AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **86(3)** (2023), Pages 387–413

An involution on derangements preserving excedances and right-to-left minima

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Abstract

We give a bijective proof of a result by Mantaci and Rakotondrajao from 2003, regarding even and odd derangements with a fixed number of excedances. We refine their result by also considering the set of right-to-left minima.

1 Introduction and preliminaries

Let $\mathfrak{S}_n$ be the symmetric group acting on the set $[n] \coloneqq \{1, 2, \dots, n\}$. An integer $i \in [n]$ is said to be a *fixed point* of a permutation $\pi \in \mathfrak{S}_n$ if $\pi(i) = i$. The set of fixed points of π is denoted by $\mathrm{FIX}(\pi)$ and we set $\mathrm{fix}(\pi) \coloneqq |\mathrm{FIX}(\pi)|$. Recall that the set of *derangements* is defined as $\mathfrak{D}_n \coloneqq \{\pi \in \mathfrak{S}_n : \mathrm{fix}(\pi) = 0\}$.

An *inversion* in a permutation π is a pair (i, j), for $1 \leq i < j \leq n$, such that $\pi(i) > \pi(j)$. The parity of the number of inversions, $\mathrm{inv}(\pi)$, in a permutation π determines the parity of the permutation; π is *even* if $\mathrm{inv}(\pi)$ is even, otherwise π is called an *odd* permutation. The *sign* of π, $\mathrm{sgn}(\pi)$ is defined as $(-1)^{\mathrm{inv}(\pi)}$. The set of even permutations in $\mathfrak{S}_n$ is denoted $\mathfrak{S}_n^e$ and the set of odd permutations is $\mathfrak{S}_n^o$. Let $\mathfrak{D}_n^e$ and $\mathfrak{D}_n^o$ be the sets of even and odd derangements, respectively, in $\mathfrak{D}_n$. Whenever $S = \{s_1, \dots, s_m\}$ is a finite set of positive integers, we shall let $\mathbf{x}_S$ denote the product $x_{s_1} x_{s_2} \cdots x_{s_m}$ of (commuting) variables. By definition, $\mathbf{x}_\emptyset \coloneqq 1$.

In order to state our results, we need to recall some standard notions and terminology. For any function $g : [n] \longrightarrow [n]$, let the set of *excedances*, *excedance*

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ISSN: 2202-3518

values, *right-to-left minima indices*, *right-to-left minima values*, and *the number of inversions* respectively, be defined as

$$\begin{aligned}
\mathrm{EXCi}(g) &\coloneqq \{j \in [n] : g(j) > j\},\\
\mathrm{EXCv}(g) &\coloneqq \{g(j) : j \in \mathrm{EXCi}(g)\},\\
\mathrm{RLMi}(g) &\coloneqq \{i \in [n] : g(i) < g(j) \text{ for all } j \in \{i+1, \dots, n\}\},\\
\mathrm{RLMv}(g) &\coloneqq \{g(i) : i \in \mathrm{RLMi}(g)\},\\
\mathrm{inv}(g) &\coloneqq |\{(i,j) : 1 \le i < j \le n \text{ such that } g(i) > g(j)\}|.
\end{aligned}$$

Moreover, we denote $\mathrm{exc}(g) \coloneqq |\,\mathrm{EXCi}(g)|$ and $\mathrm{rlm}(g) \coloneqq |\,\mathrm{RLMi}(g)| = |\,\mathrm{RLMv}(g)|$. Note that $|\,\mathrm{EXCv}(\sigma)| = |\,\mathrm{EXCi}(\sigma)| = \mathrm{exc}(\sigma)$, for any $\sigma \in \mathfrak{S}_n$, and indices which are not excedances are called *anti-excedances*. Below we show three permutations in $\mathfrak{S}_7$. The first permutation has indices 3 and 6 as fixed-points, so it is not a derangement, while the remaining two are.

Permutation, π	$\mathrm{inv}(\pi)$	$\mathrm{EXCi}(\pi)$	$\mathrm{RLMi}(\pi)$	$\mathrm{RLMv}(\pi)$
2135764	5	{1,4,5}	{2,3,7}	{1,3,4}
2153746	5	{1,3,5}	{2,4,6,7}	{1,3,4,6}
6713245	11	{1,2}	{3,5,6,7}	{1,2,4,5}

The right-to-left minima statistic and the excedance statistic behave quite differently. One can see that

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{rlm}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} t^{\mathfrak{c}(\pi)} = \sum_{k=1}^{n} S_1(n,k) t^k$$

where $\mathfrak{c}(\pi)$ is the number of cycles in cycle representation of π and $S_1(n,k)$ is the unsigned Stirling number of the first kind; see A008275. However,

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{exc}(\pi)} = \sum_{k=1}^{n} A_{n,k} t^{k-1}$$

where $A_{n,k}$ denote the Eulerian numbers, A008292.

It was shown[1] by Mantaci and Rakotondrajao [7, Proposition 4.3], that for every $n \ge 1$ and $1 \le k \le n-1$,

$$|\{\pi \in \mathfrak{D}_n^e : \mathrm{exc}(\pi) = k\}| - |\{\pi \in \mathfrak{D}_n^o : \mathrm{exc}(\pi) = k\}| = (-1)^{n-1}. \tag{1}$$

This refines a result by Chapman, stating that $|\mathfrak{D}_n^e| - |\mathfrak{D}_n^o| = (-1)^{n-1}(n-1)$; see [2]. We find that Sivasubramanian provided an alternative proof for (1) by setting a connection between determinants and signed- excedance enumeration of permutations, see [9]. In addition, a bijection proof (unlike the involution in this paper) has been provided by Ksavrelof and Zeng, see [3], for (1).

In this paper, we provide a proof for a refinement of Equation (1) in Section 3, namely:

[1]Their proof uses a recursion, rather than an explicit involution.

Theorem 1.0.1. *For $n \geq 1$, we have that*

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{RLMv}(\pi)} x_j \right) \left(\prod_{j \in \mathrm{EXCv}(\pi)} y_j \right) = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 \cdots x_j y_{j+1} \cdots y_n. \tag{2}$$

We prove Theorem 1.0.1 by exhibiting a bijection $\hat{\Psi} : \mathfrak{D}_n \to \mathfrak{D}_n$ with exactly $(n-1)$ fixed-elements, where $\hat{\Psi}$ acts as a sign-reversing involution outside the set of fixed-elements. The bijection preserves the excedance value and right-to-left minima permutation statistics, which gives the desired result. Moreover, Theorem 1.0.1 allows us to deduce Theorem 1.0.2, where we now consider indices instead of values.

Theorem 1.0.2. *For $n \geq 1$, we have that*

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{RLMi}(\pi)} x_j \right) \left(\prod_{j \in \mathrm{EXCi}(\pi)} y_j \right) = (-1)^{n-1} \sum_{j=1}^{n-1} y_1 \cdots y_j x_{j+1} \cdots x_n. \tag{3}$$

We include an alternative proof of the $x_1 = x_2 = \cdots = x_n = 1$ case of Theorem 1.0.2 in Section 4. We note that Sivasubramanian also gave a slightly more general proof in this case, see [8]. Again, Sivasubramanian used determinants as in [9].

In Section 4, we provide a proof of the right-to-left minima analog (Corollary 4.1.4) of the main result in [5].

2 Subexcedant functions

The involution we shall construct is not performed directly on permutations, but rather on so-called *subexcedant functions* which are in bijection with permutations. Our main reference is [6], where several fundamental properties are proved.

Definition 2.0.1. A *subexcedant function* f on $[n]$ is a map $f : [n] \longrightarrow [n]$ such that

$$1 \leq f(i) \leq i \text{ for all } 1 \leq i \leq n.$$

We let $\mathcal{F}_n$ denote the set of all subexcedant functions on $[n]$. The *image* of $f \in \mathcal{F}_n$ is defined as $\mathrm{IM}(f) \coloneqq \{f(i) : i \in [n]\}$.

We write subexcedant functions as words, $f(1)f(2)\ldots f(n)$. For example, the subexcedant function $f = 112352$ has $\mathrm{IM}(f) = \{1, 2, 3, 5\}$.

From each subexcedant function $f \in \mathcal{F}_{n-1}$, one can obtain n distinct subexcedant functions in $\mathcal{F}_n$ by appending any integer $i \in [n]$ at the end of the word representing f. Hence, the cardinality of $\mathcal{F}_n$ is $n!$. There is a bijection $\texttt{SEFToPerm} : \mathcal{F}_n \longrightarrow \mathfrak{S}_n$, described in [6], which is defined as the following composition (using cycle notation for permutations):

$$\texttt{SEFToPerm}(f) \coloneqq (n\ f(n)) \cdots (2\ f(2))(1\ f(1)).$$

Example 2.0.2. Let $f = 112435487 \in \mathcal{F}_9$. The permutation $\sigma = \texttt{SEFToPerm}(f)$ is

$$\begin{aligned}\sigma &= (9\ 7)(8)(7\ 4)(6\ 5)(5\ 3)(4)(3\ 2)(2\ 1)(1)\\ &= (1\ 6\ 5\ 3\ 2)(4\ 9\ 7)(8)\\ &= 6\,1\,2\,9\,3\,5\,4\,8\,7.\end{aligned}$$

For $\sigma \in \mathfrak{S}_n$ and $j \in [n]$, it is fairly straightforward to see that we can compute the j^{th} entry of $\texttt{SEFToPerm}^{-1}(\sigma)$ via the recursive formula

$$\texttt{SEFToPerm}^{-1}(\sigma)_j := \begin{cases}\sigma(n) \text{ if } j = n,\\ \texttt{SEFToPerm}^{-1}\left((n\ \sigma(n)) \circ \sigma\right)_j & \text{otherwise.}\end{cases} \tag{4}$$

Note that

$$\sigma' := (n\ \sigma(n)) \circ \sigma \tag{5}$$

is the result after interchanging n and the image of n in σ. Therefore, $\sigma'(n) = n$ and, by a slight abuse of notation, σ' can be considered as a permutation in $\mathfrak{S}_{n-1}$. Hence, the definition above is well-defined, and for simplicity, we use the shorthand $f_\sigma := \texttt{SEFToPerm}^{-1}(\sigma)$.

Example 2.0.3. We shall now show how to invert the calculation in Example 2.0.2. We start with the permutation $\sigma^{(9)} = \binom{123456789}{612935487}$ using two line notation, and for $i > 1$ we let $\sigma^{(i-1)} \in \mathfrak{S}_{i-1}$ be given by

$$\sigma^{(i-1)} := (i\ \sigma^{(i)}(i)) \circ \sigma^{(i)},$$

where we use the observation in Equation (5). Combining this recursion with Equation (4), we have

$$\begin{aligned}
\sigma^{(9)} &= \tbinom{123456789}{61293548\mathbf{7}} & f_\sigma(9) &= 7\\
\sigma^{(8)} &= \tbinom{12345678}{6127354\mathbf{8}} & f_\sigma(8) &= 8\\
\sigma^{(7)} &= \tbinom{1234567}{612735\mathbf{4}} & f_\sigma(7) &= 4\\
\sigma^{(6)} &= \tbinom{123456}{61243\mathbf{5}} & f_\sigma(6) &= 5\\
\sigma^{(5)} &= \tbinom{12345}{5124\mathbf{3}} & f_\sigma(5) &= 3\\
\sigma^{(4)} &= \tbinom{1234}{312\mathbf{4}} & f_\sigma(4) &= 4\\
\sigma^{(3)} &= \tbinom{123}{31\mathbf{2}} & f_\sigma(3) &= 2\\
\sigma^{(2)} &= \tbinom{12}{2\mathbf{1}} & f_\sigma(2) &= 1\\
\sigma^{(1)} &= \tbinom{1}{\mathbf{1}} & f_\sigma(1) &= 1.
\end{aligned}$$

Thus, $f_\sigma = 112435487$.

