AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **86(1)** (2023), Pages 206–221

On the number of orthomorphisms of the cyclic group of order nine

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Abstract

It is well-known that the number of normalized orthomorphisms of $\mathbb{Z}_9$ is 225. However, this number has not been determined theoretically. We will give a theoretical proof of this result. We will also indicate how the classification and analysis used can be applied to other groups.

1 Introduction

An *orthomorphism* of a group G is a permutation θ of G for which the mapping $x \mapsto x^{-1}\theta(x)$ is also a permutation: the mapping $x \mapsto x^{-1}\theta(x)$ is a *complete mapping* of G. An orthomorphism θ of G is *normalized* if $\theta(1) = 1$. Note that, if θ is an orthomorphism of G, then the mapping $x \mapsto \theta(x)\theta(1)^{-1}$ is a normalized orthomorphism of G and that, if G has m normalized orthomorphisms, then G has $m|G|$ orthomorphisms. Orthomorphisms of a group G correspond to transversals of the Cayley table of G, a Latin square. See [12] and [17] for the enumeration of transversals of small Latin squares, which includes the enumeration of orthomorphisms of small groups.

The number of orthomorphisms of small groups is known, mainly as the result of computer searches: see Table 13.1 in [5], Table 1 in [12], or Table 4 in [17]. Groups with non-trivial, cyclic Sylow 2-subgroups have no orthomorphisms by the Hall-Paige theorem [7]. Groups of order five or less are easily dealt with by hand, and, by enumerating the transversals of the cyclic Latin square of order seven, Euler [4] implicitly showed that $\mathbb{Z}_7$ has 19 normalized orthomorphisms. Each of the non-cyclic groups of order eight has 48 normalized orthomorphisms. A theoretical proof of this was given in [1]. For the group $GF(8)^+$, all of its orthomorphisms can be constructed using permutation polynomials, as described in Section 9.2 of [5]; or by solving systems of linear equations, as described in Sections 10.3 and 10.4 of [5].

The cyclic group of order nine is the smallest group for which the number of orthomorphisms has not been determined theoretically. In [2], using the "method of exhaustion," $\mathbb{Z}_9$ was found to have 225 normalized orthomorphisms. This same

ISSN: 2202-3518

result has been found many times using computer searches: see [10], [11], [15], and [16]. A complete list of these orthomorphisms can be found in [9] and, as presentation functions of neofields, in [8].

In Section 2 we will discuss collapsed permutation matrices, in Section 3 we will present a theoretical proof that $\mathbb{Z}_9$ has 225 normalized orthomorphisms, and in Section 4 we will indicate how the methods used in Section 3 might be applied to other groups, and, in particular, how these methods can be used to show that $GF(9)^+$, the additive group of the field of order 9, has 249 normalized orthomorphisms.

2 Collapsed permutation matrices

In this section, all groups considered will be abelian and written additively. We will introduce the concept of a collapsed permutation matrix, a concept that will prove crucial in counting the number of normalized orthomorphisms of $\mathbb{Z}_9$.

Let $G = \{g_1, \dots, g_n\}$ be an abelian group. Any permutation θ of G can be represented by a permutation matrix $M_\theta = (m_{ij})$, where

$$m_{ij} = \begin{cases} 1 & \text{if } \theta(g_j) = g_i, \\ 0 & \text{otherwise.} \end{cases}$$

The g_k*th diagonal* of M_θ is the set of cells $\{(i,j) \mid g_i - g_j = g_k\}$.

Permutation matrices corresponding to orthomorphisms are easily characterized.

Lemma 2.1 *Let θ be a permutation of an abelian group $G = \{g_1, \dots, g_n\}$. Then θ is an orthomorphism of G if and only if each row, column, and diagonal of M_θ contains exactly one* 1.

Proof: Suppose that θ is an orthomorphism of G. Then, for all $j = 1, \dots, n$, $\theta(g_j)$ is uniquely determined and so every column of M_θ contains exactly one 1. Also, for all $i = 1, \dots, n$, there is a unique $j \in \{1, \dots, n\}$ for which $\theta(g_j) = g_i$ and so every row of M_θ contains exactly one 1. Further, for all $k = 1, \dots, n$, there is a uniquely determined $j \in \{1, \dots, n\}$ for which $g_k = \theta(g_j) - g_j = g_i - g_j$ and so every diagonal of M_θ contains exactly one 1.

The converse is straightforward. □

If θ is an orthomorphism of G, then M_θ is an *orthomorphism matrix* of G, and a $(0,1)$-matrix in which each row, column, and diagonal of M contains at most one 1 is a *partial orthomorphism matrix of* G.

