3-edge-connected graph G. They proved the Petersen graph has Frank number 3, but this was the only example with this property. We show an infinite class of graphs having Frank number 3. Hörsch and Szigeti showed every 3-edge-colorable 3-edge-connected graph has Frank number at most 3. It is tempting to consider non-3-edge-colorable graphs as candidates for having Frank number greater than 2. Snarks are sometimes a good source of finding critical examples or counterexamples. One might suspect various snarks should have Frank number 3. However, we prove several candidate infinite classes of snarks have Frank number 2. This holds also for the generalized Petersen Graphs $GP(2s + 1, s)$. We formulate numerous conjectures inspired by our experience.

* Research supported by ERC Advanced Grant “GeoScape” and the National Research, Development and Innovation Office, grant K-131529.

† The author was supported by the ÚNKP-22-4-SZTE-480 New National Excellence Program of the Ministry of Human Capacities. The research was supported by the Hungarian National Research, Development and Innovation Office, OTKA grant no. SNN 132625.

ISSN: 2202-3518

1 Introduction

The graphs in this paper are finite and without loops or multiple edges. We recommend the book by Bondy and Murty [1] for the concepts used here.

A graph G is defined by its vertex set V and edge set E. An *orientation* of G is a directed graph $D = (V, A)$ such that each edge $uv \in E$ is replaced by exactly one of the arcs (u, v) or (v, u). In the entire paper, we use the concatenation $u_1u_2 \dots u_k$ for undirected paths and cycles, and use brackets and commas $(u_1, u_2, \dots, u_k)$ for directed paths and circuits from u_1 to u_k. A *circuit* of length k, denoted by $(v_1, v_2, \dots, v_k)$, is an orientation of a k-cycle, in which every vertex has in-degree 1. A graph is *cubic* if every vertex has degree 3. A *chord* of a cycle or circuit is an edge or arc connecting two non-consecutive vertices. A graph H is a *truncation* of a cubic graph G at vertex v, if we get H from G by subdividing each edge vx_i incident with v by a new vertex v_i for $i \in \{1, 2, 3\}$, deleting v and adding a triangle on v_1, v_2, v_3, thus H is also cubic. We also use the term truncation without specifying v, meaning there exists a v as above. The term truncation was first used by Kepler in constructing Platonic solids. In graph theory, probably Tutte [8] used the term truncation in connection with Hamiltonian cycles of polyhedra.

A graph is *k-edge-connected* if the removal of any $k-1$ edges leaves a connected graph. A digraph D is *strongly connected* if D contains a directed (x, y)-path for any two vertices $x, y \in V(D)$. An orientation of G is *k-arc-connected* if the removal of any $k-1$ arcs leaves a strongly connected digraph. The following theorems are fundamental results in the theory of directed graphs [7, 6].

Theorem 1.1 (Robbins)**.** *A graph has a strongly connected orientation if and only if it is* 2*-edge-connected.*

Theorem 1.2 (Nash-Williams)**.** *A graph has a k-arc-connected orientation if and only if it is $2k$-edge-connected.*

This theorem has the following consequence: If we fix a 2-arc connected orientation of a 4-edge-connected graph, then any arc can be removed and the remaining digraph is still strongly connected. This situation changes for 3-edge-connected graphs and their orientations. This motivated András Frank to raise some questions on 3-edge-connected graphs and their orientations. These concepts and Frank's question appeared first in the paper by Hörsch and Szigeti [4]. In an orientation O of G, the edge e is *deletable* if $O - e$ is strongly connected. For a 3-edge-connected graph G, Hörsch and Szigeti defined the *Frank number* $F(G)$ as the minimum k for which G admits k orientations such that every edge e of G is deletable in at least one of the k orientations. In a more general setting, DeVos, Johnson and Seymour [3] proved that any 3-edge-connected graph G satisfies $F(G) \leq 9$. Hörsch and Szigeti [4] showed that any 3-edge-connected graph G satisfies $F(G) \leq 7$. They also showed any 3-edge-colorable G has Frank number at most 3, and the Petersen graph has Frank number 3. These results lead to the question whether there are any graphs with Frank number greater than 2 besides the Petersen graph.

In Section 3, we consider a few infinite families of 3-edge-connected graphs and determine their Frank number, which turns out to be 2. We introduce a useful tool, that helps us to check the Frank number.

In Section 4, we introduce *local cubic modifications*, that generalize truncation to vertices of larger degree. More importantly, the perfect matching between the new vertices of the cycle and the neighbors of v can be chosen arbitrarily. Although the truncation of a 3-edge-connected cubic graph always remains 3-edge-connected, it is not necessarily true for local cubic modifications in general. However, we prove that even for a vertex of degree larger than 3, the perfect matching can always be chosen such that the resulting graph remains 3-edge-connected.

As an important result, we construct infinitely many graphs with Frank number 3 in Section 4. We show that performing a local cubic modification on a vertex of degree 3 preserves the Frank number. Using this operation on the Petersen graph iteratively, we can prove our main result.

Theorem 1.3. *There are infinitely many* 3*-edge-connected cubic graphs* G *such that* $F(G)$=3*.*

By investigating the properties of local cubic modifications, we are also able to show the following. If one seeks other graphs with higher Frank number, then one can restrict the search for the class of cubic, triangle-free graphs. We describe this in detail in Section 4.

The results of Hörsch and Szigeti made us think probably some other non-3-edge-colorable graphs might have Frank number larger than 2. Snarks are 4-edge-chromatic bridgeless cubic graphs with girth at least 5. The Petersen graph is the smallest snark. The next smallest are the Blanuša snarks. We show, they have Frank number 2. We also study an infinite snark family. In Section 5, we show that each flower snark has Frank number 2.

In Section 6, we prove exhaustively that indeed the Petersen graph is the only cubic 3-edge-connected graph on at most 10 vertices having Frank number 3. Using the observations from the previous sections the proof is significantly shorter than the known proofs.

Some crucial properties of the Petersen graph can be generalized to the so called generalized Petersen graphs $GP(2s{+}1, s)$. One might hope to find a graph with Frank number 3 among them. However, we prove in Section 7 that $F(GP(2s+1, s)) = 2$ for $s \geq 3$.

2 Preliminaries

If O is an orientation of G, then let $-O$ be the orientation which we get by reversing every arc in O.

Fact 2.1. *The set of deletable edges is the same for* O *and* $-O$*.*

We routinely have to check whether an edge is deletable. The following observation shows one way to do that.

Proposition 2.2. *Let G be a 2-edge-connected graph, and $e = uv \in E(G)$. Suppose that O is a strongly connected orientation of G such that the arc corresponding to e goes from u to v. The orientation $O - e$, which we get by deleting the arc (u, v) from O is strongly connected if and only if there exists a directed path in $O - e$ from u to v.*

Proof. If there is no (u, v)-path in $O - e$, then $O - e$ is not strongly connected by definition. If there is a (u, v)-path P in $O - e$, then in any (x, y)-path of O, which uses the arc (u, v), we replace (u, v) by P. Since O was strongly connected, we now find an (x, y)-walk in $O - e$ for any pair x and y. Therefore $O - e$ is strongly connected. □

Since a strongly connected directed graph does not contain any source or sink vertices, a vertex of total degree 3 can have out-degree 2 or 1. We call such a vertex red or green, respectively. The following observation gives a necessary but not sufficient condition on the deletability of an arc in a cubic graph.

Fact 2.3. *Let G be a cubic graph and O a strongly connected orientation of G. If an arc $e = (u, v)$ is deletable, then u is red and v is green.*

By Proposition 2.2, the deletability of the arc (u, v) is equivalent to the existence of a directed path from u to v in $O - e$. Therefore u must have outdegree exactly 2, and v must have indegree exactly 2. However, the example in Figure 1 shows that these degree conditions are insufficient. If there exists an edge cut containing e such that every arc except e is going in the same direction, then after deleting e, this edge cut becomes a directed cut, hence no directed (u, v)-path exists anymore regardless of the in- and outdegree of u and v.

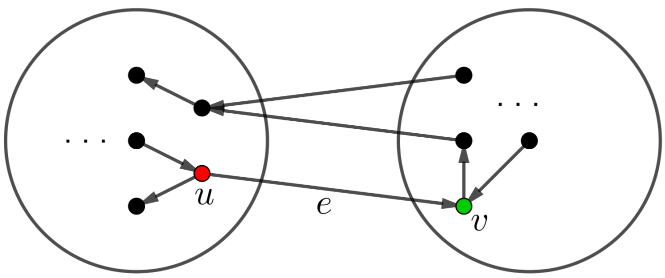


Figure 1: The arc $e = (u, v)$ is not deletable despite the fact that u is red, and v is green

We use the following observation repeatedly. If O is a strongly connected orientation of a 2-edge-connected graph and C is a circuit of O, then every chord of C is deletable regardless of its orientation. Thus if O contains a Hamiltonian circuit C, then every arc outside of C is deletable.

3 Three elementary classes

In this section, we write subscripts modulo n or k depending on the context. We use the following three results later in Section 6. Wheel graphs are a family of 3-edge-connected graphs. For a positive integer $n \geq 3$, the wheel W_n consists of a hub vertex v_0 and n other vertices forming a cycle such that v_0 is adjacent to all other vertices forming the spoke edges. Notice that W_3 is the complete graph on 4 vertices.

Lemma 3.1. *For every positive integer $n \geq 3$, the wheel W_n has Frank number 2.*

Proof. First let n be even. We give the first orientation O_1 of the edges of W_n as follows. We orient the edges of the outer n-cycle to get a circuit $(v_1, v_2, \ldots, v_n)$. We alternately orient the spoke edges: $(v_1, v_0), (v_0, v_2), \ldots, (v_0, v_n)$. For every odd i, the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the 2-path (v_i, v_0, v_{i+1}). Also for every odd i, the arc (v_i, v_0) is deletable by Proposition 2.2 using the 3-path $(v_i, v_{i+1}, v_{i+2}, v_0)$.