Proposition 2.0.4 (See [6, Prop. 3.5]). *For $f_\sigma \in \mathcal{F}_n$ we have that $[n] \setminus \mathrm{IM}(f_\sigma) = \mathrm{EXCv}(\sigma)$. In particular, $\mathrm{exc}(\sigma) = n - |\,\mathrm{IM}(f_\sigma)|$.*

Since subexcedant functions are seen as maps $g : [n] \to [n]$, we have the notion of right-to-left minima, fixed points, etc., as defined in the previous section. The following proposition is reminiscent of [1, Property 1], but they consider a different bijection (the Lehmer code) between permutations and subexcedant functions.

Proposition 2.0.5. *Let $\pi \in \mathfrak{S}_n$ and f_π be the corresponding subexcedant function. Then*

(a) $i \in \mathrm{RLMi}(\pi) \implies \pi(i) = f_\pi(i)$,

(b) $\mathrm{RLMv}(\pi) = \mathrm{RLMv}(f_\pi)$,

(c) $\mathrm{RLMi}(\pi) = \mathrm{RLMi}(f_\pi)$.

Proof. We use induction over n, where the base case for $n = 1$ is trivial. Now let $\pi^{(n)} \in \mathfrak{S}_n$ and define $\pi^{n-1} \in \mathfrak{S}_{(n-1)}$ as

$$\pi^{(n-1)}(j) := \begin{cases} \pi^{(n)}(n) & \text{if } \pi^{(n)}(j) = n \\ \pi^{(n)}(j) & \text{otherwise,} \end{cases} \quad \text{so that} \quad f_{\pi^{(n)}}(j) = \begin{cases} \pi^{(n)}(n) & \text{if } j = n \\ f_{\pi^{(n-1)}}(j) & \text{otherwise.} \end{cases} \tag{6}$$

This is the same setup as in Example 2.0.3. By induction hypothesis, $\pi^{(n-1)}$ fulfills properties *(a), (b)*, and *(c)*.

Now suppose $i \in \mathrm{RLMi}(\pi^{(n)})$. We must show that $\pi^{(n)}(i) = f_{\pi^{(n)}}(i)$.
Case $i = n$: Here, $\pi^{(n)}(i) = f_{\pi^{(n)}}(i)$, as this follows immediately Equation (6).
Case $i < n$: Now, either $\pi^{(n)}(i) = n$ or $\pi^{(n)}(i) = \pi^{(n-1)}(i)$. But $\pi^{(n)}(i) = n$ is impossible since $\pi^{(n)}(i)$ is a right-to-left minima and $i < n$. Hence

$$\pi^{(n)}(i) = \pi^{(n-1)}(i) \text{ and } f_{\pi^{(n)}}(i) = f_{\pi^{(n-1)}}(i). \tag{7}$$

Moreover, $\pi^{(n)}(i) < \pi^{(n)}(t)$ whenever $i < t \leq n$. But $\pi^{(n)}(t) = \pi^{(n-1)}(t)$ whenever $\pi^{(n)}(t) \neq n$. Thus, $\pi^{(n-1)}(i) = \pi^{(n)}(i) < \pi^{(n)}(t) = \pi^{(n-1)}(t)$ when $\pi^{(n)}(t) \neq n$.
If $\pi^{(n)}(t) = n$, then $\pi^{(n-1)}(t) = \pi^{(n)}(n) > \pi^{(n)}(i) = \pi^{(n-1)}(i)$, by the first formula in Equation (6).

In any case, $\pi^{(n-1)}(i) < \pi^{(n-1)}(t)$, for $i < n$ and whenever $i < t \leq n-1$. So $i \in \mathrm{RLMi}(\pi^{(n-1)})$. This fact, together with Equation (7) and the induction hypothesis, finally gives

$$i \in \mathrm{RLMi}(\pi^{(n)}) \text{ implies that } \pi^{(n)}(i) = \pi^{(n-1)}(i) = f_{\pi^{(n-1)}}(i) = f_{\pi^{(n)}}(i),$$

which completes the proof of property *(a)*.

We proceed with *(b)*. By definition of $\pi^{(n-1)}$, we have that

$$\begin{aligned} \mathrm{RLMv}(\pi^{(n)}) &= (\mathrm{RLMv}(\pi^{(n-1)}) \cap [\pi^{(n)}(n)]) \;\cup\; \{\pi^{(n)}(n)\}, \\ &= (\mathrm{RLMv}(f_{\pi^{(n-1)}}) \cap [f_{\pi^{(n)}}(n)]) \cup \{f_{\pi^{(n)}}(n)\} \\ &= \mathrm{RLMv}(f_{\pi^{(n)}}), \end{aligned}$$

where the second equality follows from the induction hypothesis.

By the first property and the inductive hypothesis, we have

$$\begin{aligned} \mathrm{RLMi}(\pi^{(n)}) &= \{j \in \mathrm{RLMi}(\pi^{(n-1)}) : \pi^{(n-1)}(j) \leq \pi^{(n)}(n)\} \;\cup\; \{n\} \\ &= \{j \in \mathrm{RLMi}(f_{\pi^{(n-1)}}) : f_{\pi^{(n-1)}}(j) \leq f_{\pi^{(n)}}(n)\} \;\cup\; \{n\} \\ &= \mathrm{RLMi}(f_{\pi^{(n)}}). \end{aligned}$$

This concludes the proof of property *(c)*. □

We say that f has a *strict anti-excedance* at i if $f(i) < i$. Let $\mathrm{sae}(f)$ denote the number of strict anti-excedances in f.

Proposition 2.0.6 (See [6, Prop. 4.1]). *The permutation σ is even if and only if $\mathrm{sae}(f_\sigma)$ is even.*

A *fixed point* of $f \in \mathcal{F}_n$ is an integer $i \in [n]$ such that $f(i) = i$. Moreover, i is a *multiple fixed point* of f if:

1. $f(i) = i$ and
2. there is some $j > i$ such that $f(j) = i$.

Proposition 2.0.7 (See [6, Prop. 3.8]). *We have that $\sigma \in \mathfrak{D}_n$ if and only if all fixed points of f_σ are multiple.*

3 An involution and its consequences

A subexcedant function f is *matchless* if it is of the form

$$f = 1\,1\,2\,3\,4 \dots k{-}1\ k\ k \dots k \qquad \text{for some } 1 \leq k \leq n{-}1.$$

There are $n{-}1$ matchless subexcedant functions of length n. For example, for $n = 10$, the following subexcedant functions are matchless:

$$\begin{array}{lll} 1111111111, & 1122222222, & 1123333333, \\ 1123444444, & 1123455555, & 1123456666, \\ 1123456777, & 1123456788, & 1123456789. \end{array}$$

Lemma 3.0.1 (Properties of matchless functions). *Let $f_\sigma \in \mathcal{F}_n$ be matchless. Then*

$$\sigma = (1\ k{+}1\ k{+}2\ \dots\ n\ k\ k{-}1\ \dots\ 2).$$

Moreover,

$$(-1)^{\mathrm{inv}(\sigma)} = (-1)^{n-1},\ \mathrm{EXCv}(\sigma) = [n] \setminus [k],\ \textit{and}\ \mathrm{RLMv}(\sigma) = [k].$$

Proof. The form of σ follows directly from Section 2. Since σ has only one cycle, its sign is $(-1)^{n-1}$. From the definition of f_σ, we have that

$$\mathrm{IM}(f_\sigma) = [k] \text{ which implies } \mathrm{EXCv}(\sigma) = [n] \setminus [k],$$

by Proposition 2.0.4. Similarly, the last property follows from Proposition 2.0.5. □

Note that for each $k \in [n]$, there is a unique matchless subexcedant function such that the corresponding permutation has $n{-}k$ excedances. We shall see that this property gives a combinatorial interpretation of the right-hand side of Mantaci and Rakotondrajao's identity (stated in (1) above).

3.1 The involution

Let $\mathcal{DF}_n \coloneqq \{f_\sigma : \sigma \in \mathfrak{D}_n\}$ and $\mathcal{DF}^*_n \coloneqq \{f_\sigma : \sigma \in \mathfrak{D}_n$ and f_σ is not matchless$\}$. In other words, $\mathcal{DF}_n$ is the set of subexcedant functions corresponding to derangements of $[n]$. Note that every $f \in \mathcal{DF}_n$ must have at least two 1's in its row representation. We also call σ a *matchless derangement* if $f_\sigma \in \mathcal{DF}_n$ is matchless, and we use $\mathfrak{D}^*_n$ to denote the set of non-matchless derangements.

Our goal is now to define an involution $\Psi : \mathcal{DF}_n \longrightarrow \mathcal{DF}_n$, with the following properties:

(i) The image is preserved, $\mathrm{IM}(\Psi(f_\sigma)) = \mathrm{IM}(f_\sigma)$.

(ii) The set of right-to-left minima is preserved, $\mathrm{RLMv}(\Psi(f_\sigma)) = \mathrm{RLMv}(f_\sigma)$.

(iii) The fixed-elements of Ψ consist of the matchless subexcedant functions.

(iv) The sign is reversed, $\mathrm{sgn}(\Psi(f_\sigma)) = -\,\mathrm{sgn}(f_\sigma)$, whenever $f_\sigma \in \mathcal{DF}^*_n$.

We shall define $\Psi : \mathcal{DF}_n \longrightarrow \mathcal{DF}_n$ below, where f_τ is short for $\Psi(f_\sigma)$. First, if f_σ is matchless, we set $f_\tau \coloneqq f_\sigma$. Now we fix some $f_\sigma \in \mathcal{DF}^*_n$ and let

$$\mathrm{IM}(f_\sigma) = \{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \dots, \mathbf{m}_\ell\}.$$

Note that $\mathbf{m}_1 = 1$ and since f_σ is non-matchless, we know that $\ell \geq 2$ in $\mathrm{IM}(f_\sigma)$. With these preparations, we define two auxiliary maps, $\mathtt{fix}_i$, $\mathtt{unfix}_i$ on subexcedant functions. For $i \in \{2, \dots, \ell\}$,

$$\mathtt{fix}_i(f_\sigma)(\mathbf{m}_i) \coloneqq \mathbf{m}_i, \qquad \mathtt{unfix}_i(f_\sigma)(\mathbf{m}_i) \coloneqq \mathbf{m}_{i-1}$$

while the remaining entries of f_σ are untouched. For $i \in \{2, \dots, \ell\}$, we say that f_σ satisfies $\circledast_i$ (or simply $\circledast_i$ holds if f_σ is clear from the context) if the three conditions

$$f_\sigma(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_\ell, \quad f_\sigma^{-1}(1) = \{1, 2\}, \text{ and } \{\mathbf{m}_i + 1\} \subsetneq f_\sigma^{-1}(\mathbf{m}_i), \qquad (\circledast_i)$$

hold. Note that

$$\{\mathbf{m}_i + 1\} \subsetneq f_\sigma^{-1}(\mathbf{m}_i) \text{ if and only if } f_\sigma(\mathbf{m}_i + 1) = \mathbf{m}_i \text{ and } |f_\sigma^{-1}(\mathbf{m}_i)| \geq 2.$$

Now let $i \in \{2, \dots, \ell\}$ be the *smallest* element satisfying one of the cases below, and let f_τ be given as described in each case.

Case $\mathbf{A}_i$: If $f_\sigma(\mathbf{m}_i) = \mathbf{m}_i$, then $f_\tau \coloneqq \mathtt{unfix}_i(f_\sigma)$.