If a permutation matrix M can be written as a block matrix $M = (A_{ij})$, then the corresponding *collapsed permutation matrix* is $CM(a_{ij})$, where a_{ij} is the number of 1s in A_{ij}. In particular, let G be an abelian group, $|G| = mt$, let $H = \{h_1, \dots, h_m\}$ be a subgroup of G, and let $D = \{d_1, \dots, d_t\}$ be a system of distinct coset representatives for H in G. Then

$$G = \{h_1 + d_1, \dots, h_m + d_1; h_1 + d_2, \dots, h_m + d_2; \dots; h_1 + d_t, \dots, h_m + d_t\},$$

and any permutation θ of G has a corresponding permutation matrix $M_\theta = (A_{ij})$, where A_{ij} is formed from the rows $h_1+d_i, \ldots, h_m+d_i$ and columns $h_1+d_j, \ldots, h_m+d_j$ of M_θ. The corresponding collapsed permutation matrix is $CM_\theta = (a_{ij})$, where $a_{ij} = |\{g_k \mid g_k \in H+d_j, \theta(g_k) \in H+d_i\}|$: we will call CM_θ the *collapsed permutation matrix of θ with respect to H (and D).*

When $t = 3$ the collapsed permutation matrix of θ has a simple form.

Lemma 2.2 *If H is a subgroup of an abelian group G of index 3, $|H| = m$, D a system of distinct coset representatives for H in G, and θ is an orthomorphism of G, then the collapsed permutation matrix of θ with respect to H (and D) is*

$$CM_\theta = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

for some a, b, c, $a + b + c = m$.

Proof: Let the collapsed permutation matrix of θ with respect to H be $CM_\theta = (a_{ij})$, $i, j = 1, 2, 3$. As M_θ is a permutation matrix,

$$a_{i1} + a_{i2} + a_{i3} = m \quad \text{for } i = 1, 2, 3,$$

and

$$a_{1j} + a_{2j} + a_{3j} = m \quad \text{for } j = 1, 2, 3.$$

As each diagonal of M_θ contains exactly one 1,

$$a_{11} + a_{22} + a_{33} = m,$$

$$a_{12} + a_{23} + a_{31} = m,$$

and

$$a_{13} + a_{21} + a_{32} = m.$$

Solving this system of linear equations we find that

$$a_{11} = a_{23} = a_{32} = a,$$

$$a_{12} = a_{21} = a_{33} = b,$$

and

$$a_{13} = a_{22} = a_{31} = c,$$

for some a, b, c, $a + b + c = m$. □

If θ is an orthomorphism of an abelian group G, H a subgroup of G of index 3, D a system of distinct coset representatives for H in G, and the collapsed permutation matrix of θ with respect to H (and D) is

$$CM_\theta = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

then we will say that θ is in the *class abc*, and $|abc|$ will denote the number of normalized orthomorphisms in the class abc.

3 The group $\mathbb{Z}_9$

In this section we will only consider the group $G = \mathbb{Z}_9 = \{0, \ldots, 8\}$. Now $H = \{0, 3, 6\}$ is a subgroup of G of index 3, and $(d_1, d_2, d_3) = (0, 1, 2)$ is a system of distinct coset representatives for H in G. Let us order the elements of G as $\{0, 3, 6; 1, 4, 7; 2, 5, 8\}$. Each normalized orthomorphism of G belongs to one of the classes 300, 210, 201, 120, 102, or 111, and can be represented by an orthomorphism matrix of the form

$$M_\theta = (A_{ij}) = \begin{pmatrix} A_0 & B_2 & C_1 \\ B_1 & C_0 & A_2 \\ C_2 & A_1 & B_0 \end{pmatrix},$$

$i, j = 0, 1, 2$. For X one of A, B, C, it is easy to see that each cell of X_k is on the $(h+k)$th diagonal for some $h \in H$. Let us form a matrix X'_k from X_k by making each entry in a cell of X'_k the diagonal the corresponding cell of X_k is on. Thus

$$X'_k = \begin{pmatrix} 0+k' & 2+k' & 1+k' \\ 1+k' & 0+k' & 2+k' \\ 2+k' & 1+k' & 0+k' \end{pmatrix} = X'_0 + k'J,$$

where, if $X_k = A_{ij}$, then $k' = i - j$ and $k' \in H + k$: J is the all 1s matrix. We will say that the corresponding entry of CM_θ is on the kth *diagonal* of CM_θ. Each row, column, and diagonal of X_k contains at most one 1, and, as X_0 is a partial orthomorphism matrix of H, by abuse of terminology we will refer to each matrix X_k as a partial orthomorphism matrix of H.

Among the class of mappings that permute the set of normalized orthomorphisms of a group G are the *homologies* H_α, $\alpha \in \mathrm{Aut}(G)$, defined by $H_\alpha[\theta] = \alpha\theta\alpha^{-1}$, the *translations* T_g, $g \in G$, defined by $T_g[\theta](x) = \theta(x+g) - \theta(g)$, and the *reflection* defined by $R[\theta](x) = x + \theta(-x)$: these mappings are discussed in Section 8.1.3 of [5]. The reflection and homologies prove useful in reducing the number of classes that we need to enumerate as demonstrated in the following lemma.