Now we give the second orientation O_2. We orient the edges of the outer n-cycle to get a circuit $(v_1, v_2, \ldots, v_n)$. We alternately orient the spoke edges: $(v_n, v_0), (v_0, v_1)$, $(v_2, v_0), \ldots, (v_0, v_{n-1})$. For every even i, the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the 2-path (v_i, v_0, v_{i+1}). Also for every even i, the arc (v_i, v_0) is deletable by Proposition 2.2 using the 3-path $(v_i, v_{i+1}, v_{i+2}, v_0)$. These two orientations of W_n show that the Frank number is 2.

Let n be odd now. We give the first orientation O_1 of the edges of W_n as follows. We orient the edges to get a directed path $(v_1, v_2, \ldots, v_n)$. However, we orient the last edge from v_1 to v_n. We orient the spoke edges as follows: (v_0, v_1), (v_0, v_i) if i is even and (v_i, v_0) if i is odd, except $i = 1$. Now for every odd $i \in \{1, \ldots, n-2\}$, the arcs (v_i, v_{i+1}) are deletable by Proposition 2.2 using the 2-path (v_i, v_0, v_{i+1}), and in the case $i = 1$ the 3-path (v_1, v_n, v_0, v_2). The arc (v_1, v_n) is deletable by Proposition 2.2 using the path $(v_1, v_2, \ldots, v_n)$. For every odd i larger than 1 smaller than n, the arcs (v_i, v_0) are deletable by Proposition 2.2 using the path $(v_i, v_{i+1}, v_{i+2}, v_0)$. For every even i, the arc (v_0, v_i) is deletable by Proposition 2.2 using the path $(v_0, v_{i-2}, v_{i-1}, v_i)$ for $i > 2$ and (v_0, v_1, v_2) for $i = 2$. Thus every spoke edge is deletable except (v_n, v_0) and (v_0, v_1).

In the second orientation, we orient the outer cycle $(v_n, v_{n-1}, \ldots, v_1)$ to get a circuit. We orient the spoke edges as follows: (v_i, v_0) if i is odd and (v_i, v_0) if i is even.

Now, for every even i, the arcs (v_{i+1}, v_i) are deletable by Proposition 2.2 using the path (v_{i+1}, v_0, v_i). Also (v_1, v_0) is deletable using the path (v_1, v_n, v_0) and (v_n, v_0) is deletable using the path $(v_n, \ldots, v_1, v_0)$. □

For an even integer $n \geq 4$, let the Möbius ladder M_n be defined as follows. Let $v_1 v_2 \ldots v_n$ be a cycle and we connect each opposite pair, these are edges of form $v_i v_{i+n/2}$.

Lemma 3.2. *For every positive even integer $n \geq 4$, the graph M_n has Frank number 2.*

Proof. First let $n/2$ be an odd number. We give the first orientation O_1 of the edges as follows. We orient the cycle edges consecutively (v_i, v_{i+1}) to get a circuit. This implies every diagonal edge is deletable independent of its orientation. For every odd i, we orient the diagonals as $(v_i, v_{i+n/2})$. Thereby the diagonal edges alternate in direction (this uses the oddness of $n/2$). Now, for every odd i the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the path $(v_i, v_{i+n/2}, v_{i+1+n/2}, v_{i+1})$. We construct the second orientation O_2 from O_1 by reversing the diagonals. Thereby for every even i the arc (v_i, v_{i+1}) becomes deletable. Hence every edge is deletable in at least one of the two orientations.

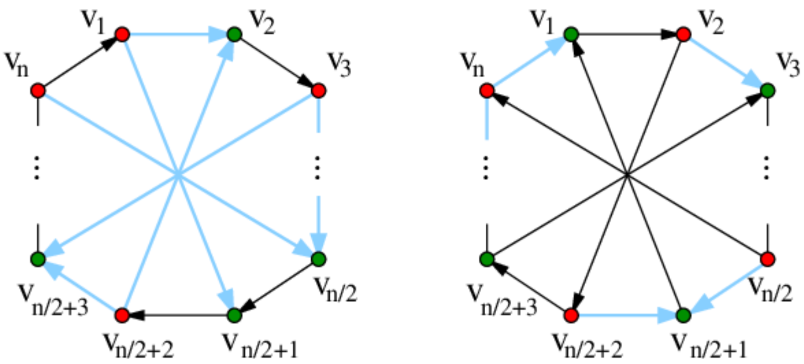


Figure 2: Two appropriate orientations of a Möbius ladder for $n/2$ even, where the blue arcs are deletable.

Let $n/2$ be an even number now. In the first orientation, we orient the cycle edges consecutively (v_i, v_{i+1}) to get a circuit. Thereby, all diagonal edges are deletable. For every odd i smaller than $n/2$, we orient the diagonals as $(v_i, v_{i+n/2})$. For every even i at most $n/2$, we orient the diagonals as $(v_{i+n/2}, v_i)$, see Figure 2. Now, for every odd i smaller than $n/2$, the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the path $(v_i, v_{i+n/2}, v_{i+1+n/2}, v_{i+1})$. For every even i greater than $n/2$, the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the path $(v_i, v_{i+n/2}, v_{i+1+n/2}, v_{i+1})$. Both arcs of the outer circuit incident with v_n and $v_{n/2+1}$ are non-deletable.

In the second orientation, we create the following circuit C, that contains every vertex but $v_{n/2+1}$: $(v_1, v_2, v_{n/2+2}, v_{n/2+3}, v_3, v_4, v_{n/2+4}, v_{n/2+5}, \ldots, v_{n/2-1}, v_{n/2}, v_n)$. We add $(v_{n/2}, v_{n/2+1})$, $(v_{n/2+2}, v_{n/2+1})$ and $(v_{n/2+1}, v_1)$ thereby making the orientation strongly connected, plus (v_n, v_{n-1}). The remaining edges of M_n have arbitrary orientation. Now, every second edge $v_2v_3, \ldots, v_{n/2-2}v_{n/2-1}$ and $v_{n/2+3}v_{n/2+4}, \ldots, v_{n-1}v_n$ are deletable independent of their orientation, since these are chords of a circuit. The arc (v_n, v_1) is deletable by Proposition 2.2 using the path $(v_n, v_{n-1}, v_{n/2-1}, v_{n/2}, i$ $v_{n/2+1}, v_1)$. The arc $(v_{n/2}, v_{n/2+1})$ is deletable by Proposition 2.2 using the path $(v_{n/2}, v_n, v_1, v_2, v_{n/2+2}, v_{n/2+1})$. The arc $(v_{n/2+2}, v_{n/2+1})$ is deletable by Proposition 2.2 using the directed path $(v_{n/2+2}, \ldots, v_{n/2})$ in C plus $(v_{n/2}, v_{n/2+1})$. We are done, since every edge is deletable in at least one of the two given orientations of M_n. □

Next, we consider the prisms, which are almost identical to Möbius ladders.

Lemma 3.3. *For every k, the prism $P_k = C_k \times K_2$ has Frank number 2, where $k \geq 3$.*

Proof. We denote the outer cycle by $v_1v_2\dots v_k$ and the inner cycle by $u_1u_2\dots u_k$ and the spoke edges by u_iv_i for every $1 \le i \le k$. Let k be even. We give the first orientation of the edges as follows. We orient the outer cycle edges consecutively (v_i, v_{i+1}) to get a circuit. We orient the inner cycle edges consecutively backwards (u_{i+1}, u_i) to get a circuit. We orient the spoke edges alternately. That is, every vertex v_i with odd index is the tail of a spoke arc, and every even-indexed vertex v_i is the head. Now for every odd i, the arc (v_i, u_i) is deletable by Proposition 2.2 using the path $(v_i, v_{i+1}, v_{i+2}, u_{i+2}, u_{i+1}, u_i)$. For every even i, the arc (u_i, v_i) is deletable by Proposition 2.2 using the path $(u_i, u_{i-1}, u_{i-2}, v_{i-2}, v_{i-1}, v_i)$. Hence every spoke edge is deletable. For every odd i, the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the path $(v_i, u_i, u_{i-1}, \dots, u_{i+1}, v_{i+1})$. For every even i, the arc (u_i, u_{i-1}) is deletable by Proposition 2.2 using the path $(u_i, v_i, v_{i+1}, \dots, v_{i-1}, u_{i-1})$. Since these arcs appear alternately, we get the second orientation by reversing the spokes (Visually rotating every arc by one.). We deduce that every edge is deletable in at least one of the two orientations.

Let k be odd. The case $k = 3$ can be checked individually, see Figure 3.

Figure 3: Two appropriate orientations of the 3-prism, where the blue arcs are deletable

Let $k \ge 5$. We give the first orientation of the edges as follows. For every i, we set (v_i, v_{i+1}) to get a circuit. For every i, we set (u_{i+1}, u_i) to get another circuit. We orient the spoke edges as follows. For every odd i we set (v_i, u_i). For every even i we set (u_i, v_i). Now every spoke edge is deletable by Proposition 2.2 as in the case k even. For every odd i smaller than k, the arc (v_i, v_{i+1}) is deletable by Proposition 2.2 using the path $(v_i, u_i, u_{i-1}, \dots, u_{i+1}, v_{i+1})$. Similarly for every odd i smaller than k, the arc (u_{i+1}, u_i) is deletable.