Case $\mathbf{B}_i$: If $f_\sigma(\mathbf{m}_i) < \mathbf{m}_i$ and $|f_\sigma^{-1}(1)| \geq 3$, then $f_\tau \coloneqq \mathtt{fix}_i(f_\sigma)$.

Case $\mathbf{C}_i$: If $\circledast_i$ holds and $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$, then $f_\tau \coloneqq \mathtt{unfix}_{i+1}(f_\sigma)$.

Case $\mathbf{D}_i$: If $\circledast_i$ holds and $f_\sigma(\mathbf{m}_{i+1}) < \mathbf{m}_{i+1}$, then $f_\tau \coloneqq \mathtt{fix}_{i+1}(f_\sigma)$.

Note that for the same i, the four cases are mutually exclusive. We emphasize that by saying that a case with subscript i holds, this particular $i \geq 2$ is the smallest i for which the conditions one of the four cases hold.

Example 3.1.1. Consider the following four subexcedant functions in $\mathcal{DF}_7$.

1. Let $f_\sigma = 1133535$. Then $\mathrm{IM}(f_\sigma) = \{1, 3, 5\}$ and 2 is the smallest index greater than 1 with $f_\sigma(\mathbf{m}_2) = f_\sigma(3) = 3$. Hence, f_σ is in case $\mathbf{A}_2$ and $f_\tau = \mathtt{unfix}_2(f_\sigma) = 1113535$.
2. Now let $f_\sigma = 1121355$. Then $\mathrm{IM}(f_\sigma) = \{1, 2, 3, 5\}$. Since $f_\sigma(2) < 2$ and $|f_\sigma^{-1}(1)| = 3$, then f_σ is in case $\mathbf{B}_2$. Thus, $f_\tau = \mathtt{fix}_2(f_\sigma) = 1221355$.
3. Suppose that $f_\sigma = 1123535$, then $\mathrm{IM}(f_\sigma) = \{1, 2, 3, 5\}$. The index 2 does not satisfy any of the four cases. So, we consider the next integer $i = 3$. We note that $\circledast_3$ holds and in addition, $f_\sigma(\mathbf{m}_4) = f_\sigma(5) = 5$. Hence, f_σ fulfills $\mathbf{C}_3$ and $f_\tau = \mathtt{unfix}_{i+1}(f_\sigma) = \mathtt{unfix}_4(f_\sigma) = 1123335$.
4. Now take $f_\sigma = 1123445$. Then $\mathrm{IM}(f_\sigma) = \{1, 2, 3, 4, 5\}$. None of the four cases for f_σ are fulfilled with $i \in \{2, 3\}$. However, f_σ satisfies $\circledast_4$ and $f_\sigma(\mathbf{m}_5) = f_\sigma(5) = 4 < \mathbf{m}_5$. Thus, we are in $\mathbf{D}_4$ and $f_\tau = \mathtt{fix}_5(f_\sigma) = 1123545$.

Remark 3.1.2. Suppose $\mathbf{B}_i$ applies for f_σ. Then, for sure $f_\sigma(\mathbf{m}_2) < \mathbf{m}_2$, since otherwise, we would be in the case $\mathbf{A}_2$. Hence, $\mathbf{B}_i$ may only apply when $i = 2$.

We have several things that need to be proved. In Lemma 3.1.3 we show that Ψ is well-defined, and in Lemma 3.1.5, we show that the range is correct. In Lemma 3.1.6, we show that Ψ preserves the image. Finally, in Lemma 3.1.7, we show that Ψ preserves the right-to-left minima set. In Lemmas 3.1.8 and 3.1.9, we show that Ψ is sign-reversing on $\mathcal{DF}_n^*$ and Ψ is indeed an involution, respectively.

It is clear from the definition of Ψ that at most one of the cases applies for any $f_\sigma \in \mathcal{DF}_n^*$. For the well-definedness of Ψ, we must also verify that at least one of the cases applies.

Lemma 3.1.3 (Well-defined)**.** *Let $f_\sigma \in \mathcal{DF}_n$ with ℓ elements in its image. If none of the four cases ($\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$) applies to f_σ, then f_σ is matchless.*

Moreover, if no $i \in \{2, \dots, t\}$ fulfills any of $\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i, \boldsymbol{D}_i$, conditions for some $t \in [\ell]$, and either $t = \ell$ or cases $\boldsymbol{A}_{t+1}$ and $\boldsymbol{B}_{t+1}$ do not hold, then

$$f_\sigma(j) = \max\{1, j-1\}, \text{ for all } j \in [t+1]. \tag{8}$$

Consequently, the prefix

$$f_\sigma(1)\ f_\sigma(2)\ \dots\ f_\sigma(t)\ f_\sigma(t+1) \tag{9}$$

is matchless. In addition, if $\ell = t$, then

$$f_\sigma = 1\ 1\ 2\ 3\ \dots\ \ell{-}1\ \ell\ \ell\ \dots\ \ \ell, \tag{10}$$

which is matchless. Otherwise,

$$\{f_\sigma(t+2), \dots, f_\sigma(n)\} = \{\mathbf{m}_{t+1}, \dots, \mathbf{m}_\ell\}. \tag{11}$$

Proof. We first note that (9) follows immediately from (8) and $\mathbf{m}_i = i$, for $i \in [t]$ by (9). The main statement follows from considering $t = \ell$ in (9). We shall use induction on t in order to prove (8), (10), and (11).

Base case $t = 1$: In this case, $\{2, \dots, t\}$ is empty. If $\ell = t = 1$, then $f_\sigma = 1\,1\,1 \cdots 1\,1$ (which is matchless). Otherwise, suppose that the cases $\mathbf{A}_{t+1}$ and $\mathbf{B}_{t+1}$ do not hold.

Since case $\mathbf{A}_2$ is not fulfilled, then $f_\sigma(\mathbf{m}_2) < \mathbf{m}_2$ so $f_\sigma(\mathbf{m}_2) = 1$. Hence, $f_\sigma(1) = 1$ and $f_\sigma(2) = 1$, otherwise $f_\sigma(2) = 2$ which would violate our assumption.

Since case $\mathbf{B}_2$ is not fulfilled, although $f_\sigma(\mathbf{m}_2) < \mathbf{m}_2$, then $|f_\sigma^{-1}(1)| < 3$. Thus, $f_\sigma^{-1}(1) = \{1, 2\}$. Consequently, $\mathbf{m}_2 = 2$, since else $f_\sigma(3) = 3$ and $\mathbf{A}_2$ would be fulfilled. Hence, (11) follows.

Induction hypothesis: Suppose the statements hold for $t = k$, for some $k \geq 1$. We shall prove that they hold for $t = k + 1$.

For this purpose suppose that none of the cases ($\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$) holds for $i \in \{2, 3, \dots, k+1\}$ and either $\ell = t = k + 1$ or cases $\mathbf{A}_{k+2}$ and $\mathbf{B}_{k+2}$ are not satisfied. Then, by the induction hypothesis, $f_\sigma(j) = \max\{1, j - 1\}$, for $j \in [k + 1]$ and f_σ starts with $1\,1\,2\,3 \cdots k{-}1\,k$, which is matchless. Since $\ell > k$ and the two cases ($\mathbf{A}_{k+1}$, $\mathbf{B}_{k+1}$) are not fulfilled, (by the induction hypothesis) none of the elements in $[k]$ belongs to $\{f_\sigma(k+2), \dots, f_\sigma(\ell)\}$. So, $f_\sigma(k+2) \in \{k+1, k+2\}$. We also have $\mathbf{m}_i = i$, for $i \in [k]$. We claim that $\mathbf{m}_{k+1} = k + 1$. Otherwise, $\mathbf{m}_{k+1} > k + 1$ and then $f_\sigma(\mathbf{m}_{k+1}) = \mathbf{m}_{k+1}$, which would satisfy case $\mathbf{A}_{k+1}$.

If $\ell = k + 1$, then $f_\sigma(k + 2) = \mathbf{m}_{k+1} = k + 1$ and

$$f_\sigma = 1\,1\,2\,3 \cdots k{-}1\,k\,k{+}1\,k{+}1 \cdots k{+}1,$$

indeed (8) and (10) holds.

Else, $f_\sigma(k + 2) = k + 1$ and f_σ starts with $1\,1\,2\,3 \cdots k{-}1\,k\,k{+}1$ since cases $\mathbf{A}_{k+2}$ and $\mathbf{B}_{k+2}$ are not fulfilled. Thus, (8) holds. And since neither of the two cases ($\mathbf{C}_i$, $\mathbf{D}_i$) holds for $i \in [k+1]$, at least one of the conditions in $\circledast_i$ is not fulfilled. However, $f_\sigma(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_\ell$ (since $\ell > k{+}1$) and $f_\sigma(\mathbf{m}_i{+}1) = \mathbf{m}_i$, for all $i \in [k{+}1]$. Moreover, $f_\sigma^{-1}(1) = \{1, 2\}$. Thus, $|f_\sigma^{-1}(\mathbf{m}_i)| = 1$, for all $i \in \{2, \dots, k + 1\}$. Hence, none of the elements in $[k + 1]$ belongs to $\{f_\sigma(k + 3), \dots, f_\sigma(\ell)\}$, which proves (11). □

Remark 3.1.4. If either $\mathbf{C}_i$ or $\mathbf{D}_i$ holds, then $\ell > i$, and both (8) and (11) hold for $t = i - 1$. If, in particular, case $\mathbf{D}_i$ is fulfilled, then $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_i$ since $\mathbf{m}_{i+1} > f_\sigma(\mathbf{m}_{i+1}) \in \{\mathbf{m}_i, \dots, \mathbf{m}_\ell\}$.

Lemma 3.1.5 (Correct range)**.** *If $f_\sigma \in \mathcal{DF}_n$, then $f_\tau := \Psi(f_\sigma) \in \mathcal{DF}_n$.*

Proof. If f_σ is matchless, then we are done. Suppose that $f_\sigma \in \mathcal{DF}_n^*$ and $i \geq 2$ satisfies one the cases in ($\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$). By Proposition 2.0.7, it suffices to show that all fixed-points of f_τ are multiple.

In the case of either $\mathbf{A}_i$ or $\mathbf{C}_i$, there will be no new fixed point created in f_τ since $f_\tau = \mathtt{unfix}_r(f_\sigma)$, for $r \in \{i, i + 1\}$. So all the fixed points of f_σ remain multiple in f_τ too except for $\mathbf{m}_r$, which is not fixed in f_τ.

If the case $\mathbf{B}_i$ is fulfilled, then $i = 2$ by Remark 3.1.2, and $f_\sigma(\mathbf{m}_2) = 1$. Moreover, $f_\tau(\mathbf{m}_2) = \mathbf{m}_2$ and there is some $j > \mathbf{m}_2$ such that $f_\tau(j) = f_\sigma(j) = \mathbf{m}_2$. That is, $\mathbf{m}_2$ is a multiple fixed point in f_τ. And so is 1 since $|f_\sigma^{-1}(1)| \geq 3$ implies $|f_\tau^{-1}(1)| \geq 2$.