Lemma 3.1 $|210| = |201|$ *and* $|120| = |102|$.

Proof: Let $\alpha \in \mathrm{Aut}(G)$ be defined by $\alpha(g) = 2g$. Note that $\alpha(h) = \alpha^{-1}(h) = -h$ for all $h \in H$.

If θ is a normalized orthomorphism in the class 210, then, for some $h_1, h_2 \in H$, $h_1 \neq 0$, $\theta(0) = 0$, $\theta(h_1) = -h_1$, and $\theta(-h_1) = h_2 + 1$. Now $H_\alpha[\theta](0) = 0$, $H_\alpha[\theta](h_1) = -h_2 + 2$, and $H_\alpha[\theta](-h_1) = h_1$. Hence $H_\alpha[\theta]$ is in the class 201 and thus $|210| \leq |201|$. If θ is a normalized orthomorphism in the class 201, then, for some $h_1, h_2 \in H$, $h_1 \neq 0$, $\theta(0) = 0$, $\theta(h_1) = -h_1$, and $\theta(-h_1) = h_2 + 2$. Now $H_\alpha[\theta](0) = 0$, $H_\alpha[\theta](h_1) = (-h_2 + 3) + 1$, and $H_\alpha[\theta](-h_1) = h_1$. Hence $H_\alpha[\theta]$ is in the class 210 and thus $|201| \leq |210|$. Hence $|210| = |201|$.

If θ is a normalized orthomorphism in the class 120, then, for some $h_1, h_2, h_3 \in H$, $h_1 \neq 0$, $\theta(0) = 0$, $\theta(h_1) = h_2 + 1$, and $\theta(-h_1) = h_3 + 1$. Now $H_\alpha[\theta](0) = 0$,

$H_\alpha[\theta](h_1) = -h_3 + 2$, and $H_\alpha[\theta](-h_1) = -h_2 + 2$. Hence $H_\alpha[\theta]$ is in the class 102 and thus $|120| \le |102|$. If θ is a normalized orthomorphism in the class 102, then, for some $h_1, h_2, h_3 \in H$, $h_1 \ne 0$, $\theta(0) = 0$, $\theta(h_1) = h_2 + 2$, and $\theta(-h_1) = h_3 + 2$. Now $H_\alpha[\theta](0) = 0$, $H_\alpha[\theta](h_1) = (-h_3 + 3) + 1$, and $H_\alpha[\theta](-h_1) = (-h_2 + 3) + 1$. Hence $H_\alpha[\theta]$ is in the class 120 and thus $|102| \le |120|$. Hence $|120| = |102|$. □

For the class 111, set $C(h_1, h_2, h_3) = \{\theta \in \text{ class } 111 \mid \theta(0) = 0, \theta(h_1) = h_2 + 1, \theta(-h_1) = h_3 + 2\}$, $h_1, h_2, h_3 \in H$, $h_1 \ne 0$. Homologies, translations, and the reflection can be used to reduce the problem of determining $|111|$ to that of determining $|C(3, 0, 0)|$.

Lemma 3.2 $|111| = 18 \times |C(3, 0, 0)|$.

Proof: The sets $C(h_1, h_2, h_3)$, $h_1, h_2, h_3 \in H$, $h_1 \ne 0$, partition the set of normalized orthomorphisms in the class 111.

We will show that $|C(h_1, h_2, h_3)| = |C(3, 0, 0)|$, for all $h_1, h_2, h_3 \in H$, $h_1 \ne 0$. To do this we will first show that $|C(3, h_2, h_3)| = |C(3, 0, 0)|$, for all $h_2, h_3 \in H$.

Let $\alpha \in \mathrm{Aut}(G)$ be defined by $\alpha(x) = 2x$. Simple computation shows that:

$H_\alpha[C(3, 0, 3)] = C(3, 0, 0)$,	$T_6[C(3, 0, 6)] = C(3, 0, 0)$,
$H_{\alpha^{-1}}[C(3, 3, 0)] = C(3, 0, 0)$,	$T_6 H_\alpha[C(3, 3, 3)] = C(3, 0, 0)$,
$T_3 H_\alpha[C(3, 3, 6)] = C(3, 0, 0)$,	$T_6 H_{\alpha^{-1}}[C(3, 6, 0)] = C(3, 0, 0)$,
$H_{\alpha^{-1}} T_6[C(3, 6, 3)] = C(3, 0, 0)$,	and $T_3[C(3, 6, 6)] = C(3, 0, 0)$.

Hence $|C(3, h_2, h_3)| = |C(3, 0, 0)|$, for all $h_2, h_3 \in H$.

Computation also shows that $R[C(6, h_2, h_3)] = C(3, h_2 + 3, h_3 + 6)$, for all $h_2, h_3 \in H$, from which the result follows. □

The following elementary observation will prove useful.