In the second orientation, for every i, we set (v_{i+1}, v_i) except (v_k, v_1). For every i, we set (u_i, u_{i+1}) except (u_1, u_k). For every odd i smaller than k, we set (v_i, u_i) and put (u_k, v_k). For every even i, we set (u_i, v_i). In particular, there are two circuits of length $k + 2$, which we use in the next part of the proof: $C = (v_k, v_{k-1}, \dots, v_1, u_1, u_k)$ and $C' = (u_1, u_2, \dots, u_k, v_k, v_1)$. We use Proposition 2.2 again to show deletable edges. The directed paths $(v_k, v_{k-1}, \dots, v_1)$ and $(u_1, u_2, \dots, u_k)$ guarantee that (v_k, v_1) and (u_1, u_k) are deletable. For every odd i between 3 and k, the arc (v_i, v_{i-1}) is deletable by Proposition 2.2 using the path $(v_i, u_i, P'_i, u_{i-1}, v_{i-1})$, where P'_i is the subpath of C' from u_i to u_{i-1}. Similarly the arc (u_{i-1}, u_i) is deletable by Proposition 2.2 using the path $(u_{i-1}, v_{i-1}, P_i, v_i, u_i)$, where P_i is the subpath of C from v_{i-1} to v_i. Hence every edge is deletable in at least one of the two orientations. □

4 Local cubic modification

Let us introduce the following local operation on a graph G of minimum degree at least 3. For $d \geq 3$, let v be a vertex of degree d, and let the neighbors of v be $x_1, \ldots, x_d$. We remove v and add new vertices $v_1, \ldots, v_d$. We add a cycle $C_v = v_1 v_2 \ldots v_d$. We replace each edge vx_i by an edge $v_j x_i$ (see Figure 4) so that each of the new vertices has exactly one neighbor from $\{x_1, x_2, \ldots, x_d\}$. The resulting graph G_v is a *local cubic modification* of G at v. Let us remark there may be several local cubic modifications at the same vertex v. It depends on the chosen perfect matching M between $\{x_1, x_2, \ldots, x_d\}$ and $\{v_1, v_2, \ldots, v_d\}$. Denote by $G_v(M)$ the local cubic modification of G at v with the chosen perfect matching M. Note that truncation is a well-known subcase of local cubic modifications.

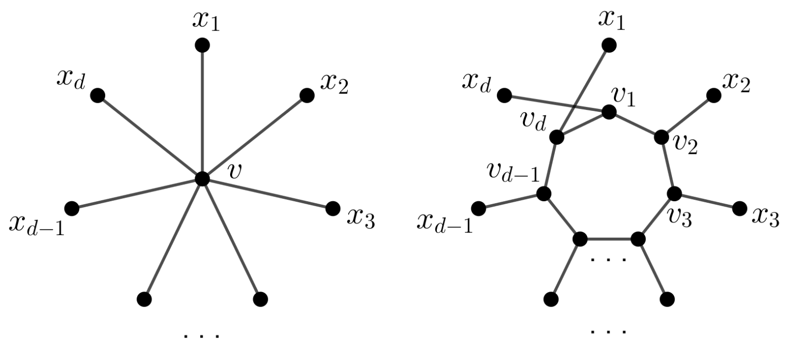


Figure 4: A local cubic modification at v

Let us emphasize that at this point it may happen that after performing a local cubic modification the edge-connectivity decreases (see Figure 5).

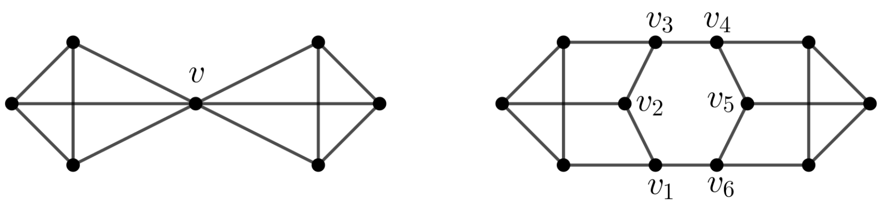


Figure 5: The edge-connectivity may decrease by performing a local cubic modification at v

In this paper, we focus on 3-edge-connected graphs. We show for any 3-edge-connected graph, there exists a local cubic modification at any vertex, which is still 3-edge-connected.

Lemma 4.1. *For any* 3*-edge-connected graph* G *and for an arbitrary vertex* $v \in V(G)$ *of degree* d *there exists a perfect matching* M *such that* $G_v(M)$ *remains* 3*-edge-connected.*

Proof. Assume first, that v is not a cut vertex of G. Suppose to the contrary there is an edge cut of size 2 in $G_v(M)$ for every choice of M. For an arbitrary perfect matching M, let $C(M) = \{e, f\}$ be an edge cut of size 2, and $A(M)$ and $B(M)$ denote different non-empty connected components of $G_v(M) - C(M)$.

Suppose $A(M) \subseteq \{v_1, v_2, \dots, v_d\}$ and consists of $\{v_{j_1}, v_{j_2}, \dots v_{j_\ell}\}$. Now $|A(M)| \leq 2$ since e and f must contain the edges of M incident with v_{j_i} for all $i \in \{1, 2, \dots, \ell\}$. On the other hand, e and f must belong to the new cycle C_v otherwise all the vertices $\{v_1, v_2, \dots, v_d\}$ would be in the same connected component of $G_v(M) - C(M)$. This cannot happen since in that case $|A(M)| = d \geq 3$, which is a contradiction. By symmetry, the same argument also works for $B(M)$.

Hence we can assume that both $A(M)$ and $B(M)$ have a vertex outside of $\{v_1, v_2, \dots, v_d\}$. Choose a vertex of both $A(M) \setminus \{v_1, v_2, \dots, v_d\}$ and $B(M) \setminus \{v_1, v_2, \dots, v_d\}$, y and z say. Since G is 3-edge-connected, there are at least 3 edge-disjoint paths between y and z in G by Menger's theorem. Therefore there exists at least one path P^{yz} disjoint from $\{e, f\}$ in G. If P^{yz} can be chosen such that it does not go through v, then the same path exists in $G_v(M)$, a contradiction. On the other hand, if all possible P^{yz} paths in $G - \{e, f\}$ passes through v, then we can complete any such P^{yz} to P_v^{yz} in $G_v(M)$ by connecting the corresponding vertices v_i and v_j (i and j are not necessarily distinct) using the cycle C_v. This completion can be done unless both e and f are edges of C_v. But in that case, v would be a cut vertex in G, which is a contradiction.

If v is a cut vertex of G, then denote the non-empty connected components of $G-v$ by $K_1, K_2, \dots, K_k$, where $k \geq 2$. Since G is 3-edge-connected, $|K_i \cap N_G(v)| \geq 3$ for any $i \in \{1, 2, \dots, k\}$. We interpret the choice of M as an assignment of the vertices of C_v to the corresponding connected components of $G - v$. Imagine this assignment in the following way. For every $i \in \{1, 2, \dots, k\}$, assign all vertices of K_i consecutively on the cycle C_v, and then swap those k pair of consecutive vertices which correspond to different components. The assignment is illustrated in Figure 6.

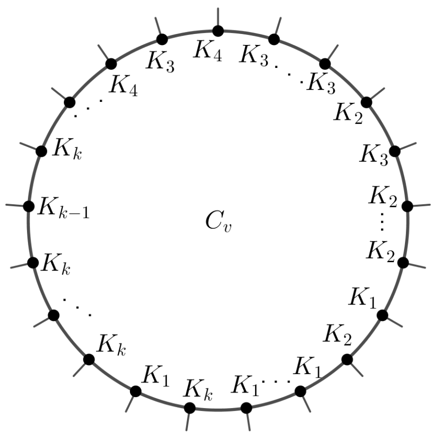


Figure 6: This assignment shows that M can be chosen such that $G_v(M)$ remains 3-edge-connected even if v is a cut vertex.

We claim that for such a perfect matching M, the local cubic modification $G_v(M)$ is 3-edge-connected. Suppose to the contrary there is an edge cut $C = \{e, f\}$ of size 2 in $G_v(M)$. Observe that both edges of the edge cut C must belong to the cycle

C_v. Otherwise the vertices $v_1, v_2, \dots, v_d$ would be in the same connected component. Thus C leads to an edge-cut of size at most 2 in G, a contradiction.

Assume both e and f belong to C_v; then deleting C from C_v results in two arcs. If there exists an $i \in \{1, 2, \dots, k\}$ such that the vertices of $K_i \cap N_G(v)$ are adjacent to at least one vertex from both arcs then again the vertices $v_1, v_2, \dots, v_d$ would be in the same connected component which leads to a contradiction. Hence $G_v(M) - C$ is disconnected if and only if M is chosen such that for all $i \in \{1, 2, \dots, k\}$ the vertices of $K_i \cap N_G(v)$ are adjacent to vertices from the same arc of the two arcs of $C_v - C$. Due to the carefully chosen assignment shown in Figure 6, one can check that no matter how we select two edges $\{e', f'\}$ of C_v there exists an $i \in \{1, 2, \dots, k\}$ such that the vertices assigned to K_i intersect both arcs of $C_v - \{e', f'\}$. □

Let us call a local cubic modification $G_v(M)$ *proper* if it is 3-edge-connected. In the rest of the paper, we assume that G_v always denotes a proper local cubic modification of the 3-edge-connected graph G at vertex v.

Let us recall that Hörsch and Szigeti [4] introduced the notion of *cubic extensions* of a graph with minimum degree at least 3 in Subsection 2.3. It is a global modificationwhich replaces every vertex v of degree at least 4 with a cycle of size $\deg(v)$, leave the vertices of degree 3 intact, and substitute every edge with an edge between the corresponding objects in such a way that this not necessarily unique graph is cubic.