If the case $\mathbf{D}_i$ is fulfilled, then $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_i$ by Remark 3.1.4, and there is some $j > \mathbf{m}_{i+1}$ such that $f_\tau(j) = f_\sigma(j) = \mathbf{m}_{i+1}$. Consequently, $f_\tau(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$ is a multiple fixed point in f_τ while $\mathbf{m}_i$ is not a fixed point in both f_σ and f_τ. □

Lemma 3.1.6 (Image-set preserving)**.** *For $f_\tau = \Psi(f_\sigma)$, we have*

$$\mathrm{IM}(f_\sigma) = \mathrm{IM}(f_\tau) \textit{ and } \mathrm{EXCv}(\sigma) = \mathrm{EXCv}(\tau). \tag{12}$$

Proof. First note that $\mathrm{IM}(f_\tau) \subseteq \mathrm{IM}(f_\sigma)$, which clearly follows from the definition of Ψ. Now suppose that one of the cases in $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)$ is satisfied for $i \geq 2$. Recall that the map Ψ first removes an element in position $\mathbf{m}_r$, for $r \in \{i, i+1\}$, in f_σ and then insert another element on the same position to obtain f_τ. So, it is enough to show that the removed element is in $\mathrm{IM}(f_\tau)$ for $\mathrm{IM}(f_\sigma) = \mathrm{IM}(f_\tau)$ to hold.

In the case of $\mathbf{A}_i$ or $\mathbf{C}_i$, $f_\tau = \mathtt{unfix}_r(f_\sigma)$ and there is some $j > \mathbf{m}_r$ such that $\mathbf{m}_r = f_\sigma(j) = f_\tau(j)$ since $\mathbf{m}_r$ is a multiple fixed point in f_σ in these cases. So $\mathbf{m}_r \in \mathrm{IM}(f_\tau)$.

If $\mathbf{B}_i$ holds, then $f_\sigma(\mathbf{m}_i) = 1$, since $i = 2$ (by Remark 3.1.2). Moreover, $f_\tau(\mathbf{m}_i) = \mathbf{m}_i$. However, $|f_\sigma^{-1}(1)| \geq 3$. So, $|f_\tau^{-1}(1)| \geq 2$ and then $1 \in \mathrm{IM}(f_\tau)$.

Finally, suppose case $\mathbf{D}_i$ is fulfilled. Then by Remark 3.1.4, $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_i$. We can now conclude that $\mathbf{m}_i \in \mathrm{IM}(f_\tau)$, since $|f_\sigma^{-1}(\mathbf{m}_i)| \geq 2$.

Therefore, the first equality in (12) is proved, while the second follows from Proposition 2.0.4. □

Lemma 3.1.7. *For $f_\tau = \Psi(f_\sigma)$ we have*

$$\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau) \textit{ and } \mathrm{RLMv}(\sigma) = \mathrm{RLMv}(\tau). \tag{13}$$

Proof. Let $f_\sigma \in \mathcal{DF}_n$. If f_σ is matchless, then $f_\tau = f_\sigma$ and $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau)$. Suppose f_σ is non-matchless, so that one of the four cases applies.

Case $\mathbf{A}_i$: Then $f_\sigma(\mathbf{m}_i) = \mathbf{m}_i$ and $f_\tau(\mathbf{m}_i) = \mathbf{m}_{i-1}$. Moreover, there is some $j > \mathbf{m}_i$ such that $f_\tau(j) = f_\sigma(j) = \mathbf{m}_i$. The property of $\mathbf{m}_i$ being a right-to-left minimum in f_σ as well as f_τ is determined either at the position j or to the right of j. Hence, replacing $\mathbf{m}_i$ by $\mathbf{m}_{i-1}$ at position $\mathbf{m}_i$, preserves $\mathbf{m}_i$ being (or not) a right-to-left minimum.

- If $i \geq 3$, then $\mathbf{m}_i$ is the leftmost occurrence of $\mathbf{m}_{i-1}$ in f_τ, since i is the smallest such that $f_\sigma(\mathbf{m}_i) = \mathbf{m}_i$. Since $\mathrm{IM}(f_\sigma) = \mathrm{IM}(f_\tau)$, there is some $k > \mathbf{m}_i$ such that $f_\tau(k) = f_\sigma(k) = \mathbf{m}_{i-1}$. So, $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau)$.
- If $i = 2$, then $\mathbf{m}_{i-1} = \mathbf{m}_1 = 1 \in \mathrm{RLMv}(f_\tau)$. Since $1 \in \mathrm{RLMv}(f_\sigma)$, then $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau)$.

Case B$_i$: Then $i = 2$ and $f_\sigma(\mathbf{m}_2) = \mathbf{m}_1 = 1$ and $f_\tau(\mathbf{m}_2) = \mathbf{m}_2$. Moreover, there is some $k > \mathbf{m}_2$ such that $f_\tau(k) = f_\sigma(k) = \mathbf{m}_2$. Since the right-to-left minimum property of $\mathbf{m}_2$ is determined at or to the right of the k^{th} position and $1 \in \mathrm{RLMv}(f_\tau)$, we have $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau)$.

Case C$_i$: Then $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$ and $f_\tau(\mathbf{m}_{i+1}) = \mathbf{m}_i$. We claim that $\mathbf{m}_i \in \mathrm{RLMv}(f_\sigma)$. Otherwise, there are $r < i$ and $s > j$, such that $f_\sigma(s) = \mathbf{m}_r$, where j is the rightmost position of $\mathbf{m}_i$ in f_σ. But now, $|f_\sigma^{-1}(\mathbf{m}_r)| \geq 2$ (by Lemma 3.1.3) and so $\circledast_r$ is fulfilled and case $\mathbf{D}_r$ holds. This contradicts the choice of i being minimal, and the claim follows.

Since $\mathbf{m}_{i+1}$ being a right-to-left minimum is determined at some other position $k > \mathbf{m}_{i+1}$ where $f_\tau(k) = f_\sigma(k) = \mathbf{m}_{i+1}$, we can conclude that $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau)$.

Case D$_i$: There is some $i \geq 2$ such that $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_i < \mathbf{m}_{i+1}$ (by Remark 3.1.4) and $f_\tau(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$. We also know that $\mathbf{m}_i \in \mathrm{RLMv}(f_\sigma)$. Since $\mathbf{m}_{i+1} > \mathbf{m}_i$, we have $\mathbf{m}_i \in \mathrm{RLMv}(f_\tau)$.

Now, $\mathbf{m}_{i+1}$ being a right-to-left minimum is determined at some position $k > \mathbf{m}_{i+1}$ where $f_\tau(k) = f_\sigma(k) = \mathbf{m}_{i+1}$. Hence, we can conclude that $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(f_\tau)$.

The second equality in (13) follows from Proposition 2.0.5. □

Lemma 3.1.8. *If $f_\sigma \in \mathcal{DF}_n^*$, then*

$$\mathrm{sae}(\Psi(f_\sigma)) \in \{\mathrm{sae}(f_\sigma) - 1,\ \mathrm{sae}(f_\sigma) + 1\}.$$

Moreover, if $f_\tau = \Psi(f_\sigma)$, then σ and τ have different parity.

Proof. The first statement follows from the fact that $\mathtt{fix}_i$ and $\mathtt{unfix}_i$ decreases and increases, respectively, the number of strict anti-excedances by one. The second statement follows from the first by Proposition 2.0.6. □

Lemma 3.1.9. *The map $\Psi : \mathcal{DF}_n \longrightarrow \mathcal{DF}_n$ is an involution.*

Proof. Let $f_\sigma \in \mathcal{F}_n$ with $\mathrm{IM}(f_\sigma) = \{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \dots, \mathbf{m}_\ell\}$ and set $f_\tau = \Psi(f_\sigma)$. For matchless f_σ, there is nothing to show. Now assume that one of ($\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$) holds for f_σ and one of ($\mathbf{A}_{i'}$, $\mathbf{B}_{i'}$, $\mathbf{C}_{i'}$, $\mathbf{D}_{i'}$) holds f_τ, for $i, i' \in \{2, \dots, l\}$.

Case A$_i$: We have $f_\sigma(\mathbf{m}_i) = \mathbf{m}_i$, and $f_\tau = \mathtt{unfix}_i(f_\sigma)$.

- If $|f_\tau^{-1}(1)| \geq 3$, then $i = 2$.

  Suppose $i \geq 3$. Then $|f_\sigma^{-1}(1)| < 3$ since otherwise f_σ would be in case $\mathbf{B}_j$ for some $j < i$. Thus, $|f_\sigma^{-1}(1)| < |f_\tau^{-1}(1)|$ and then there is some $r \in [n]$ such that $f_\tau(r) = 1 \neq f_\sigma(r)$. However, $r = \mathbf{m}_i$ since f_τ and f_σ differs only on position $\mathbf{m}_i$. So, $f_\tau(\mathbf{m}_i) = 1 = \mathbf{m}_{i-1}$ and this only happens if $i = 2$.

  Hence, $i = 2$ and $\mathbf{m}_2 \in f_\tau^{-1}(1) \setminus f_\sigma^{-1}(1)$. There is now some $h > \mathbf{m}_2$ such that $f_\tau(h) = f_\sigma(h) = \mathbf{m}_2$ since $f_\sigma \in \mathcal{DF}_n$. So, f_τ satisfies the conditions for case $\mathbf{B}_{i'}$ with $i' = i$. It follows that $\Psi(f_\tau) = f_\sigma$.

- If $|f_\tau^{-1}(1)| = 2$, then $f_\tau^{-1}(1) = f_\sigma^{-1}(1)$. Because we always have $f_\sigma^{-1}(1) \subseteq f_\tau^{-1}(1)$ in $\mathbf{A}_i$ and $\mathbf{C}_i$, and since $|f_\sigma^{-1}(1)| \geq 2$, we must have equality. Moreover, $i \geq 3$ since otherwise $f_\tau(\mathbf{m}_2) = 1$ and then $|f_\tau^{-1}(1)| > |f_\sigma^{-1}(1)|$.

  By applying Lemma 3.1.3 for $t = i - 2$, the first $i - 1$ entries of f_σ and f_τ are

  $$1, 1, 2, 3, \dots, i{-}3,\ i{-}2,$$

  and $\mathbf{m}_j = j$, for all $j \in [i-2]$. In addition, $|f_\sigma^{-1}(j)| = 1$ otherwise f_σ would lie in case $\mathbf{D}_j$ holds for some $j \in [i-2]$. Hence, $f_\sigma(i) \in \{i{-}1, i\}$.

  If $f_\sigma(i) = i$, then $\mathbf{m}_i = i$ (since f_σ is in case $\mathbf{A}_i$) and $f_\tau(i) = \mathbf{m}_{i-1} = i - 1$ since $i - 2 = \mathbf{m}_{i-2}$. Thus, there exists some $s > i$ such that $f_\tau(s) = f_\sigma(s) = i - 1$. Then

  $$\begin{aligned} f_\sigma &= 1\,1\,2\,3 \cdots i{-}3\ i{-}2\ i \cdots i{-}1 \cdots i \cdots \\ f_\tau &= 1\,1\,2\,3 \cdots i{-}3\ i{-}2\ i{-}1 \cdots i{-}1 \cdots i \cdots \\ &\text{or} \\ f_\sigma &= 1\,1\,2\,3 \cdots i{-}3\ i{-}2\ i \cdots i \cdots i{-}1 \cdots \\ f_\tau &= 1\,1\,2\,3 \cdots i{-}3\ i{-}2\ i{-}1 \cdots i \cdots i{-}1 \cdots . \end{aligned}$$

  Now we can see that f_τ satisfies the conditions in $\circledast_{i'}$, for $i' = i - 1$:

  $$f_\tau(\mathbf{m}_{i-1}) = f_\tau(i-1) = i - 2 < \mathbf{m}_{i-1} < \mathbf{m}_\ell \text{ (since } \ell \geq i),\ \ f_\tau^{-1}(1) = \{1, 2\},$$

  $$f_\tau(\mathbf{m}_{i-1} + 1) = f_\tau(i) = i - 1 = \mathbf{m}_{i-1}, \text{ and } |f_\tau^{-1}(\mathbf{m}_{i-1})| \geq 2.$$

  Hence, f_τ fulfills case $\mathbf{D}_{i'}$ for $i' = i - 1$ and $\Psi(f_\tau) = f_\sigma$.