Lemma 3.3 *For $a, b, c \in GF(3)$, $a + b + c = 0$ if and only if either $a = b = c$ or a, b, and c are distinct.*

Theorem 3.1 *$\mathbb{Z}_9$ has 225 normalized orthomorphisms.*

Proof: We will compute $|abc|$ for each class abc, using Lemmas 3.1 and 3.2 to reduce the amount of computation.

The class 300. If θ is a normalized orthomorphism in the class 300, then A_0, A_1, and A_2 are orthomorphism matrices of H. A_0 is uniquely determined as θ is normalized, and there are three choices for each of A_1 and A_2. Hence $|300| = 1 \times 3 \times 3 = 9$.

The classes 210 and 201. If θ is a normalized orthomorphism in the class 210, then, for some $h_1, h_2, h_3 \in H$, $h_1 \ne 0$, $\theta(0) = 0$, $\theta(h_1) = -h_1$, $\theta(-h_1) = h_2 + 1$, and $\theta(h_3 + 1) = h_1$. We will show that, for each choice of h_1, h_2, h_3, θ, equivalently M_θ, is uniquely determined. We have specified the partial orthomorphism matrices A_0,

B_1, and B_2, and $C_0 = C_1 = C_2 = O$, the all 0s matrix. It remains to determine A_1, A_2, and B_0. By Lemma 2.1, each row, column, and diagonal of M_θ contains exactly one 1. So far we have determined the positions of 1s in rows 0, $-h_1$, h_2+1, and h_1; columns 0, h_1, $-h_1$, and h_3+1; and diagonals 0, h_1, h_2+h_1+1, and h_1-h_3+6+2. All other entries in these rows, columns, and diagonals must be 0.

In Figures 1 and 2 we depict A_1' and A_2' with row and column headings: the cells that are marked with $\times$s are the cells that we know have entry 0 in A_1 and A_2. The two cells, containing 1s, in A_1 cannot be on the (h_2+h_1+1)th diagonal, and $h_2+h_1 = -h_3+u$ for some $u \in H$. It is easy to see from Figure 1 that, for each u, the cells in A_1 containing 1s, are the cell in row $u+h_1+2$ and column $u+h_1-h_2+1$ on diagonal h_2+1, and the cell in row $u-h_1+2$ and column $u-h_2+1$ on diagonal $-h_1+h_2+1$.

	h_3-h_1+1	h_3+h_1+1	h_3+1
2	$-h_3+h_1+1$	$-h_3-h_1+1$	$\times$
$-h_1+2$	$-h_3+1$	$-h_3+h_1+1$	$\times$
h_1+2	$-h_3-h_1+1$	$-h_3+1$	$\times$

,

Figure 1: A_1' for the class 210

	2	$-h_1+2$	h_1+2
h_2-h_1+1	h_2-h_1+6+2	h_2+6+2	h_2+h_1+6+2
h_2+h_1+1	h_2+h_1+6+2	h_2-h_1+6+2	h_2+6+2
h_2+1	$\times$	$\times$	$\times$

.

Figure 2: A_2' for the class 210

Similarly, the two cells, containing 1s, in A_2 cannot be on the (h_1-h_3+6+2)th diagonal and $h_1-h_3 = h_2+v$ for some $v \in H$. It is easy to see from Figure 2 that, for each v, the cells in A_2 containing 1s, are the cell in row $-v-h_3+1$ and column $-v+h_1+2$ on diagonal $-h_3-h_1+6+2$, and the cell in row $-v-h_3-h_1+1$ and column $-v-h_1+2$ on diagonal $-h_3+6+2$.

All cells of B_0 have entry 0 except one cell that has entry 1. M_θ is an orthomorphism matrix if and only if the cell in row $u+2$ and column $-v+2$ is not on diagonal 0 or h_1. But this cell is on the diagonal $u+v = (h_1+h_2+h_3)+(h_1-h_2-h_3) = -h_1$, and so M_θ is an orthomorphism matrix uniquely determined by h_1, h_2, h_3. Thus $|210| = 2 \times 3 \times 3 = 18$ and, hence, by Lemma 3.1, $|201| = 18$.

The classes 120 and 102. If θ is a normalized orthomorphism in the class 120, then, for some $h_1, h_2 \in H$, $\theta(0) = 0$, $\theta(h_1+1) = 3$, and $\theta(3) = h_2+1$. We will show that, for each choice of h_1, h_2, θ, equivalently M_θ, is uniquely determined. By Lemma 2.1, each row, column, and diagonal of M_θ contains exactly one 1. So far we have determined the positions of 1s in rows 0, 3, and h_2+1; columns 0, 3, and h_1+1; and diagonals 0, $-h_1+2$, and h_2+6+1. All other entries in these rows, columns, and diagonals must be 0.