However, it is true that every cubic extension of a graph can be realized as a series of proper local cubic modifications. In the other direction, if we perform a series of proper local cubic modifications of a graph at all vertices of degree at least 4, then we get a cubic extension. Consequently, Lemma 4.1 implies that one can find a 3-edge-connected cubic extension of a 3-edge-connected graph even if there are cut vertices.

The following general observation plays a key role in the next proofs, when applied to proper local cubic modifications.

Fact 4.2. *Let G_v be a proper local cubic modification of G at v, and an orientation O_v be given such that there exists a directed (y, z)-path P_v^{yz} in O_v for $\{y, z\} \nsubseteq \{v_1, v_2, \dots, v_d\}$. Now a directed (y, z)-path also exists for the inherited orientation O of G if $|\{y, z\} \cap \{v_1, \dots, v_d\}| = 0$ and a directed (y, v)-path also exists in O if $z \in \{v_1, \dots, v_d\}$.*

Now we are ready to show that a proper local cubic modification cannot decrease the Frank number. Moreover, if the vertex v has degree 3, then it cannot increase either. Hence in that case, the Frank number remains the same.

Lemma 4.3. *Let G be a 3-edge-connected graph. If G_v is a proper local cubic modification of G at v, then $F(G_v) \geq F(G)$.*

Proof. Suppose to the contrary that $F(G_v) = k < F(G)$ witnessed by the strongly connected orientations $O_1^v, \dots, O_k^v$. Let $O_1, \dots, O_k$ be the orientations of Gwhich coincide with $O_1^v, \dots, O_k^v$ on identical edges. Also let the direction of $v_j x_i$ be copied to vx_i in each orientation. Since each O_j^v was strongly connected, for any pair of vertices y, z there exists a directed path between them in both directions. By Fact 4.2, we can deduce that O_j also has the same property hence it is strongly connected.

We claim each edge $e = yz$ of G is deletable in at least one orientation. Let O_j^v be the orientation of G_v, where e with the appropriate orientation (say (y, z)) was deletable. We know that O_j^v is strongly connected and contains a directed (y, z)-path P_{yz}^v in $O_j^v - \{e\}$. Consequently, similarly to the proof of Fact 4.2, $O_j - \{e\}$ contains a directed (y, z)-path P_{yz} where P_{yz} can be obtained from P_{yz}^v by contracting the subgraph between the first and last vertex of $C_v \cap P_{yz}^v$. Therefore e is deletable in O_j by Proposition 2.2. □

Corollary 4.4. *Let G be a 3-edge-connected graph. Every 3-edge-connected cubic extension H of G satisfies $F(H) \geq F(G)$.*

By Lemma 4.3, we can create an infinite family $\mathcal{G}$ of cubic graphs with $F(G) \geq 3$ for any $G \in \mathcal{G}$ starting from the Petersen graph in the following way. Hörsch and Szigeti [4] showed the Petersen graph has Frank number 3. Pick a vertex v of the Petersen graph, and consider a proper local cubic modification G_v of G at v. Since the Petersen graph is cubic and 3-edge-connected and G_v is 3-edge-connected as well, hence by Lemma 4.3, we get $F(G_v) \geq F(G)$. After iterating this proper local cubic modification procedure with an arbitrary vertex of the always cubic current graph, the Frank number never decreases. Thus we created an infinite family of 3-edge-connected graphs with Frank number at least 3.

In Theorem 1.3, we claimed the existence of an infinite family of cubic graphs with Frank number equal to 3. So far we have seen that the Frank number cannot decrease performing a proper local cubic modification at an arbitrary vertex v. In the next lemma, we show that the Frank number cannot increase if $\deg(v) = 3$.

Lemma 4.5. *Let G be a 3-edge-connected graph and v a vertex of degree 3. If G_v is a proper local cubic modification of G at v, then $F(G_v) \leq F(G)$.*

Proof. Suppose the orientations $\mathcal{O} = \{O_1, O_2, \ldots, O_k\}$ are the witnesses of $F(G) = k$. We create k orientations $\mathcal{O}^v = \{O_1^v, O_2^v, \ldots, O_k^v\}$ of G_v to prove $F(G_v) \leq k$. Let us focus on the modified part of G_v, we just copy the orientations from the corresponding O_i outside of the modified part. Since every O_i is a strong orientation, the 3-edge-cut formed by the edges $\{av, bv, cv\}$ cannot be a directed cut.

By Fact 2.1, we might assume that in every orientation O_i, exactly two edges leave v. For convenience, instead of referring to a, b, c as the concrete neighbors of v, let us permute their roles. We may assume that a denotes the tail of the unique arc entering v. In Figure 7, we introduce the four orientations we use later in this proof. Note that the first two orientations become the same if we interchange the roles of b and c, and so do the last two orientations. Hence there are essentially two types of extensionswhich we use on the modified part of G_v.

Firstly, observe that no matter which extensions we use from Figure 7, the orientation O_i^v we get is also strongly connected. Indeed, we can enter the triangle v_a, v_b, v_c only from a and we can leave in both directions through b or c, hence every directed path of O_i can be extended even if it goes through v in G. Moreover, there exists a directed path between any pair of new vertices in O_i^v.

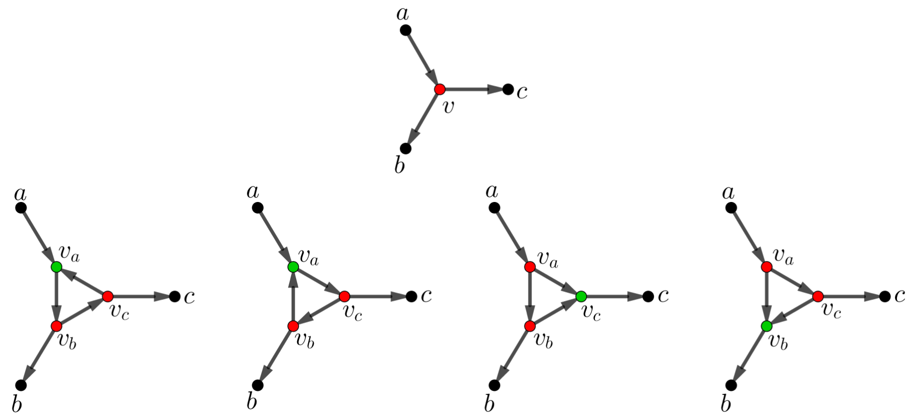


Figure 7: The four orientations we use on the new arcs (essentially two different types)

An arc of O_i not incident with v is deletable if and only if the same arc is deletable in O_i^v. By Proposition 2.2, it is enough to show a directed path between its endpoints in the modified graph as well. As we discussed in the previous paragraph, this can be done and it does not depend on the choice of the orientation of the triangle at the modified vertex v as long as we use the four orientations above. Therefore for every edge not incident with v, there exists an orientation O_i^v of G_v so that the corresponding arc is deletable in O_i^v.

Choose a smallest subset $\mathcal{S} = \{O_{j_1}, O_{j_2}, \ldots, O_{j_\ell}\}$ of $\mathcal{O}$ such that all of the edges incident with v is deletable in at least one of the orientations in $\mathcal{S}$. Here $1 < \ell \leq 3$ holds.

If $|\mathcal{S}| = 2$, then in at least one of these orientations both arcs leaving v are deletable and in the other orientation the third edge incident with v is not just outgoing but also deletable. In Figure 8, we show how the orientations $\{O_{j_1}^v, O_{j_2}^v\}$ look like at the modified vertex v (remember that the role of b and c are interchangeable). Notice that the blue color indicate which arcs are deletable.

Indeed, the arcs of type (v_x, x) are deletable in $O_{j_i}^v$ if and only if (v, x) was deletable in O_{j_i}. The arcs inside the triangle of type (v_x, v_y) are deletable either trivially or because of the fact that O_{j_i} is strongly connected.

If $|\mathcal{S}| = 3$, then for each of the edges incident with v, there is a unique orientation of $\mathcal{S}$ so that the corresponding arc is deletable. By choosing the appropriate roles for the three neighbors we can use one of the last two orientations in Figure 7 that results in three orientations for which every arc of the triangle is also deletable in at least one of them. Indeed, the arc opposite to the deletable one which leaves v is always deletable by Proposition 2.2 since there is a directed path within the triangle. Naturally, we can use any of the orientations described in Figure 7 in any of those orientations of $\mathcal{O}$ which have not yet been touched. Hence we proved $F(G_v) \leq F(G)$. □

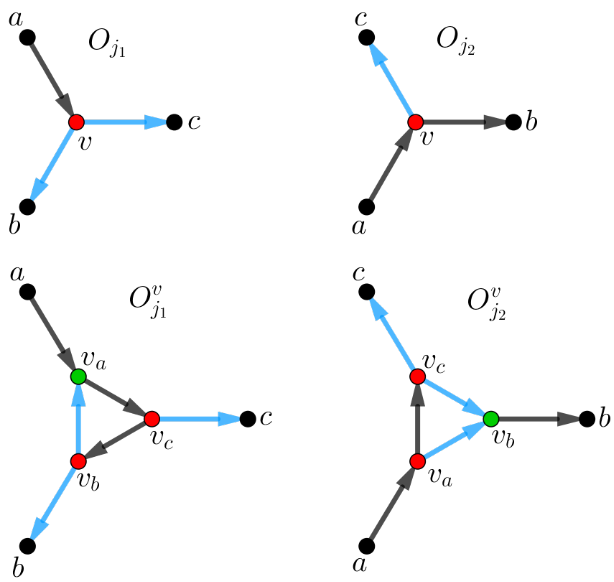


Figure 8: The orientations $\{O^v_{j_1}, O^v_{j_2}\}$, if $|\mathcal{S}| = 2$ and the blue arcs are deletable.