  If $f_\sigma(i) = i - 1$, then $\mathbf{m}_{i-1} = i - 1$ and $\mathbf{m}_i > i$. Thus, $f_\tau(\mathbf{m}_i) = i - 1$. Moreover, $|f_\sigma^{-1}(i-1)| = 1$. Otherwise, f_σ would satisfy $\circledast_i$ whence either $\mathbf{C}_i$ or $\mathbf{D}_i$ would be fulfilled. This implies that $f_\tau^{-1}(i-1) = \{i, \mathbf{m}_i\}$. Now, it is easy to see that f_τ satisfies the conditions in $\circledast_{i'}$ for $i' = i - 1$. Therefore, f_τ fulfills the case $\mathbf{D}_{i'}$ for $i' = i - 1$ and $\Psi(f_\tau) = f_\sigma$.

Case $\mathbf{B}_i$: Then $i = 2$ and $f_\tau = \mathtt{fix}_2(f_\sigma)$. We have that $f_\tau(\mathbf{m}_2) = \mathbf{m}_2$ and f_τ belongs to $\mathbf{A}_{i'}$ for $i' = i$, so $\Psi(f_\tau) = f_\sigma$.

Case $\mathbf{C}_i$: In this case, $\circledast_i$ hold, which state that

$$f_\sigma(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_\ell,\ f_\sigma^{-1}(1) = \{1, 2\},\ f_\sigma(\mathbf{m}_i + 1) = i, \text{ and } |f_\sigma^{-1}(\mathbf{m}_i)| \geq 2.$$

And also $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$.

Since $f_\tau = \mathtt{unfix}_{i+1}(f_\sigma)$ and $i \geq 2$, we have $f_\tau(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_\ell$ and $f_\tau^{-1}(1) = \{1, 2\}$. And also since $f_\sigma(\mathbf{m}_i + 1) = \mathbf{m}_i$ and $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$, we have $\mathbf{m}_i + 1 \neq \mathbf{m}_{i+1}$. This implies that $f_\tau(\mathbf{m}_i + 1) = f_\sigma(\mathbf{m}_i + 1) = \mathbf{m}_i$. We also have that $|f_\tau^{-1}(\mathbf{m}_i)| \geq 3$, since $|f_\sigma^{-1}(\mathbf{m}_i)| \geq 2$ and $f_\tau(\mathbf{m}_{i+1}) = \mathbf{m}_i$. Hence, f_τ satisfies the conditions in $\circledast_{i'}$ for $i' = i$ and then it belongs to $\mathbf{D}_{i'}$. It follows that $\Psi(f_\tau) = f_\sigma$.

Case $\mathbf{D}_i$: Again, we have $\circledast_i$ for f_σ, and $f_\sigma(\mathbf{m}_{i+1}) < \mathbf{m}_{i+1}$. Moreover, $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_i$ (from Remark 3.1.4). Recall, $f_\tau = \mathtt{fix}_{i+1}(f_\sigma)$. We also have that $f_\tau(\mathbf{m}_i) = f_\sigma(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_l$, $f_\tau^{-1}(1) = \{1, 2\}$, and $f_\tau(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$. We shall now consider two subcases.

- *Suppose* $\mathbf{m}_i+1 < \mathbf{m}_{i+1}$ *and* $|f_\sigma^{-1}(\mathbf{m}_i)| \geq 3$. We have that $f_\tau(\mathbf{m}_i+1) = f_\sigma(\mathbf{m}_i + 1) = \mathbf{m}_i$. Then $|f_\tau^{-1}(\mathbf{m}_i)| \geq 2$ (then $\circledast_i$ is satisfied for f_τ) and f_τ belongs to case $\mathbf{C}_{i'}$ for $i' = i$.
- *Otherwise,* $|f_\tau^{-1}(\mathbf{m}_i)| = 1$ if $\mathbf{m}_i + 1 < \mathbf{m}_{i+1}$ and $|f_\sigma^{-1}(\mathbf{m}_i)| = 2$. On the other hand, $f_\tau(\mathbf{m}_i + 1) = f_\tau(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$ if $\mathbf{m}_i + 1 = \mathbf{m}_{i+1}$. Then $\circledast_i$ will not be satisfied for f_τ in both cases. Therefore, f_τ lies in case $\mathbf{A}_{i'}$ for $i' = i + 1$.

In both cases, $\Psi(f_\tau) = f_\sigma$.

□

Remark 3.1.10. In Table 1, we give an overview under what circumstances a subexcedant function belonging to a case, is mapped to a different case.

	$f_\sigma \in \mathbf{A}_i$	$f_\sigma \in \mathbf{B}_i$	$f_\sigma \in \mathbf{C}_i$	$f_\sigma \in \mathbf{D}_i$
$f_\tau \in \mathbf{A}_{i'}$	$\emptyset$	Always, $(i' = i)$	$\emptyset$	f_τ does not fulfill Equation $(\circledast_i)$, $(i'=i)$
$f_\tau \in \mathbf{B}_{i'}$	$\lvert f_\tau^{-1}(1)\rvert \geq 3$, $(i' = i)$	$\emptyset$	$\emptyset$	$\emptyset$
$f_\tau \in \mathbf{C}_{i'}$	$\emptyset$	$\emptyset$	$\emptyset$	f_τ fulfills Equation $(\circledast_i)$, $(i' = i + 1)$
$f_\tau \in \mathbf{D}_{i'}$	$\lvert f_\tau^{-1}(1)\rvert = 2$, $(i' = i - 1)$	$\emptyset$	Always, $(i' = i)$	$\emptyset$

Table 1: When $f_\tau = \Psi(f_\sigma)$, we have the combinations under the conditions described in the cells of the table.

Example 3.1.11. Consider the following subexcedant functions.

1. $f_\sigma = 1133535$ satisfies case $\mathbf{A}_2$. However, its image, $f_\tau = 1113535$, lies in case $\mathbf{B}_2$.
2. $f_\sigma = 1124545$ satisfies case $\mathbf{A}_3$. Then, its image, $f_\tau = 1122545$, lies in case $\mathbf{D}_2$.
3. $f_\sigma = 1121355$, which is in case $\mathbf{B}_2$, mapped to $f_\tau = 1221355$ that belongs to case $\mathbf{A}_2$.
4. $f_\sigma = 1123535$ satisfies case $\mathbf{C}_3$. Nevertheless, the image $f_\tau = 1123335$ appears in case $\mathbf{D}_3$.

5. $f_\sigma = 11233353$ is in case $\mathbf{D}_3$. The image, $f_\tau = 11235353$, is in case $\mathbf{C}_3$.

6. $f_\sigma = 1123445$ is in case $\mathbf{D}_4$. However, its image, $f_\tau = 1123545$, is in case $\mathbf{A}_5$.

We conclude this subsection by listing all properties proved for Ψ.

Corollary 3.1.12. *The map $\Psi : \mathcal{DF}_n \longrightarrow \mathcal{DF}_n$ is an involution with the following properties.*

(i) The image is preserved, $\mathrm{IM}(\Psi(f_\sigma)) = \mathrm{IM}(f_\sigma)$.

(ii) The set of right-to-left minima is preserved, $\mathrm{RLMv}(\Psi(f_\sigma)) = \mathrm{RLMv}(f_\sigma)$.

(iii) Whenever $f_\sigma \in \mathcal{DF}^*_n$,
$$\mathrm{sae}(\Psi(f_\sigma)) = \mathrm{sae}(f_\sigma) \pm 1.$$

We now have an involution on derangements $\hat{\Psi} : \mathfrak{D}_n \to \mathfrak{D}_n$ by setting

$$\hat{\Psi}(\sigma) := (\texttt{SEFToPerm} \circ \Psi \circ \texttt{SEFToPerm}^{-1})(\sigma), \text{ for } \sigma \in \mathfrak{D}_n.$$

Corollary 3.1.13. *The involution $\hat{\Psi}$ satisfies the properties below:*

(i) The excedance value set is preserved, $\mathrm{EXCv}(\hat{\Psi}(\sigma)) = \mathrm{EXCv}(\sigma)$.

(ii) The set of right-to-left minima is preserved, $\mathrm{RLMv}(\hat{\Psi}(\sigma)) = \mathrm{RLMv}(\sigma)$.

(iii) Whenever $\sigma \in \mathfrak{D}^*_n$, $\mathrm{sgn}(\hat{\Psi}(\sigma)) = -\mathrm{sgn}(\sigma)$.

3.2 Consequences

Before stating the main theorem, we shall first introduce two auxiliary involutions on $\mathfrak{S}_n$, and prove some of their properties. Let $\texttt{flip} : \mathfrak{S}_n \to \mathfrak{S}_n$ be the map

$$\texttt{flip}(\sigma)(k) := n + 1 - \sigma(k) \quad \text{for} \quad k \in [n],$$

and let $\zeta : \mathfrak{S}_n \to \mathfrak{S}_n$ be the composition $\zeta := \texttt{flip}^{-1} \circ (\,\cdot\,)^{-1} \circ \texttt{flip}$. In other words,

$$\zeta(\sigma)(k) := n + 1 - \sigma^{-1}(n + 1 - k) \quad \text{for} \quad k \in [n].$$

Lemma 3.2.1. *The map ζ is an involution, and*

$$\begin{aligned}
\mathrm{EXCv}(\pi) &= \{n + 1 - k : k \in \mathrm{EXCi}(\zeta(\pi))\},\\
\mathrm{RLMv}(\pi) &= \{n + 1 - k : k \in \mathrm{RLMi}(\zeta(\pi))\},\\
\mathrm{FIX}(\pi) &= \{n + 1 - k : k \in \mathrm{FIX}(\zeta(\pi))\},\\
\mathrm{inv}(\pi) &= \mathrm{inv}(\zeta(\pi)).
\end{aligned}$$

In particular, ζ restricts to a sign-preserving involution $\zeta : \mathfrak{D}_n \to \mathfrak{D}_n$.

Proof. It follows immediately from the definition that ζ is an involution. For the first property, let $i \in [n]$ and set $j \coloneqq n+1-i$. We then see that

$$i \in \mathrm{EXCv}(\pi) \text{ if and only if } n+1-\pi^{-1}(i) > n+1-i.$$

Replacing i by $n+1-j$, we have that

$$n+1-\pi^{-1}(n+1-j) > j.$$

That is, $\zeta(\pi)(j) > j$, which is equivalent with

$$i \in \{n+1-k : k \in \mathrm{EXCi}(\zeta(\pi))\}.$$

Now for the right-to-left minima, again with $j \coloneqq n+1-i$, we have

$$i \in \{n+1-k : k \in \mathrm{RLMi}(\zeta(\pi))\} \text{ if and only if } \zeta(\pi)(j) < \zeta(\pi)(t) \\ \text{whenever } j < t \leq n.$$

By definition of ζ, we have

$$\pi^{-1}(n+1-j) > \pi^{-1}(n+1-t) \text{ whenever } j < t \leq n.$$

And then change variables to get

$$\pi^{-1}(i) > \pi^{-1}(k) \text{ whenever } k \in [i-1].$$

That is, every $k \in [i-1]$ lies to the left of i in π. Thus, $i \in \mathrm{RLMv}(\pi)$.

Similarly, with $i \in [n]$, $j \coloneqq n+1-i$,

$$i \in \mathrm{FIX}(\pi) \text{ if and only if } \pi^{-1}(i) = i,$$

which is equivalent with

$$\pi^{-1}(n+1-j) = n+1-j.$$

That is, $\zeta(\pi)(j) = j$ by definition of ζ. Hence, $j \in \mathrm{FIX}(\zeta(\pi))$ and so is i.