There must be a second 1 in B_1 in column 6 and row $1+x$ and on diagonal $x+3+1$, for some $x \in H$. As this cell cannot be in row h_2+1 or on diagonal h_2+6+1, it must be that $x = h_2+6$, placing the second 1 in B_1 in row h_2+6+1, column 6 and on diagonal h_2+1. Similarly, the second 1 in B_2 is in row 6, column h_1+6+1, and on diagonal $-h_1+6+2$.

The 1 in A_1 must be in column h_1+3+1, row $y+2$, and on diagonal $y-h_1+6+1$, for some $y \in H$. As this cell cannot be on diagonals h_2+6+1 or h_2+1, it must be that $y = h_1+h_2+6$, placing the 1 in A_1 in row h_1+h_2+6+2, column h_1+3+1, and on diagonal h_2+3+1. Similarly, the 1 in A_2 is in row h_2+3+1, column h_1+h_2+6+2, and on diagonal $-h_1+3+2$.

In Figure 3 we depict B_0' with row and column headings: the cells that are marked with ×s are the cells that we know have entry 0. As no cell in B_0 can be on diagonal 0, the 1s in B_0 must be in row h_2+h_1+2, column h_2+h_1+3+2, and on diagonal 6; and in row h_2+h_1+3+2, column h_2+h_1+2, and on diagonal 3. Hence M_θ is an orthomorphism matrix uniquely determined by h_1, h_2. Thus $|120| = 3 \times 3 = 9$ and, hence, by Lemma 3.1, $|102| = 9$.

	h_2+h_1+2	h_2+h_1+3+2	h_2+h_1+6+2
h_2+h_1+2	0	6	×
h_2+h_1+3+2	3	0	×
h_2+h_1+6+2	×	×	×

Figure 3: B_0' for the class 120

The class 111. If $\theta \in C(3,0,0)$, then, for some $h_1, h_2, h_3 \in H$, $h_1 \neq 0$, $\theta(0) = 0$, $\theta(3) = 1$, $\theta(6) = 2$, $\theta(h_2+1) = h_1$, and $\theta(h_3+2) = -h_1$. We will see that the choice of h_1, h_2, h_3 need not uniquely determine θ, equivalently M_θ. We have specified the partial orthomorphism matrices A_0, B_1, B_2, C_1, and C_2. It remains to determine A_1, A_2, B_0, and C_0. By Lemma 2.1, each row, column, and diagonal of M_θ contains exactly one 1. So far we have determined the positions of 1s in rows 0, 1, 2, h_1, and $-h_1$; columns 0, 3, 6, h_2+1, and h_3+2; and diagonals 0, $6+1$, $3+2$, h_1-h_2+6+2, and $-h_1-h_3+6+1$. All other entries in these rows, columns, and diagonals must be 0. As there can only be one 1 on a diagonal $h_1-h_2+6 \neq 3$ and $-h_1-h_3+6 \neq 6$. Thus $h_2 = h_1+3+k_1$ and $h_3 = -h_1+k_2$ for some $k_1, k_2 \in H$, $k_1, k_2 \neq 0$.

Figure 4 shows A_1', A_2', B_0', and C_0' with row and column headings, and rows and columns, for which the matrices A_1, A_2, B_0, and C_0 have only 0 entries, removed.

	h_2+3+1	h_2+6+1	h_3+3+2	h_3+6+2
$3+1$	$(-h_2)$	$(-h_2)+6$	$(-h_3+6+2)$	$(-h_3+6+2)+6$
$6+1$	$(-h_2)+3$	$(-h_2)$	$(-h_3+6+2)+3$	$(-h_3+6+2)$
$3+2$	$(-h_2+1)$	$(-h_2+1)+6$	$(-h_3)$	$(-h_3)+6$
$6+2$	$(-h_2+1)+3$	$(-h_2+1)$	$(-h_3)+3$	$(-h_3)$

Figure 4: The matrices A_1', A_2', B_0', and C_0'

For some $c_0, c_1, c_2, c_3 \in H$, there is an entry 1 in a cell of C_0 on the diagonal $(-h_2)+c_0$, an entry 1 in a cell of A_2 on the diagonal $(-h_3+6+2)+c_1$, an entry 1 in a cell of B_0 on the diagonal $(-h_3)+c_2$, and an entry 1 in a cell of A_1 on the diagonal $(-h_2+1)+c_3$.

The entries on diagonal 0 of CM_θ sum to three and, as the corresponding cells in M_θ are on distinct diagonals, by Lemma 3.3

$$0+(-h_2+c_0)+(-h_3+c_2)=0. \tag{1}$$

Similarly

$$3+(h_1-h_2+6)+(-h_3+6+c_1)=0, \tag{2}$$

and

$$6+(-h_1-h_3+6)+(-h_2+c_3)=0. \tag{3}$$

If the equations (1), (2), and (3) are satisfied, then, by Lemma 3.3, all cells of M_θ are on distinct diagonals if and only if

$$h_2=h_1+3+k_1, \tag{4}$$

$$h_3=-h_1+k_2, \tag{5}$$

and

$$-h_2+c_0=-h_3+c_2+k_3, \tag{6}$$

for some $k_3 \in H$, $k_3 \neq 0$.