Corollary 4.6. *If a 3-edge-connected graph G contains at least one vertex of degree 3, then there is an infinite family $\mathcal{H}$ of 3-edge-connected graphs such that $F(G) = F(H)$ for every $H \in \mathcal{H}$.*

Proof. Lemma 4.3 and Lemma 4.5 together imply the following: if a 3-edge-connected graph G contains at least one vertex of degree 3, then by successively performing a proper local cubic modification at vertices of degree 3, we get a family of graphs with the same Frank number as G. Notice that in each step, the newly introduced vertices have degree 3. □

Thus if we start with the Petersen graph, we can build a family of graphs with Frank number exactly 3, concluding the proof of Theorem 1.3.

However, if a graph H is 3-edge-connected and distinct from K_4 but contains a triangle T, then we can contract the vertices of T into a new vertex v_T (or in other words identify these vertices) such that v_T is adjacent to the other neighbors of the three vertices of T, thus the resulting graph H/T is simple (since H was cubic) and cubic. What can we say about the relation between the Frank number of H and H/T?

Since H is a proper local cubic modification of H/T at v_T, we get $F(H) \geq F(H/T)$ by Lemma 4.3. On the other hand, Lemma 4.5 yields that $F(H/T) \leq F(H)$ since v_T is a vertex of degree 3 in H/T. Hence $F(H) = F(H/T)$. Consequently, we can contract triangles starting from H until the resulting graph H^* is either triangle-free or $H^* \simeq K_4$ while the Frank number remains the same. We know that $F(K_4) = 2$, and $F(H^*) \geq 2$ if H^* is a 3-edge-connected cubic triangle-free graph.

Theorem 4.7. *Let G be a 3-edge-connected graph such that $F(G) \geq 3$. There exists a cubic, triangle-free, 3-edge-connected graph H^* such that $F(H^*) \geq F(G) \geq 3$.*

Proof. By Corollary 4.4, we can consider the cubic extension H of G for which $F(H) \geq F(G)$. After contracting triangles, the resulting graph H^* is either triangle-free or it is K_4 while $F(H^*) = F(H)$ by the previous paragraph. Notice that contracting triangles in a 3-edge-connected cubic graph does not create parallel edges. Since $F(G) \geq 3$ thus $H^* = K_4$ is a contradiction, hence we get a 3-edge-connected cubic triangle-free graph H^* such that $F(H^*) \geq F(G) \geq 3$. □

Theorem 4.7 may help the computer aided search for 3-edge-connected graphs with Frank number greater than 2. For every graph G with Frank number 3, that we have constructed so far the graph H^* is the Petersen graph.

5 Snarks

Snarks are bridgeless cubic graphs with chromatic index 4 and girth at least 5. The Petersen graph is the smallest such graph. Hörsch and Szigeti [4] proved each 3-edge-connected, 3-edge-colorable graph has Frank number at most 3, and the Petersen graph has Frank number 3. Therefore, we expected to find other examples with Frank number 3 among snarks.

In this section, we investigate the second smallest snarks that are the Blanuša snarks and an infinite family of snarks, the so-called flower snarks. For every odd $n \geq 5$ let J_n denote the flower snark on $4n$ vertices. One can construct this graph starting with n copies of stars on 4 vertices with centers $v_1, v_2, \ldots, v_n$ and outer vertices denoted by $\{a_i, b_i, c_i\}$ for $1 \leq i \leq n$. Then add an n-cycle on the vertices $(a_1, a_2, \ldots, a_n)$, and a $2n$-cycle on $(b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n)$. It turns out that the Frank number of each of these snarks is 2.

Theorem 5.1. *Both Blanuša snarks have Frank number* 2.

Proof. Since these snarks are not 4-edge-connected, therefore they do not admit a 2-arc-connected orientation by Theorem 1.2. Hence their Frank number must be greater than 1. On the other hand, we construct two strongly connected orientations $\{O_1, O_2\}$ of these snarks and then verify that they are strongly connected and every arc is deletable in at least one of these orientations which concludes the proof. The two orientations of the Blanuša snarks are explicitly given in Figures 9, 10.

The first thing is to check that these orientations are indeed strongly connected. To see this, observe in each orientation, each vertex is covered by a circuit. For any two vertices there is a chain of intersecting circuits covering these vertices, hence there is a directed path between them in both directions.

To prove that an arc (u, v) is deletable, it is enough to find a directed path from u to v after the deletion of (u, v) by Proposition 2.2.

In Figures 9, 10 the blue arcs indicate the deletable arcs of the corresponding orientations. For the two Blanuša snarks, there is no general rule (other than using the still intact circuits) for deciding whether an arc is deletable or not, one should manually check them. But finding the appropriate directed path after the deletion is usually straightforward due to the small degrees of the vertices. □

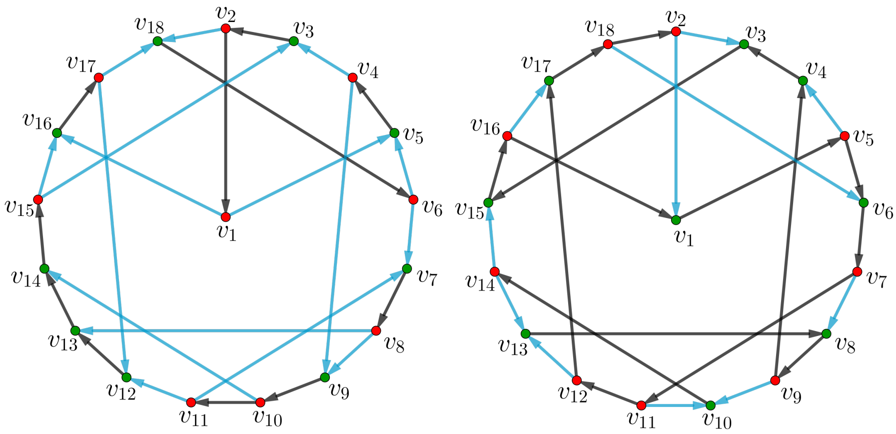


Figure 9: The first Blanuša snark has Frank number 2, the blue arcs are deletable.

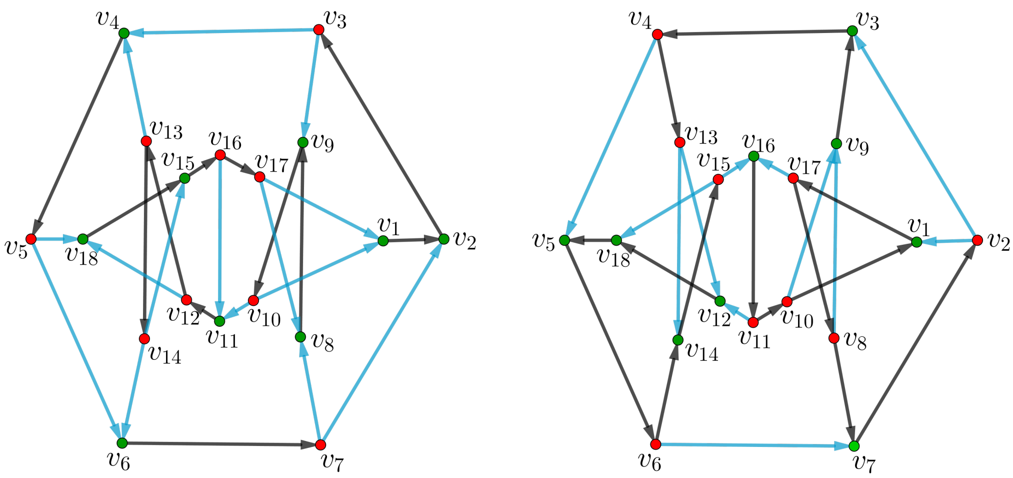


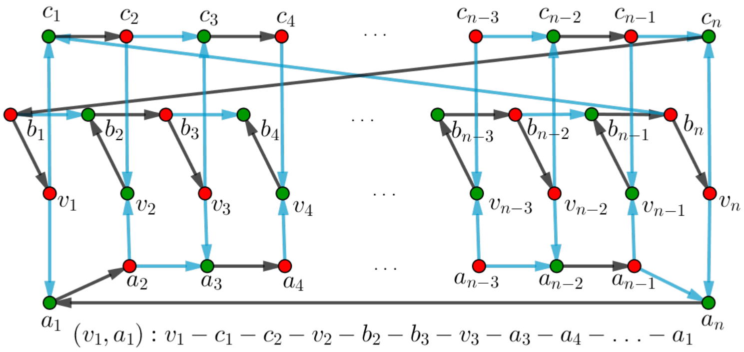
Figure 10: The second Blanuša snark has Frank number 2, the blue arcs are deletable.

Theorem 5.2. *Every member of the infinite family of flower snarks has Frank number 2.*

Proof. Since the flower snarks are also not 4-edge-connected, therefore they do not admit a 2-arc-connected orientation by Theorem 1.2. Hence their Frank number must be also greater than 1. Again, we construct two strongly connected orientations $\{O_1, O_2\}$ of J_n and then verify that these orientations are indeed witness that $F(J_n) = 2$.