This last property also shows that ζ is an involution on $\mathfrak{D}_n$.

Finally, we have that

$$\begin{aligned}
\mathrm{inv}(\zeta(\pi)) &= |\{(i,j) : 1 \leq i < j \leq n \text{ such that } \zeta(\pi)(i) > \zeta(\pi)(j)\}| \\
&= |\{(i,j) : 1 \leq i < j \leq n \text{ such that } \pi^{-1}(n+1-i) < \pi^{-1}(n+1-j)\}| \\
&= |\{(n+1-k, n+1-l) : 1 \leq l < k \leq n \text{ such that } \pi^{-1}(k) < \pi^{-1}(l)\}| \\
&= |\{(l', k') : 1 \leq l' < k' \leq n \text{ such that } \pi^{-1}(k') < \pi^{-1}(l')\}| \\
&= \mathrm{inv}(\pi),
\end{aligned}$$

so ζ preserves the number of inversions. In particular, ζ preserves the sign. $\square$

We are now ready to prove the main theorems in this paper.

Proof. (**Proof of Theorems 1.0.1 and 1.0.2**) By applying the involution $\hat{\Psi}$ and using all the properties listed in Corollary 3.1.13, all terms in the left-hand side of Equation (2) cancel, except the terms with $\pi \notin \mathfrak{D}_n^*$. Thus, the left-hand side of Equation (2) equals

$$\sum_{\pi \notin \mathfrak{D}_n^*} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMv}(\pi)} \mathbf{y}_{\mathrm{EXCv}(\pi)}.$$

By Lemma 3.0.1, this sum is equal to

$$\sum_{k=1}^{n-1} (-1)^{n-1} \mathbf{x}_{[k]} \mathbf{y}_{[n] \setminus [k]},$$

which is the right-hand side of Equation (2).

Let $\rho(S) \coloneqq \{n+1-s : s \in S\}$ whenever $S \subseteq [n]$. By applying the change of variables $x_i \mapsto x_{n+1-i}$, $y_j \mapsto y_{n+1-j}$ on both sides of Equation (2), we get

$$\begin{aligned}\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\rho(\mathrm{RLMv}(\pi))} \mathbf{y}_{\rho(\mathrm{EXCv}(\pi))} &= (-1)^{n-1} \sum_{j=1}^{n-1} x_n \cdots x_{n+1-j} \cdot y_{n-j} \cdots y_1 \\ &= (-1)^{n-1} \sum_{j'=1}^{n-1} x_{j'+1} \cdots x_n \cdot y_1 \cdots y_{j'}.\end{aligned}$$

Now by Lemma 3.2.1,

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\rho(\mathrm{RLMv}(\pi))} \mathbf{y}_{\rho(\mathrm{EXCv}(\pi))} = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\zeta(\pi))} \mathbf{x}_{\mathrm{RLMi}(\zeta(\pi))} \mathbf{y}_{\mathrm{EXCi}(\zeta(\pi))}.$$

Since ζ sends $\mathfrak{D}_n$ to $\mathfrak{D}_n$, the last sum must be exactly the left-hand side of Equation (3), and we are done. $\square$

Corollary 3.2.2. *By letting $x_j \to 1$ and $y_j \to t$, we have that*

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{exc}(\pi)} = (-1)^{n-1} (t + t^2 + \cdots + t^{n-1}).$$

By comparing coefficients of t^k, we get Equation (1). In a similar manner,

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{rlm}(\pi)} = (-1)^{n-1} (t + t^2 + \cdots + t^{n-1}).$$

4 A simpler proof in the excedance case

We shall first define an involution $\iota : \mathfrak{S}_n \to \mathfrak{S}_n$ such that for $\pi \in \mathfrak{S}_n$,

1. $\mathrm{EXCi}(\iota(\pi)) = \mathrm{EXCi}(\pi)$,
2. $\mathrm{sgn}(\iota(\pi)) = -\mathrm{sgn}(\pi)$ if $\iota(\pi) \neq \pi$,

3. $\operatorname{sgn}(\pi) = (-1)^{\operatorname{exc}(\pi)}$ if $\iota(\pi) = \pi$,

4. for each $E \subseteq [n-1]$, there is a unique π with $\iota(\pi) = \pi$ such that $\operatorname{EXCi}(\pi) = E$.

We shall now describe ι, which is essentially the one given in [5].

Definition 4.0.1. Define a mapping $\iota : \mathfrak{S}_n \to \mathfrak{S}_n$ by $\iota(\pi) = \pi'$, where π' is obtained from π by swapping $\pi(l)$ and $\pi(m)$, where

$$(l, m) = \max\{(i, j) : 2 \leq i < j \leq n \text{ and either } i, j \in \operatorname{EXCi}(\pi) \text{ or } i, j \notin \operatorname{EXCi}(\pi)\}$$

with respect to lexicographical order, so that $\operatorname{EXCi}(\pi') = \operatorname{EXCi}(\pi)$. It can be defined as $\iota(\pi) = (l, m)\pi$ if π is in cycle form. If there is no such (l, m), then $\iota(\pi) = \pi$ and we say that π is *critical.*

From the definition, it is clear that (1) and (2) hold. We must show that (3) and (4) hold as well, which are done in Proposition 4.0.4.

Lemma 4.0.2. *Suppose $\pi \in \mathfrak{S}_n$, and that $i, j \in \operatorname{EXCi}(\pi)$ with $i < j$. If $\pi(i) > j$, then π is not critical.*

Similarly, if $i', j' \notin \operatorname{EXCi}(\pi)$ such that $i' < j'$ with $\pi(j') \leq i'$, then π is not critical.

Proof. After swapping $\pi(i)$ and $\pi(j)$, both i and j remain excedances since $\pi(j) > j > i$ and $\pi(i) > j$. Hence, π is not critical as there is at least one pair of entries where we can perform a swap as in the definition of ι. A similar argument proves the second statement. □

Corollary 4.0.3. *Suppose $\pi \in \mathfrak{S}_n$ with $\operatorname{EXCi}(\pi) = \{j_1, \dots, j_k\}$ and $[n] \setminus \operatorname{EXCi}(\pi) = \{i_1, \dots, i_{n-k}\}$. Then, π is critical iff*

$$\begin{aligned} &j_1 < \pi(j_1) \leq j_2 < \pi(j_2) \leq j_3 < \pi(j_3) \leq \cdots \leq j_k < \pi(j_k) \text{ and} \\ &\pi(i_1) \leq i_1 < \pi(i_2) \leq i_2 < \pi(i_3) \leq i_3 < \cdots < \pi(i_{n-k}) \leq i_{n-k} = n. \end{aligned} \tag{14}$$

Moreover, if $i_{n-k-1} < j_1$, then

$$i_1 < i_2 < \cdots < i_{n-k-1} < j_1 < j_2 < \cdots < j_k < i_{n-k}$$

and it follows directly that

$$\pi = (n-k \quad n-k+1 \quad \dots \quad n-1 \quad n) \tag{15}$$

with $\operatorname{inv}(\pi) = \operatorname{exc}(\pi)$.

Proof. The forward statement follows directly from Corollary 4.0.3. Now suppose that (14) holds. Then

$$\pi(j_s) \leq j_{s'} \text{ and } i_r < \pi(i_{r'}), \text{ for } s < s' \text{ and } r < r'.$$

However, after swapping $\pi(j_s)$ and $\pi(j_{s'})$, j_s remains an excedance while $j_{s'}$ is not since $\pi(j_{s'}) > j_{s'} > j_s$ and $\pi(j_s) \leq j_{s'}$. Similarly, swapping $\pi(i_r)$ and $\pi(i_{r'})$ preserves $i_{r'}$ being an anti-excedance but not i_r since $\pi(i_r) < i_r < i_{r'}$ and $\pi(i_{r'}) > i_r$. Thus, π is critical. □

The following is similar to an argument in [5], where a slightly different[2] approach is taken.

Proposition 4.0.4. *Let* $E = \{j_1, j_2, \dots, j_k\} \subseteq [n-1]$*, and define* $\pi_E \in \mathfrak{S}_n$ *with excedance set* E *via*

$$\begin{aligned} \pi_E(j_s) &= j_s + 1 && \text{for each } j_s \in E, \\ \pi_E(i_r) &= i_{r-1} + 1 && \text{for each } i_r \in [n] \setminus E,\ (i_0 := 0). \end{aligned}$$

Then π_E *is the unique critical permutation in* $\mathfrak{S}_n$ *with* $\mathrm{EXCi}(\pi_E) = E$*, and* $\mathrm{inv}(\pi_E) = |E|$.

Proof. We first show that π_E is critical, so assume it is not. Then swapping $\pi_E(j_s)$ and $\pi_E(j_{s'})$, for $s < s'$, produces a π'_E from π_E with the same set of (anti)excedances. However, $\pi'_E(j_{s'}) = j_s + 1 < j_{s'} + 1$ implies $\pi'_E(j_{s'}) \leq j_{s'}$. So, the set of excedances is not preserved. A similar argument shows that we cannot swap a pair of anti-excedances either.

Now we have established that π_E is critical, we must show that there are no other critical permutations in $\mathfrak{S}_n$ with E as excedance set.

We proceed by (strong) induction over n. The base case $n = 1$ is trivial. And if $\pi(n) = n$, then the statement follows easily by induction hypothesis. From now on we consider $\pi^{-1}(n) < n$.

First we handle the case $|E| = n - 1$ where $E = \{1, 2, \dots, n-1\}$. There is only one permutation with E as excedance set, namely $\pi = (1\ 2\ \dots\ n)$ in cycle form, and this is exactly π_E, where $\mathrm{inv}(\pi_E) = |E|$.

Suppose now that $|E| = k < n-1$, $E = \{j_1, \dots, j_k\}$, and $\{i_1, \dots, i_{n-k}\}$ be the set of anti-excedances of some critical permutation π. Let $h \in [n]$ be the largest integer such that $j_h < i_{n-k-1}$. If there is no such h, then $E = \{n-k, \dots, n-2, n-1\}$ and π is of the form given in (15) and the statement holds. So now, there is some $m \in [n-k]$ such that $j_h + 1 = i_{n-k-m}$, and the permutation π has the following structure:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & j_h & i_{n-k-m} & \cdots & i_{n-k-1} & j_{h+1} & \cdots & j_k & n \\ \pi(1) & \pi(2) & \cdots & \pi(j_h) & \pi(i_{n-k-m}) & \cdots & \pi(i_{n-k-1}) & \pi(j_{h+1}) & \cdots & \pi(j_k) & \pi(n) \end{pmatrix}.$$

We have that

$$j_{h+1} < \pi(j_{h+1}), \quad j_{h+2} < \pi(j_{h+2}), \quad j_k < \pi(j_k),$$

and taking (14) into account, this is only possible if

$$\pi(j_{h+1}) = j_{h+2}, \quad \pi(j_{h+2}) = j_{h+3}, \quad \dots, \quad \pi(j_k) = n,$$

or $h = k$ and $\pi(j_h) = n$.

Suppose now $\pi(j_h) > i_{n-k-1}$. Then by (14), either $\pi(j_h) = j_{h+1}$, or $h = k$ and $\pi(j_h) = n$. In either case, we must have that $\pi(i_{n-k}) \leq j_{h+1} - 1 = i_{n-k-1}$ then $\pi(n) = i_{n-k-1}$ by (14), and $\pi(i_{n-k-1}) < i_{n-k-1}$. But this is not possible, as π would

[2] We note that the original proof has a few typographical errors.

not be critical (we could swap the values at positions $n-k-1$ and n). Hence, $\pi(j_h) \leq i_{n-k-1}$.