From equations (1),…, (6) we derive the following four equations

$$c_0=h_1+3+(k_1-k_3), \tag{7}$$

$$c_1=-h_1+6+(k_1+k_2), \tag{8}$$

$$c_2=-h_1+(k_2+k_3), \tag{9}$$

and

$$c_3=h_1+(k_1+k_2). \tag{10}$$

Note that $c_0+c_1+c_2+c_3=0$.

If we specify h_1, k_1, k_2, and k_3, then c_0, c_1, c_2, and c_3 are determined by equations $(7), \ldots, (10)$, h_2 is determined by equation (4), and h_3 by equation (5). It is easy to then verify that equations (1), (2), and (3) are satisfied. To determine whether these solutions yield orthomorphisms or not, we need to check that no row or column of M_θ contains more than one 1. We will see that, in $C(3,0,0)$, the solution $c_0 = c_1 = c_2 = c_3 = 0$ will never occur. If, for some $i \in \{0,1,2,3\}$, $c_i = c_{i+1} \neq 0$, subscripts added modulo 4, then some row or column of M_θ must contain two 1s, and hence this solution yields no orthomorphisms. Any other solution yields exactly one normalized orthomorphism in $C(3,0,0)$. We will determine $|C(3,0,0)|$ case by case. There are two cases, $h_1 = 6$ and $h_1 = 3$.

Case 1: $h_1 = 6$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 0$, $c_1 = -k_1$, $c_2 = 3 - k_1$, and $c_3 = 6 - k_1$. If $k_1 = 3$, then $(c_0, c_1, c_2, c_3) = (0, 6, 0, 3)$, and if $k_1 = 6$, then $(c_0, c_1, c_2, c_3) = (0, 3, 6, 0)$. This yields two normalized orthomorphisms in $C(3,0,0)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = -k_1 = c_1$. This yields no normalized orthomorphisms in $C(3,0,0)$.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = 0$, $c_1 = 0$, $c_2 = 3$, and $c_3 = 6$. This yields one normalized orthomorphism in $C(3,0,0)$ for each value of k_1.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = -k_1$, $c_1 = 0$, $c_2 = 3 + k_1$, and $c_3 = 6$. If $k_1 = 3$, then $(c_0, c_1, c_2, c_3) = (6, 0, 6, 6)$, and if $k_1 = 6$, then $(c_0, c_1, c_2, c_3) = (3, 0, 0, 6)$ yielding one normalized orthomorphism in $C(3,0,0)$.

Case 1 yields five normalized orthomorphisms in $C(3,0,0)$.

Case 2: $h_1 = 3$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 6$, $c_1 = 3 - k_1$, $c_2 = 6 - k_1$, and $c_3 = 3 - k_1$. If $k_1 = 3$, then $(c_0, c_1, c_2, c_3) = (6, 0, 3, 0)$, and if $k_1 = 6$, then $(c_0, c_1, c_2, c_3) = (6, 6, 0, 6)$ yielding one normalized orthomorphism in $C(3,0,0)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 6 - k_1$, $c_1 = 3 - k_1$, $c_2 = 6$, and $c_3 = 3 - k_1$. If $k_1 = 3$, then $(c_0, c_1, c_2, c_3) = (3, 0, 6, 0)$, and if $k_1 = 6$, then $(c_0, c_1, c_2, c_3) = (0, 6, 6, 6)$ yielding one normalized orthomorphism in $C(3,0,0)$.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = 6$, $c_1 = 3$, $c_2 = 6$, and $c_3 = 3$ yielding one normalized orthomorphism in $C(3,0,0)$ for each value of k_1.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = 6 - k_1$, $c_1 = 3$, $c_2 = 6 + k_1$, and $c_3 = 3$. If $k_1 = 3$, then $(c_0, c_1, c_2, c_3) = (3, 3, 0, 3)$, and if $k_1 = 6$, then $(c_0, c_1, c_2, c_3) = (0, 3, 3, 3)$ yielding no normalized orthomorphisms in $C(3,0,0)$.

Case 2 yields four normalized orthomorphisms in $C(3,0,0)$.