The two orientations of the infinite family of flower snarks are illustrated in Figure 11. The first orientation of the flower snark uses circuits $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n)$. For every odd i $(1 \leq i \leq n)$ v_i has out-degree 2 and the only incoming arc is from b_i, and for every even i vertex v_i has in-degree 2 and the only outgoing arc is to b_i. The second orientation of the flower snark comes from the first orientation by reversing some of its arcs and directed paths: paths

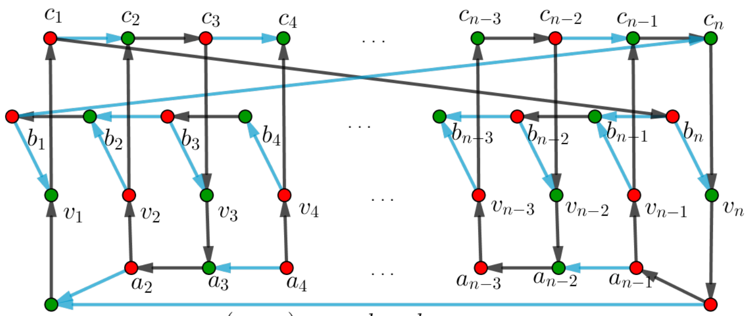
$(a_1, a_2, \ldots, a_n)$ and $(c_n, b_1, b_2, \ldots, b_n, c_1)$ are reversed along with the arcs between v_i and c_i except for (v_1, c_1) which remains the same, and also (v_1, a_1) is reversed.



$(v_1, a_1) : v_1 - c_1 - c_2 - v_2 - b_2 - b_3 - v_3 - a_3 - a_4 - \ldots - a_1$

$(a_2, v_2) : a_3 - a_4 - v_4 - b_4 - b_5 - \ldots - b_n - c_1 - c_2$

$(c_2, v_2) : c_3 - c_4 - v_4 - b_4 - b_5 - v_5 - a_5 - a_6 - \ldots - a_2$



$(a_2, a_1) : v_2 - b_2 - b_1 - c_n - v_n - a_n$

$(b_3, v_3) : b_2 - \ldots - b_1 - c_n - v_n - a_n - a_{n-1} - \ldots - a_2 - v_2 - c_2 - \ldots - c_3$

$(v_2, b_2) : c_2 - \ldots - c_n - v_n - a_n - a_{n-1} - v_{n-1} - b_{n-1} - \ldots - b_2$

$(v_{n-1}, b_{n-1}) : c_{n-1} - c_n - v_n - a_n - a_1 - v_1 - c_1 - b_n$

$(c_1, c_2) : b_n - v_n - a_n - a_{n-1} - \ldots - a_2 - v_2$

$(c_3, c_4) : v_3 - a_3 - a_2 - v_2 - b_2 - b_1 - c_n - v_n - a_n - a_{n-1} - \ldots - a_4 - v_4$

$(b_1, c_n) : v_1 - c_1 - b_n - v_n - a_n - a_{n-1} - v_{n-1} - c_{n-1}$

$(b_1, v_1) : c_n - v_n - a_n - a_1$

Figure 11: $F(J_n) = 2$, for any odd n, the blue arcs are deletable.

It is again easy to check that these orientations are indeed strongly connected. As before, observe in each orientation, each vertex is covered by a circuit. For any two vertices there is a chain of intersecting circuits covering these vertices, hence there is a directed path between them in both directions.

To prove that an arc (u, v) is deletable, it is enough to find a directed path from u to v after the deletion of (u, v) by Proposition 2.2.

In Figure 11, the blue arcs indicate the deletable arcs of the corresponding orientations. Some hints are included in Figure 11 which can be generalized for an arbitrary flower snark J_n. However, there is no general rule (other than using the still intact circuits) for deciding whether an arc is deletable or not, one should manually check them. But finding the appropriate directed path after the deletion is usually straightforward due to the small degrees of the vertices. □

6 Small graphs

Hörsch and Szigeti [4] showed the Petersen graph has Frank number 3. We complement their result with the following.

Theorem 6.1. *If G is a 3-edge-connected cubic graph on at most 10 vertices different from the Petersen graph, then G has Frank number at most 2.*

Proof. We used the House of Graphs [2] and nauty [5] to determine all candidates. There are two such graphs on 6 vertices. One of them is M_6, and the other one is the 3-prism. By Lemma 3.2, M_6 has Frank number 2. The 3-prism is the truncation of K_4, therefore has Frank number 2.

There are four 3-edge-connected, cubic graphs on 8 vertices: the cube, M_8, the 3-prism with a handle, and one more, call it S depicted in Figure 12. By Lemma 3.2, M_8 has Frank number 2. The 3-prism with a handle is a double truncation of K_4, therefore it has Frank number 2. The graph S is a truncation of M_6, therefore it has Frank number 2. Figure 13 shows that the Frank number is 2 for the cube.

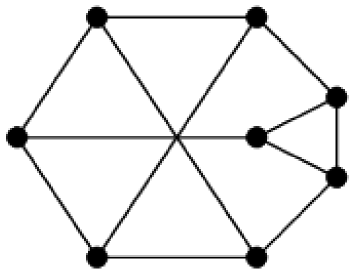


Figure 12: The graph S.

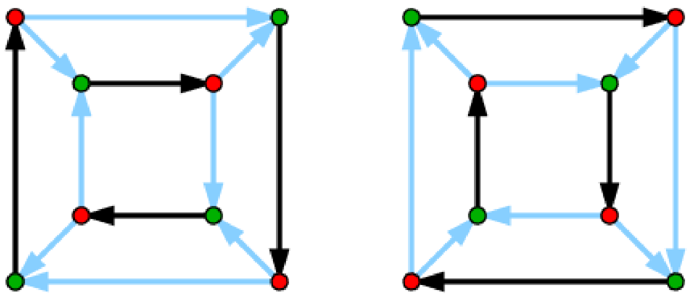


Figure 13: Orientations of the cube showing Frank number 2; the blue arcs are deletable.

There are fourteen 3-edge-connected cubic graphs on 10 vertices. Among them, the graphs containing a triangle can be constructed from K_4, M_6 or the cube by consecutive truncations. We show the triangle-free, 3-edge-connected, cubic graphs on 10 vertices in Figure 14.

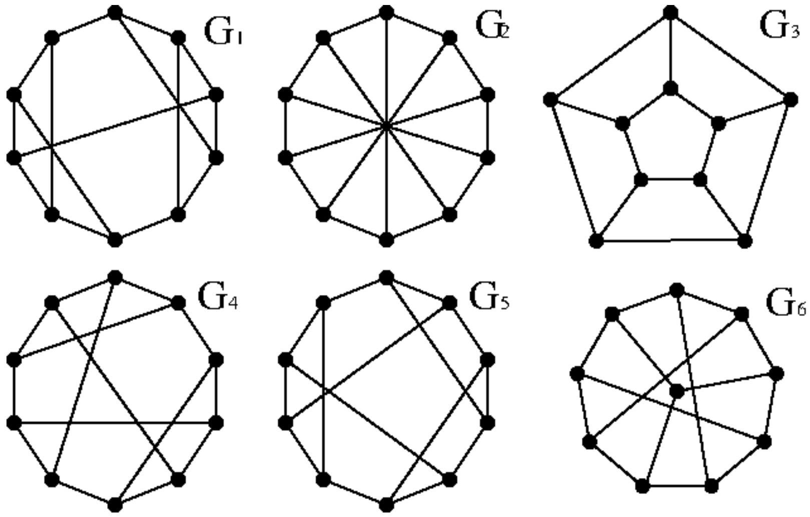


Figure 14: The triangle-free 3-edge-connected cubic graphs on 10 vertices

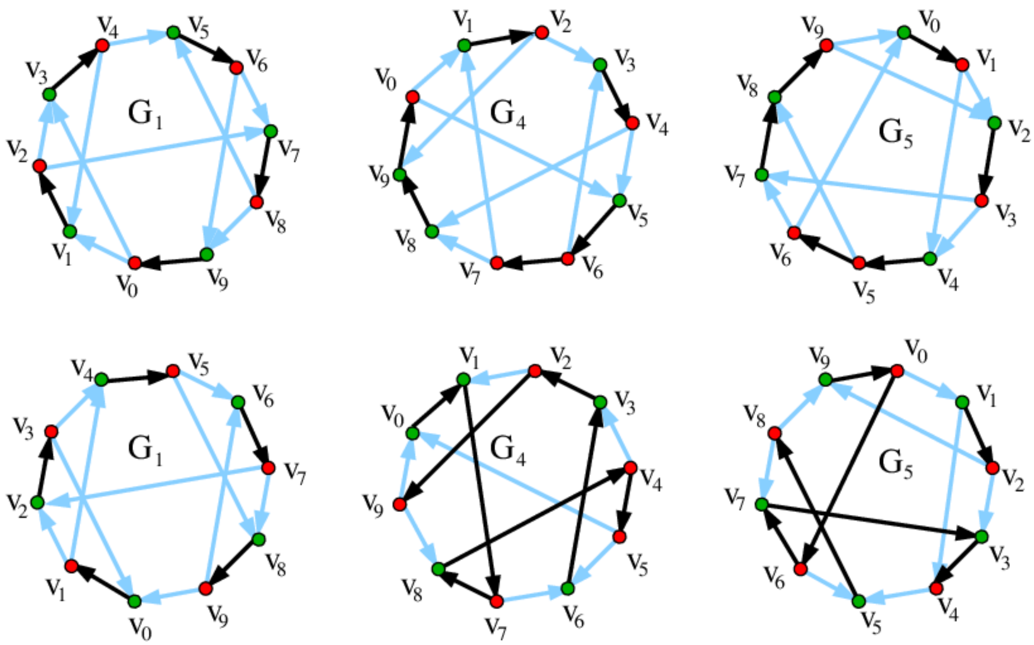


Figure 15: Orientations showing Frank number 2, the blue arcs are deletable.

The graphs G_2 and G_3 were examined in Section 2, The Petersen graph, G_6 is the only graph with Frank number 3. For the remaining graphs G_1, G_4, G_5, we give the two orientations on Figure 15. In each case, the blue edges are deletable.