It follows that π sends $\{1, 2, \dots, i_{n-k-1}\}$ to itself and that $\pi(n) = i_{n-k-1} + 1$. Hence, the value of $\pi(s)$ is uniquely determined for all $s > i_{n-k-1}$. So, π restricts to a critical permutation π' acting on $[i_{n-k-1}]$. By induction, π' is uniquely determined by $E \cap [i_{n-k-1}]$ with so it follows that π is unique and of the form π_E. Also by induction, $\mathrm{inv}(\pi') = h = |E \cap [i_{n-k-1}]|$, and finally

$$\mathrm{inv}(\pi) = \mathrm{inv}(\pi') + (k-h) = h + (k-h) = |E|.$$

□

Example 4.0.5. Let $n = 4$ and consider all permutations with two excedances. We have 7 even permutations with two excedances, and 4 odd permutations. The sign-reversing involution should therefore have $\binom{3}{2} = 3$ fixed-points, all with even sign $(-1)^2$.

Even	Odd
1342	
2143	
2314	
2431	2413
3241	3142
3412	3421
4321	4312

Proposition 4.0.4 now immediately gives a bijective proof of the following result, which is essentially due to Mantaci in [5] (albeit stated in terms of anti-excedances instead of excedances).

Proposition 4.0.6. *Let $n \geq 1$, then*

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \prod_{j \in [n-1]} (1 - x_j) = \sum_{E \subseteq [n-1]} (-1)^{|E|} \mathbf{x}_E. \tag{16}$$

In particular, by setting all x_i equal to t, we have

$$\sum_{\pi \in \mathfrak{S}_n^e} t^{\mathrm{exc}(\pi)} - \sum_{\pi \in \mathfrak{S}_n^o} t^{\mathrm{exc}(\pi)} = (1-t)^{n-1}.$$

Proposition 4.0.7. *Let $n \geq 1$ and let $T \subseteq [n]$. Let $m \leq n$ be the largest integer not in T and set $E = \{1, 2, \dots, m-1\} \setminus T$. Then*

$$\sum_{\substack{\pi \in \mathfrak{S}_n \\ T \subseteq \mathrm{FIX}(\pi)}} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \prod_{j \in E} (1 - x_j), \tag{17}$$

where the empty product has value 1.

Setting all x_i to be t, we have

$$\sum_{\substack{\pi\in\mathfrak{S}_n^e \\ T\subseteq \mathrm{FIX}(\pi)}} t^{\mathrm{exc}(\pi)} - \sum_{\substack{\pi\in\mathfrak{S}_n^o \\ T\subseteq \mathrm{FIX}(\pi)}} t^{\mathrm{exc}(\pi)} = \begin{cases} 1 & \text{if } |T| = n \\ (1-t)^{n-1-|T|} & \text{otherwise.} \end{cases} \tag{18}$$

Proof. First note that $E = \emptyset$ if $T = [n]$ and (17) is easy to verify, so from now on, we may assume $|T| < n$.

By definition of m, we have that $T = T_1 \cup T_2$ where $T_1 \subseteq \{1, 2, \dots, m-1\}$, and $T_2 = \{m+1, m+2, \dots, n\}$. Hence, $|E| + |T_1| = m - 1$ and $|T_2| = n - m$, and

$$|E| = n - 1 - |T_1| - |T_2| = n - 1 - |T|.$$

Now suppose $\pi \in \mathfrak{S}_n$ is a permutation such that $T \subseteq \mathrm{FIX}(\pi)$. We then construct $\pi' \in \mathfrak{S}_{n-|T|}$, by only considering the positions not in T, and the relative ordering of the entries at these positions. For example, if $n = 9$, $T = \{2, 4, 6, 8, 9\}$, $[n] \setminus T = \{1, 3, 5, 7\}$ and

$$\pi = \underline{1}2\underline{7}4\underline{3}6\underline{5}89, \text{ we have } \pi' = 1423$$

since the relative ordering of $1, 3, 5$ and 7 in π is 1423. Observe that $\mathrm{exc}(\pi) = \mathrm{exc}(\pi')$ and $(-1)^{\mathrm{inv}(\pi)} = (-1)^{\mathrm{inv}(\pi')}$.

Hence, the sum in the left-hand side of Equation (17) can be taken as a sum over permutations $\pi' \in \mathfrak{S}_{n-|T|}$, but with a reindexing of the variables using values in $[n] \setminus T$. Now, this sum can be computed using Proposition 4.0.6 which finally gives Equation (17). Note that m is the largest member of $[n] \setminus T$, so we do not get any variable with this index — this corresponds to the fact that the right-hand side of Equation (16) only uses elements in $[n-1]$. □

A generalized proof of the following theorem is provided in [8, Thm. 7].

Theorem 4.0.8. *Let $n \geq 1$. Then*

$$\sum_{\pi\in\mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 x_2 \cdots x_j. \tag{19}$$

Proof. By inclusion-exclusion, we have the two identities:

$$\sum_{\substack{\pi\in\mathfrak{S}_n^e \\ \mathrm{FIX}(\pi)=\emptyset}} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \sum_{T\subseteq[n]} (-1)^{|T|} \sum_{\substack{\pi\in\mathfrak{S}_n^e \\ T\subseteq \mathrm{FIX}(\pi)}} \mathbf{x}_{\mathrm{EXCi}(\pi)},$$

$$\sum_{\substack{\pi\in\mathfrak{S}_n^o \\ \mathrm{FIX}(\pi)=\emptyset}} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \sum_{T\subseteq[n]} (-1)^{|T|} \sum_{\substack{\pi\in\mathfrak{S}_n^o \\ T\subseteq \mathrm{FIX}(\pi)}} \mathbf{x}_{\mathrm{EXCi}(\pi)}.$$

By taking the difference of these two identities, we get

$$\sum_{\pi\in\mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{EXCi}(\pi)} = \sum_{T\subseteq[n]} (-1)^{|T|} \left(\sum_{\substack{\pi\in\mathfrak{S}_n^e \\ T\subseteq \mathrm{FIX}(\pi)}} \mathbf{x}_{\mathrm{EXCi}(\pi)} - \sum_{\substack{\pi\in\mathfrak{S}_n^o \\ T\subseteq \mathrm{FIX}(\pi)}} \mathbf{x}_{\mathrm{EXCi}(\pi)} \right).$$

By Proposition 4.0.7, the difference in the right-hand side is equal to

$$\prod_{j \in [m_T-1] \setminus T} (1-x_j)$$

where $m_T \leq n$ is the largest integer not in T. We group the terms depending on the value of m_T. If $m_T = 0$ then $T = [n]$ and the product is empty, so its value is 1. In general, the left hand side of Equation (19) is equal to

$$(-1)^n + \sum_{k=1}^{n} \sum_{\substack{T \subseteq [n] \\ m_T = k}} (-1)^{|T|} \prod_{j \in [k-1] \setminus T} (1-x_j).$$

By using $E = [k-1] \setminus T$, this can then be expressed as

$$(-1)^n + \sum_{k=1}^{n} \sum_{E \subseteq [k-1]} (-1)^{n-1-|E|} \prod_{j \in E} (1-x_j).$$

Canceling the $k = 1$ case with $(-1)^n$, and then shifting the index, we get

$$(-1)^{n-1} \sum_{k=1}^{n-1} \sum_{E \subseteq [k]} \prod_{j \in E} (x_j - 1).$$

Now,

$$\begin{aligned}(-1)^{n-1} \sum_{k=1}^{n-1} \sum_{E \subseteq [k]} \prod_{j \in E} (x_j - 1) &= (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{E \subseteq [k]} \sum_{F \subseteq E} (-1)^{|E|-|F|} \mathbf{x}_F \\ &= (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{F \subseteq [k]} (-1)^{|F|} \mathbf{x}_F \sum_{F \subseteq E \subseteq [k]} (-1)^{|E|}.\end{aligned}$$

The last sum vanish unless $F = [k]$, and we have that

$$(-1)^{n-1} \sum_{k=1}^{n-1} \sum_{E \subseteq [k]} \prod_{j \in E} (x_j - 1) = (-1)^{n-1} \sum_{k=1}^{n-1} \mathbf{x}_{[k]}, \tag{20}$$

which is exactly the right-hand side in Equation (19). □

Corollary 4.0.9. *For $n, k \geq 1$, we have that*

$$|\{\pi \in \mathfrak{D}_n^e : \operatorname{exc}(\pi) = k\}| - |\{\pi \in \mathfrak{D}_n^o : \operatorname{exc}(\pi) = k\}| = (-1)^{n-1}.$$

Proof. This follows directly by comparing coefficients of degree k in Equation (19). □

4.1 A right-to-left minima analog

Our goal with this subsection is to prove a right-to-left minima analog of Proposition 4.0.6. We shall use the same type of proof, i.e., exhibit an involution on $\mathfrak{S}_n$, such that all fixed-elements with the same set of right-to-left minima, also have the same sign.

Definition 4.1.1. Let $\kappa : \mathfrak{S}_n \to \mathfrak{S}_n$ be defined as follows. Given $\pi \in \mathfrak{S}_n$, let $i \in [n]$ be the smallest *odd* integer such that $\pi(i\ i{+}1)$ and π have the same sets of right-to-left minima, if such an i exists. That is, we swap the entries at positions i and $i+1$ in π. We then set $\kappa(\pi) \coloneqq \pi(i\ i{+}1)$, and $\kappa(\pi) \coloneqq \pi$ otherwise. We say that π is *decisive*[3] if it is a fixed-point of κ.

Example 4.1.2. In $\mathfrak{S}_7$, there are 8 decisive permutations:

$$1234567,\ 1234657,\ 1243567,\ 1243657,\ 2134567,\ 2134657,\ 2143567,\ 2143657.$$

Note that $\{1, 3, 5, 7\}$ are always right-to-left minima (but there might be more).

Lemma 4.1.3. *The map $\kappa : \mathfrak{S}_n \to \mathfrak{S}_n$ has the following properties;*

(i) κ is an involution,

(ii) κ preserves the number of right-to-left minima,

(iii) κ changes sign of non-fixed elements,

(iv) for each subset $T \subseteq [n] \cap \{2, 4, 6, \dots\}$, there is a unique decisive permutation with $\{1, 3, 5, \dots\} \cup T$ as right-to-left minima set, and

(v) there are $\binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil}$ decisive permutations with exactly k right-to-left minima, and they all have sign $(-1)^{n-k}$.

Proof. Items (i)–(iii) are clear from the definition of κ. It remains to prove (iv) and (v). Let us use O to denote the odd integers in $[n]$, and let E be the even integers in $[n]$. In order to prove (iv), we must construct a decisive permutation π, such that $\mathrm{RLMv}(\pi) = O \cup T$. We construct π from T according to the following rules:

- if n is odd, then $\pi(n) = n$.
- if $j \in O$, $j < n$ we have that

$$\begin{cases} \pi(j) = j \text{ and } \pi(j+1) = j+1 & \text{if } j+1 \in T \\ \pi(j) = j+1 \text{ and } \pi(j+1) = j & \text{if } j+1 \notin T. \end{cases}$$

[3] As a nod to the word *critical*.

In short, π is constructed by first placing 1 and 2, in the order determined by T, then 3 and 4, etc. By construction, $\mathrm{RLMv}(\pi) = O \cup T$. Now we must show that a permutation is decisive if and only if it is of this form. From the construction, it is clear that $\pi(i\, i+1)$ and π do not have the same set of right-to-left minima, for any choice of $i \in O$. Hence, all permutations with this structure are decisive.