Hence, $|C(3,0,0)| = 9$. By Lemma 3.2, $|111| = 18 \times |C(3,0,0)| = 18 \times 9 = 162$. It follows that the number of normalized orthomorphisms of $\mathbb{Z}_9$ is $|300| + 2 \times |210| + 2 \times |120| + |111| = 9 + 2 \times 18 + 2 \times 9 + 162 = 225$. □

4 Applications to other groups

The group $GF(9)^+$, the additive group of the field of order 9, has 249 normalized orthomorphisms. This was shown in [2] using the "method of exhaustion". This number has been confirmed by computer searches: see. [11], [15], and [16]. The normalized orthomorphisms of $GF(9)^+$ have been determined theoretically using permutation polynomials. Each mapping $GF(9) \to GF(9)$ can be represented by a polynomial of reduced degree at most 8, a permutation polynomial if the mapping is a permutation, and an orthomorphism polynomial if the mapping is an orthomorphism. Each orthomorphism polynomial of $GF(9)^+$ has reduced degree at most 6. Orthomorphism polynomials of degree at most 5 were described in [13], and the orthomorphism polynomials of $GF(9)^+$ of degree 6 were described in [14]. There are 81 normalized orthomorphisms of $GF(9)^+$ represented by orthomorphism polynomials of degree at most 5, and 168 normalized orthomorphisms of $GF(9)^+$ represented by orthomorphism polynomials of degree 6, yielding a total of 249 normalized orthomorphisms: see Sections 9.2.2 and 13.3.1 in [5]. The methods of Section 3 are easily adapted to give another theoretical proof that $GF(9)^+$ has 249 normalized orthomorphisms.

Let $G = GF(9)^+ = \{ij \mid i, j \in \mathbb{Z}_3\}$, and let $H = \{00, 01, 02\}$ be a subgroup of index 3. The set $D = \{00, 10, 20\}$ is a system of distinct coset representatives for H in G. We will order the elements of G as $\{00, 01, 02; 10, 11, 12; 20, 21, 22\}$. As in the case $G = \mathbb{Z}_9$, each normalized orthomorphism of G belongs to one of the classes 300, 210, 201, 120, 102, or 111, and can be represented by an orthomorphism matrix of the form

$$M_\theta = (A_{i0,j0}) = \begin{pmatrix} A_{00} & B_{20} & C_{10} \\ B_{10} & C_{00} & A_{20} \\ C_{20} & A_{10} & B_{00} \end{pmatrix},$$

$i, j = 0, 1, 2$. As in the case $G = \mathbb{Z}_9$, for X one of A, B, C, let us form a matrix X'_{k0} from X_{k0} by making each entry in a cell of X'_{k0} the diagonal the corresponding cell of X_{k0} is on.

As in Lemma 3.1, we can reduce the number of classes that we need to deal with.

Lemma 4.1 $|210| = |201|$ *and* $|120| = |102|$.

Proof: Let $\alpha \in \mathrm{Aut}(G)$ be defined by $\alpha(ij) = (-i)j$. Then, as in Lemma 3.1, H_α acts as a bijection between the set of normalized orthomorphisms in the class 210 and the set of normalized orthomorphisms in the class 201, and also acts as a bijection between the set of normalized orthomorphisms in the class 120 and the set of normalized orthomorphisms in the class 102. □

For the class 111, set $C(h_1) = \{\theta \in \text{ class } 111 \mid \theta(00) = 00,\ \theta(0h_1) = 1h_2,\ \theta(0(-h_1)) = 2h_3 \text{ for some } h_2, h_3 \in \mathbb{Z}_3\}$, $h_1 \in \mathbb{Z}_3$, $h_1 \neq 0$. The reflection can be used to reduce the problem of determining $|111|$ to that of determining $|C(1)|$.

Lemma 4.2 $|111| = 2 \times |C(1)|$.

Proof: The reflection R acts as a bijection between $C(1)$ and $C(2)$. □

Theorem 4.1 *$GF(9)^+$ has 249 normalized orthomorphisms.*

Proof: As in the proof of Theorem 3.1, we will compute $|abc|$ for each class abc, using Lemmas 4.1 and 4.2 to reduce the amount of computation.

The class 300. The proof that $|300| = 9$ is identical to the proof in the proof of Theorem 3.1.

The classes 210 and 201. If θ is a normalized orthomorphism in the class 210, then, for some $h_1, h_2, h_3 \in \mathbb{Z}_3$, $h_1 \neq 0$, $\theta(00) = 00$, $\theta(0h_1) = 0(-h_1)$, $\theta(0(-h_1)) = 1h_2$, and $\theta(1h_3) = 0h_1$. For each choice of h_1, h_2, h_3, θ, equivalently M_θ, is uniquely determined: the proof of this is identical to the proof in the proof of Theorem 3.1. Thus $|210| = 2 \times 3 \times 3 = 18$ and, hence, by Lemma 4.1, $|201| = 18$.

The classes 120 and 102. If θ is a normalized orthomorphism in the class 120, then, for some $h_1, h_2 \in \mathbb{Z}_3$, $\theta(00) = 00$, $\theta(1h_1) = 01$, and $\theta(01) = 1h_2$. For each choice of h_1, h_2, θ, equivalently M_θ, is uniquely determined: the proof of this is very similar to the proof in the proof of Theorem 3.1. Thus $|120| = 3 \times 3 = 9$ and, hence, by Lemma 4.1, $|102| = 9$.