□

7 The Petersen family

Hörsch and Szigeti [4] proved the Petersen graph has Frank number 3. We give a simple, short, hybrid proof of the fact that the Petersen graph has Frank number larger than 2.

Proposition 7.1. *The Frank number of the Petersen graph is larger than* 2.

Proof. We can determine all non-isomorphic strongly connected orientations of the Petersen graph using nauty [5]. We find there are only 18 such orientations. We list the adjacency matrices of these directed graphs in the Appendix. In our notation, a 1 in row i and column j denotes the arc (i, j). If we suppose the Frank number is 2, then we should find two orientations such that every edge is deletable in at least one of the orientations. Since there are 15 edges, one of the orientations must contain at least 8 deletable arcs. However, there are at most 8 deletable arcs in any of the 18 orientations. Therefore, we collect the orientations of the Petersen graph with 7 or 8 deletable arcs. We find that 8 of the 18 orientations satisfy this condition. We also know, there are at most two deletable arcs incident with any vertex. Therefore, in the first orientation, every vertex must be incident with at least 1 deletable arc. We find only 4 orientations having this property out of the 8. These are G(12), G(15), G(17), G(18) in the Appendix. Each of the four graphs has 8 deletable arcs. It remains to see whether we can combine two sets of 8 arcs to cover all edges at least once. We find the deletable arcs of G(15) and G(17) form a path with 7 edges plus an independent edge (type 1), see Figure 16.

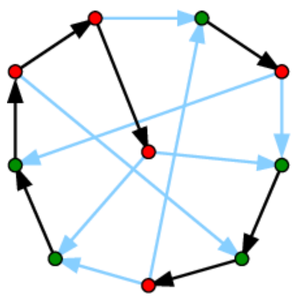


Figure 16: The orientation G(15) of the Petersen graph with 8 blue deletable arcs

In G(12) and G(18) we find a 5-path and a 3-path (type 2) of deletable edges. The non-deletable edges in any of G(12), G(15), G(17), G(18) form three paths of lengths 1,2 and 4. These edges cannot be covered by the deletable edges of G(12) or G(18). This observation shows we cannot combine two orientations of different types or two of type 2. It is left to check whether any of the pairings G(15)-G(15), G(15)-G(17) or G(17)-G(17) works. However, G(15) and G(17) differ only by the direction of a single arc, which is deletable in both cases. It remains to check if we can map the vertices of the non-deletable arcs of G(15) to the deletable arcs of the 7-path of G(15) or G(17). (The third edge of the 7-path must be the overlap in the two set of deletable arcs.) Since this is not the case, we conclude the Frank number of the Petersen graph is larger than 2. □

The generalized Petersen graph $GP(2s+1,t)$ has vertex set $\{u_1,\dots,u_{2s+1},v_1,\dots,v_{2s+1}\}$. The edges are u_iv_i, u_iu_{i+1} and v_iv_{i+t} for $i=1,\dots,2s+1$ modulo $2s+1$. In this notation, the Petersen graph is isomorphic to $GP(5,2)$.

We investigated the most natural generalized Petersen graphs in the hope of finding another example of a 3-edge-connected graph with Frank number at least 3. As it turned out, the generalized Petersen graph $G(2s+1,s)$ admits two appropriate orientations, consequently its Frank number is 2.

Theorem 7.2. *If $GP(2s+1,s)$ denotes the generalized Petersen graph for $s\geq 3$, then $F(GP(2s+1,s))=2$.*

Proof. By Theorem 1.2, the graph $GP(2s+1,s)$ does not admit a 2-arc-connected orientation since it is not 4-edge-connected. Thus $F(GP(2s+1,s))>1$. Note that $GP(2s+1,s)$ is not Hamiltonian. However, it contains a cycle of length $n-1$, where $n=4s+2$ is the number of vertices of $GP(2s+1,s)$.

In both cases O_1 admits the long cycle as a circuit, namely $(u_1,u_2,\dots,u_{2s+1},v_{2s+1},v_s,v_{2s},v_{s-1},v_{2s-1},\dots,v_2,v_{s+2},v_1)$, and the arc (u_{2s+1},u_1) completes another circuit on the outer cycle. The only vertex outside of this circuit is v_{s+1} and regardless of the parity of s the arcs (v_1,v_{s+1}) and (v_{2s+1},v_{s+1}) are added. The only difference between the odd and even case arises at the spoke edges. If s is even, then O_1 contains (u_{2i},v_{2i}) and (v_{2i-1},u_{2i-1}) for $1\leq i\leq s$ and (u_{2s+1},v_{2s+1}). But if s is odd, then O_1 contains (v_{2i},u_{2i}) and (u_{2i+1},v_{2i+1}) for $1\leq i\leq s$ and (v_1,u_1).

In both cases the orientation of the spoke edges of O_2 remains the same as they were in O_1. In both cases the orientation on the outer cycle is reversed almost completely. If s is even then only (u_{2s-1},u_{2s}) remains the same as in O_1, and for s odd (u_2,u_3) is the only arc not changing direction. In both cases the second orientations of the inner cycle can be described easily by stating that for s even $\{v_{s+1},v_{s+2},\dots,v_{2s}\}$ and for s odd $\{v_1,v_2,\dots,v_{s+1}\}$ are the red vertices. The orientation of the edges of the inner cycle in O_2 can be determined similarly in the two cases. In both cases if a spoke edge is directed from the inner cycle and if the corresponding inner vertex is supposed to be red then the only incoming arc of that vertex should come from that neighbor of the inner cycle which has a smaller index (one should think about the indices cyclically).

The constructions are very similar regardless of the parity of s. Orientation O_1 is basically the same and O_2 should be rotated in opposite directions for the two cases. In Figure 17 and 18, we illustrate two orientations of $GP(2s+1,s)$ for s even and odd respectively. The proofs are also very similar. Therefore, we only give the detailed argument for the even case. Thus suppose from now on that s is even.

First we have to show that these are strong orientations. O_1 contains a circuit containing all but one vertex and the cut, formed by the arcs incident with the missing vertex v_{s+1}, is not a directed cut. Hence O_1 is a strong orientation. O_2 has a long directed path $(u_{2s-1},u_{2s-2},\dots,u_1,u_{2s+1},u_{2s})$, and there is a circuit $(u_{2s},v_{2s},v_{s-1},u_{s-1},u_{s-2},v_{s-2},v_{2s-1},\ u_{2s-1})$, therefore all vertices u_i belong to the same strongly connected component. One can rotate the following circuit $(u_{2s+1},v_{2s+1},v_{s+1},v_1,u_1)$ clockwise by two positions, confirming that O_2 is also a strong orientation.

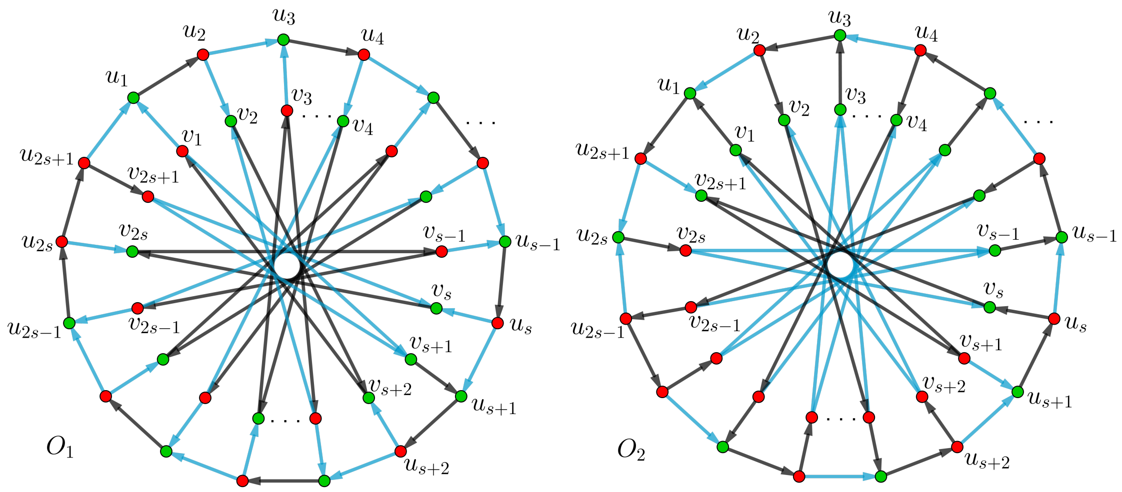


Figure 17: $F(G(2s+1, s)) = 2$ for $s \geq 3$, s even (illustrated for $s = 8$), the blue arcs are deletable.

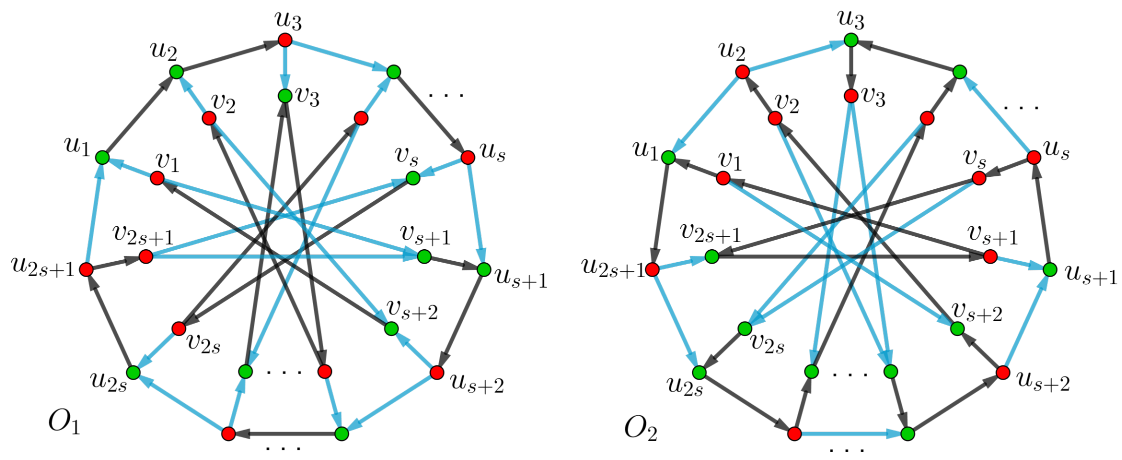


Figure 18: $F(G(2s+1, s)) = 2$ for $s \geq 3$, s odd (illustrated for $s = 5$); the blue arcs are deletable.