Claim: *Every decisive permutation has the structure described above.*

First note that the claim is true for $n = 1$ and $n = 2$, so suppose $n \geq 3$. Now, if n does not appear among the last two entries of π, then n is not a right-to-left minima. Moreover, we can swap n with the entry either to its right or to its left, and preserve the set of right-to-left minima. In particular, if n does not appear among the last two positions, then π is not decisive. Now, if $\pi(n) = n$, we can remove the last entry and use induction. Otherwise, suppose $\pi(n-1) = n$ and $\pi(n) < n-1$. In particular, $n-1 \notin \mathrm{RLMv}(\pi)$. It is then possible to swap $n-1$ with one of its neighbors and preserve $\mathrm{RLMv}(\pi)$ in a manner, which shows that π is not decisive. We conclude that if $\pi(n) \neq n$, then $\pi(n-1) = n$ and $\pi(n) = n-1$ in order for π to be decisive. Now, we may remove the last two entries, and proceed by induction. This ends the proof of the claim.

From (iv), we know that in order to construct a decisive permutations in $\mathfrak{S}_n$ with k right-to-left minima, we must include all $\lceil n/2 \rceil$ odd integers in $[n]$, and pick a subset of size $k - \lceil n/2 \rceil$, from the set of even integers in $[n]$. The subset of even integers has cardinality $\lfloor n/2 \rfloor$. Hence, we get the advertised formula in (v), so it suffices to show that all such decisive permutations have the same sign. By the claim above, it is evident that the sign only depends on the number of right-to-left minima in π, and from here, it is straightforward to show that it is indeed $(-1)^{n-k}$. □

Corollary 4.1.4. *We have that for any $n \geq 1$*

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMv}(\pi)} = \Big(\prod_{\substack{i \in [n] \\ i \text{ odd}}} x_i \Big) \Big(\prod_{\substack{j \in [n] \\ j \text{ even}}} (x_j - 1) \Big). \tag{21}$$

In particular, for any $k = 1, \dots, n$ we have that

$$|\{\pi \in \mathfrak{S}_n^e : \mathrm{rlm}(\pi) = k\}| - |\{\pi \in \mathfrak{S}_n^o : \mathrm{rlm}(\pi) = k\}| = (-1)^{n-k} \binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil}.$$

Proof. This follows directly from Lemma 4.1.3, where the first product in Equation (21) corresponds to the fact that all odd integers in $[n]$ must be right-to-left minima for decisive permutations, and the second product corresponds to choosing a subset T among the even numbers in $[n]$. It remains to check that the signs are chosen correctly, which is straightforward as well.

The second statement also follows from Lemma 4.1.3, or by simply comparing coefficients of t^k in Equation (21), after letting $x_j \to t$. □

We conclude this section with the following problem.

Problem 4.1.5. Is it possible to state an analog of Proposition 4.0.7? In particular, for $T \subseteq [n]$, is there a nice expression for the sum

$$\sum_{\substack{\pi \in \mathfrak{S}_n \\ T \subseteq \mathrm{FIX}(\pi)}} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{rlm}(\pi)}?$$

Computer experiments suggest that this sum is either 0 or of the form $\pm t^a(t+1)^b(t-1)^c$, where a, b, and c depend on T in some manner.

5 Further ideas and conjectures

5.1 Multiderangements

Let $B_n \coloneqq (1,1,2,2,3,3,\dots,n,n)$ be fixed. A *biderangement* of B_n, is a permutation, $\mathbf{w}$, of the entries in B_n, such that $\mathbf{w}(j) \neq B_n(j)$ for all $j \in [2n]$. The set of biderangements of B_n is denoted $\mathfrak{BD}_n$. The cardinality of $\mathfrak{BD}_n$ is given by A000459, which starts as $0, 1, 10, 297, 13756, \dots$.

We compute the number of *inversions* of a biderangements as for words in general. Moreover, we say that $\mathbf{w}(j)$ is an *excedance value* of $\mathbf{w} \in \mathfrak{BD}_n$ if $\mathbf{w}(j) > B_n(j)$. This defines the multi-set valued statistic $\mathrm{EXCv}(\mathbf{w})$. We also define the *right-to-left minima values* as the set

$$\mathrm{RLMv}(\mathbf{w}) \coloneqq \{\mathbf{w}_i : \mathbf{w}(i) < \mathbf{w}(j) \text{ for all } j \in \{i, i+1, \dots, 2n\}\}.$$

Example 5.1.1. In the table below, we show the ten elements in $\mathfrak{BD}_3$, together with the corresponding inversion and excedance statistics.

Biderangement	inv	EXCv	RLMv	Biderangement	inv	EXCv	RLMv
223311	8	{2,2,3,3}	{1}	231312	7	{2,3,3}	{1,2}
231321	8	{2,3,3}	{1}	233112	8	{2,3,3}	{1,2}
233121	9	{2,3,3}	{1}	321312	8	{2,3,3}	{1,2}
321321	9	{2,3,3}	{1}	323112	9	{2,3,3}	{1,2}
323121	10	{2,3,3}	{1}	331122	8	{3,3}	{1,2}

Proposition 5.1.2. *For $n \geq 1$, we have that*

$$\sum_{\mathbf{w} \in \mathfrak{BD}_n} (-1)^{\mathrm{inv}(\mathbf{w})} \mathbf{x}_{\mathrm{EXCv}(\mathbf{w})} \mathbf{y}_{\mathrm{RLMv}(\mathbf{w})} = \sum_{\pi \in \mathfrak{D}_n} \mathbf{x}_{\mathrm{EXCv}^2(\pi)} \mathbf{y}_{\mathrm{RLMv}(\pi)}, \tag{22}$$

where $\mathrm{EXCv}^2(\pi)$ is the multiset obtained from $\mathrm{EXCv}(\pi)$ by repeating each element twice.

Proof. Define a mapping $\beta : \mathfrak{BD}_n \to \mathfrak{BD}_n$ as $\beta(\mathbf{w}) = \mathbf{w}'$, where $\mathbf{w}'$ is obtained from $\mathbf{w}$ by switching $\mathbf{w}(j)$ and $\mathbf{w}(j+1)$ for the smallest *odd* $j \in [2n-1]$ such that $\mathbf{w}(j) \neq \mathbf{w}(j+1)$. If there is no such j, then $\beta(\mathbf{w}) \coloneqq \mathbf{w}$. Since $\mathbf{w}'(j) = \mathbf{w}(j+1) \neq B_n(j+1) = B_n(j)$ and $\mathbf{w}'(j+1) = \mathbf{w}(j) \neq B_n(j) = B_n(j+1)$, β is a sign-reversing involution which preserves the excedance set values; $\mathrm{EXCv}(\mathbf{w}) = \mathrm{EXCv}(\mathbf{w}')$. Now, each number appears twice in a biderangement, so $\mathbf{w}(j)$ and $\mathbf{w}(j+1)$ each appear again somewhere to the right of j and $j+1$, respectively. Hence, the positions j and $j+1$ in $\mathbf{w}$ and $\mathbf{w}'$ cannot be right-to-left minima indices, and it follows that $\mathrm{RLMv}(\mathbf{w}) = \mathrm{RLMv}(\mathbf{w}')$.

The elements fixed by β are in bijection with the derangements; we send $\pi \in \mathfrak{D}_n$ to the biderangement $\mathbf{v} = \pi(1)\pi(1)\pi(2)\pi(2)\cdots\pi(n)\pi(n)$. Consequently, $\mathrm{EXCv}(\mathbf{v})$ is the multiset obtained from $\mathrm{EXCv}(\pi)$ by repeating each element twice, and $\mathrm{RLMv}(\mathbf{v}) = \mathrm{RLMv}(\pi)$. This concludes the proof of (22). □

5.2 A generalization

The set of derangements are permutations where cycles of length 1 are disallowed. With this in mind, it is reasonable to explore what happens if we add restrictions to the length of the cycles. Recall that the *type*, $\mathrm{type}(\pi)$ of a permutation is the integer partition $(\mu_1, \mu_2, \dots, \mu_\ell)$ where the parts are the cycle lengths of π arranged in decreasing order.

Conjecture 5.2.1. *Let k be a fixed positive integer such that $2 \leq k \leq n$. Then*

$$(-1)^{n-1} \sum_{\substack{\pi\in\mathfrak{S}_n \\ \min(\mathrm{type}(\pi))\geq k}} (-1)^{\mathrm{inv}(\pi)} \mathbf{x}_{\mathrm{RLMv}(\pi)} \mathbf{y}_{\mathrm{EXCv}(\pi)},$$

where $\mathrm{type}(\pi)$*, the cycle type of* π*, is an element in* $\mathbb{N}[x_1, \dots, x_n, y_1, \dots, y_n]$.

The case $k = 2$ follows from Theorem 1.0.1 (the conjecture is not true for $k = 1$). For the case $k = n$, all permutations have the same sign, so the statement is trivial. Interestingly, summing over all permutations consisting of a single cycle of length n,

$$\sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{type}(\pi)=(n)}} \mathbf{x}_{\mathrm{RLMv}(\pi)} \mathbf{y}_{\mathrm{EXCv}(\pi)} \tag{23}$$

is a multivariate polynomial where the number of terms (not counting multiplicity!) seems to be given by the sequence A124302, which starts as

$$1, 1, 2, 5, 14, 41, 122, 365, 1094, \dots,$$

[10]. For $n = 3$, (23) is equal to

$$x_1x_2x_3y_4 + x_1x_3y_2y_4 + x_1y_3y_4 + 2x_1x_2y_3y_4 + x_1y_2y_3y_4$$

which has 5 terms.

	1	**2**	**3**	**4**	**5**	**6**	**7**
2	1						
3	1	1					
4	3	5	1				
5	11	21	11	1			
6	53	113	79	19	1		
7	309	715	589	211	29	1	
8	2119	5235	4835	2141	461	41	1

Table 2: The number of derangements of $[n]$ with exactly k right-to-left minima.

5.3 Right to left minima and derangements

Let $a_{n,k}$ be defined via

$$\sum_{\pi \in \mathfrak{D}_n} t^{\mathrm{rlm}(\pi)} = \sum_{k=1}^{n} a_{n,k} t^k, \tag{24}$$

so that $a_{n,k}$ is the number of derangements with exactly k right-to-left minima. For example, the data for $a_{n,k}$ is shown in the table below.

It is straightforward from the definition to see that $a_{n,1}$ is the number of derangements ending with a 1. The sequence $a_{n,1}$ shows up in [10] as A000255, which hints at the recursion

$$a_{n,1} = (n-2) \cdot a_{n-1,1} + (n-3) \cdot a_{n-2,1}, \qquad a_{1,1} = 1, \qquad a_{2,1} = 1.$$

Moreover, it seems that $a_{n,n-1} = (n-2) + (n-1)^2$. The data in Table 2 is not in the OEIS, and we leave it as an open problem to describe the entries via recursions or closed-form formulas.

A straightforward recursion for the number of elements in $\mathfrak{D}_n$ with k excedances, can be found in [6].

Remark 5.3.1. In a recent preprint, [4] Pei and Zeng improve our results and give a refined and shorter proof of Theorem 1.0.1 and Theorem 1.0.2. They also prove a type B analog of Proposition 4.0.6 and Corollary 4.1.4. Moreover, they provided a proof for the recursion of $a_{n,1}$.

Acknowledgements

The authors have benefited greatly from the discussions with Jörgen Backelin of Stockholm University. The Online Encyclopedia of Integer sequences [10] has been an invaluable resource in our project.

The second author is grateful for the financial support extended by the cooperation agreement between International Science Program at Uppsala University and Addis Ababa University.

Finally, we thank the anonymous referees for the constructive comments and pointing out additional relevant references.

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(Received 29 Mar 2022; revised 29 Sep 2022, 3 Nov 2022, 13 May 2023)