In O_1 the arcs between u_i and v_i are deletable (except for $i = s+1$) regardless of their orientation since they are chords of the long circuit. By Proposition 2.2, the blue arcs on the outer part are deletable since there is a directed path between its endvertices using the inner circuit. Similarly, the inner blue arcs are deletable since there is a directed path using the outer circuit. For O_2, one can also find the directed paths switching between the use of the inner and outer circuits. Since every arc is deletable in at least one of these orientations, we proved $F(GP(2s+1, s)) = 2$. $\square$

8 Discussion

If there exist three orientations of G such that every vertex is incident with a deletable arc in each orientation (and these 3 arcs are different for each vertex), then the graph has Frank number at most 3. Therefore, it is a first step to show that at least one such orientation exists. In the proof, we use considerations of Hörsch and Szigeti [4].

Proposition 8.1. *For every cubic 3-edge-connected graph G, there exists a strongly connected orientation D of G such that for every vertex v, there exists an arc a_v incident with v such that $D - a_v$ is strongly connected.*

Proof. The key statement is the following: for every edge e of G, there exists a perfect matching $M(e)$ of G such that $e \in M(e)$ and every 3-edge-cut of G contains precisely 1 edge of G. We prove this by induction on the number of vertices n. The statement holds for $n = 4$, and also for any essentially 4-edge-connected graph G. For the latter, we use the strong form of Petersen's Theorem, where we can prescribe an edge of the perfect matching. For a larger n and assuming G is not essentially 4-edge-connected, consider a 3-edge-cut C that divides $G - C$ into G_1 and G_2. We may assume that both sides have more than 1 vertex. For $i = 1, 2$, we create G_i^* by contracting G_{3-i} in G.

First assume $e \in C$. We apply the induction hypothesis on G_i^* for $i = 1, 2$ and prescribing the image of e of C for $M_i(e)$. This gives us two perfect matchings: $M_1(e)$ in G_1^* and $M_2(e)$ in G_2^* with the required properties. Now we consider G. We identify the edges of $M_1(e)$ and $M_2(e)$ that correspond to the edge e and copy the edges of $M_1(e)$ and $M_2(e)$ otherwise. This defines a perfect matching M of G with the required properties, since C is the only new 3-edge-cut. Other edge cuts inherit the required property by the induction hypothesis.

Secondly assume $e \notin C$ and $e \in G_1^*$. We apply the induction hypothesis on G_1^* prescribing e. We get $M_1(e)$, that contains precisely one of the edges corresponding to C, the edge f say. Now we apply the induction hypothesis on G_2^* prescribing the edge f', that corresponds to f. We get $M_2(f')$. Now we define the perfect matching M of G to be the union of $M_1(e)$ and $M_2(f')$ identifying f and f'. Thereby C also satisfies the required property, and every other cut of G inherits this by the inductive hypothesis.

Now we apply Lemma 4 of [4] to derive that the edges of M are deletable. Since M was a perfect matching, every vertex of G is incident with one edge. □

We pose the following conjectures, each of which is connected to the Frank number. If $F(G) = 2$, then every vertex v is incident with two edges $e_1(v)$ and $e_2(v)$ such that $e_1(v)$ is deletable in one orientation and $e_2(v)$ in the other orientation. We think this phenomenon might always hold.

Conjecture 8.2. *For every cubic 3-edge-connected graph G, there exist two strongly connected orientations D_1 and D_2 of G such that for every vertex v, there exist two different arcs $a_1(v)$ and $a_2(v)$ incident with v such that $D_1 - a_1(v)$ and $D_2 - a_2(v)$ are strongly connected.*

Studying the Frank number, we found that the following was always true. A counterexample would immediately give us a graph with Frank number larger than 2.

Conjecture 8.3. *For every cubic* 3*-edge-connected graph* G*, there exists a strongly connected orientation* D *of* G *such that for at least half of the arcs* $D - a$ *is strongly connected.*

Finally, we experienced that Hamiltonicity adds extra structure, which helps to bound the Frank number.

Conjecture 8.4. *If a* 3*-edge-connected cubic graph* G *admits a Hamiltonian cycle, then* G *has Frank number* 2*.*

This is true up to 12 vertices.
Permutation snarks on $2k$ vertices are similar to k-prisms. There are two k-cycles $v_1v_2 \dots v_k$ and $u_1u_2 \dots u_k$, and there is a permutation $\phi : [k] \to [k]$ such that $v_i u_{\phi(i)}$ are the remaining edges.

Question 8.5. *Does every permutation snark* G *have Frank number* 2*?*

Acknowledgements

The authors are grateful to the anonymous reviewer for providing insightful comments including the proof of Proposition 8.1, that was only conjectured by the authors.

Appendix

We fix the following incidence matrix of the Petersen graph:

```
0111000000
1000110000
1000001001
1000000110
0100001010
0100000101
0010100100
0001011000
0001100001
0010010010
```

The following matrices correspond to the strongly connected orientations of the Petersen graph. In our notation, a 1 in row i and column j denotes the arc (i, j).

```
G(1)        G(2)        G(3)        G(4)        G(5)
0110000000  0110000000  0110000000  0110000000  0110000000
0000110000  0000110000  0000110000  0000110000  0000110000
0000001001  0000001001  0000001001  0000001001  0000001001
1000000100  1000000100  1000000100  1000000100  1000000000
0000001010  0000001000  0000000010  0000000010  0000001010
0000000001  0000000001  0000000101  0000000001  0000000100
0000000100  0000000100  0000100000  0000100000  0000000100
0000010000  0000010000  0000001000  0000011000  0001000000
0001000000  0001100000  0001000000  0001000000  0001000001
0000000010  0000000010  0000000010  0000000010  0000010000

G(6)        G(7)        G(8)        G(9)        G(10)
0110000000  0110000000  0110000000  0110000000  0110000000
0000110000  0000110000  0000110000  0000110000  0000110000
0000001001  0000001001  0000001001  0000001000  0000001000
1000000000  1000000000  1000000000  1000000100  1000000100
0000001010  0000001010  0000001000  0000001010  0000001000
0000000100  0000000001  0000000001  0000000001  0000000001
0000000100  0000000100  0000000100  0000000100  0000000100
0001000000  0001010000  0001010000  0000010000  0000010000
0001000000  0001000000  0001100000  0001000000  0001100000
0000010010  0000000010  0000000010  0010000010  0010000010

G(11)       G(12)       G(13)       G(14)       G(15)
0110000000  0110000000  0110000000  0110000000  0110000000
0000110000  0000110000  0000110000  0000110000  0000110000
0000001000  0000001000  0000001000  0000001000  0000001000
1000000100  1000000100  1000000010  1000000010  1000000010
0000000010  0000000010  0000001010  0000000010  0000000010
0000000101  0000000001  0000000100  0000000101  0000000100
0000100000  0000100100  0000000100  0000100100  0000100100
0000001000  0000010000  0001000000  0001000000  0001000000
0001000001  0001000000  0000000001  0000000001  0000000001
0010000000  0010000010  0010010000  0010000000  0010010000

G(16)       G(17)       G(18)
0110000000  0110000000  0110000000
0000110000  0000110000  0000110000
0000001000  0000001000  0000001000
1000000000  1000000000  1000000000
0000001000  0000000010  0000000010
0000000001  0000000100  0000000100
0000000100  0000100100  0000100000
0001010000  0001000000  0001001000
0001100000  0001000001  0001000001
0010000010  0010010000  0010010000
```

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, *Springer-Verlag*, London, XII+663 pp., 2008.

[2] G. Brinkmann, K. Coolsaet, J. Goedgebeur and H. Mélo,. House of Graphs: a database of interesting graphs, *Discrete Appl. Math.* 161(1-2) (2013), 311–314; also available at `http://hog.grinvin.org`.

[3] M. DeVos, T. Johnson and P. D. Seymour, Cut Coloring and Circuit Covering, *manuscript*; `http://www.sfu.ca/~mdevos/papers/courteous.pdf`

[4] F. Hörsch and Z. Szigeti, Connectivity of orientations of 3-edge-connected graphs, *European J. Combin.* **94** (2021), Paper No. 103292, 18 pp.

[5] B. D. McKay and A. Piperno, Practical graph isomorphism II, *J. Symbolic Comput.* **60** (2014), 94–112.

[6] C. St.J. A. Nash-Williams, On orientations, connectivity, and odd vertex pairings in finite graphs, *Canad. J. Math.* **12** (1960), 555–567.

[7] H. E. Robbins, A theorem on graphs, with an application to a problem on traffic control, *Amer. Math. Monthly* **46**(5) (1939), 281–283.

[8] W. T. Tutte, Hamiltonian circuits, *Colloquio Intemazionale sulle Theorie Combinatoria*, Atti Convegni Lincei 17, Accad. Naz. Lincei, Roma I (1976), 193–199.

(Received 18 Oct 2022; revised 25 Aug 2023, 17 Oct 2023, 3 Nov 2023)