The class 111. If $\theta \in C(1)$, then, for some $h_1, h_2, h_3, h_4, h_5 \in \mathbb{Z}_3$, $h_5 \neq 0$, $\theta(00) = 00$, $\theta(01) = 1h_1$, $\theta(02) = 2h_2$, $\theta(1h_3) = 0h_5$, and $\theta(2h_4) = 0(-h_5)$.

We will see that the choice of h_1, h_2, h_3 need not uniquely determine θ, equivalently M_θ. We have specified the partial orthomorphism matrices A_{00}, B_{10}, B_{20}, C_{10}, and C_{20}. It remains to determine A_{10}, A_{20}, B_{00}, and C_{00}. By Lemma 2.1, each row, column, and diagonal of M_θ contains exactly one 1. So far we have determined the positions of 1s in rows 00, $1h_1$, $2h_2$, $0h_5$, and $0(-h_5)$; columns 00, 01, 02, $1h_3$, and $2h_4$; and diagonals 00, $2(h_5 - h_3)$, $1(-h_5 - h_4)$, $1(h_1 - 1)$, and $2(h_2 + 1)$. All other entries in these rows, columns, and diagonals must be 0. Note that, as there can only be one 1 on a diagonal $h_1 - 1 \neq -h_5 - h_4$ and $h_2 + 1 \neq h_5 - h_3$.

Figure 5 shows A'_{10}, A'_{20}, B'_{00}, and C'_{00} with row and column headings, where the rows and columns, for which the matrices A_{10}, A_{20}, B_{00}, and C_{00} have only 0 entries, removed.

For some $c_0, c_1, c_2, c_3 \in \mathbb{Z}_3$, there is an entry 1 in a cell of C_{00} on the diagonal $0(h_1 - h_3 + c_0)$, an entry 1 in a cell of A_{20} on the diagonal $2(h_1 - h_4 + c_1)$, an entry 1 in a cell of B_{00} on the diagonal $0(h_2 - h_4 + c_2)$, and an entry 1 in a cell of A_{10} on the diagonal $1(h_2 - h_3 + c_3)$.

The entries on diagonal 0 of CM_θ sum to three and, as the corresponding cells in M_θ are on distinct diagonals, by Lemma 3.3

$$0 + (h_1 - h_3 + c_0) + (h_2 - h_4 + c_2) = 0. \tag{11}$$

	$1(h_3+h_5)$	$1(h_3-h_5)$	$2(h_4+h_5)$	$2(h_4-h_5)$
$1(h_1+h_5)$	$0(h_1-h_3)$	$0(h_1-h_3-h_5)$	$2(h_1-h_4)$	$2(h_1-h_4-h_5)$
$1(h_1-h_5)$	$0(h_1-h_3+h_5)$	$0(h_1-h_3)$	$2(h_1-h_4+h_5)$	$2(h_1-h_4)$
$2(h_2+h_5)$	$1(h_2-h_3)$	$1(h_2-h_3-h_5)$	$0(h_2-h_4)$	$0(h_2-h_4-h_5)$
$2(h_2-h_5)$	$1(h_2-h_3+h_5)$	$1(h_2-h_3)$	$0(h_2-h_4+h_5)3$	$0(h_2-h_4)$

Figure 5: The matrices A'_{10}, A'_{20}, B'_{00}, and C'_{00}

Similarly

$$(h_2+1)+(h_5-h_3)+(h_1-h_4+c_1)=0, \tag{12}$$

and

$$(h_1-1)+(-h_5-h_4)+(h_2-h_3+c_3)=0. \tag{13}$$

If the equations (11), (12), and (13) are satisfied, then, by Lemma 3.3, all cells of M_θ are on distinct diagonals if and only if

$$h_1-h_3+c_0=k_1, \tag{14}$$

$$h_5-h_3=h_2+1+k_2, \tag{15}$$

and

$$-h_5-h_4=h_1-1+k_3, \tag{16}$$

for some $k_1,k_2,k_3\in\mathbb{Z}_3\setminus\{0\}$.

From equations (11),. . . , (16) we derive the following four equations

$$c_0=-k+h_5-1+(k_1-k_2), \tag{17}$$

$$c_1=k-h_5-1-(k_2+k_3), \tag{18}$$

$$c_2=-k-h_5+1-(k_1+k_3), \tag{19}$$

and

$$c_3=k+h_5+1-(k_2+k_3), \tag{20}$$

where $k=h_1+h_2$. Note that, as in Theorem 3.1, $c_0+c_1+c_2+c_3=0$.

As in the proof of Theorem 3.1, if we specify h_1, h_2, h_5, k_1, k_2, and k_3, then c_0, c_1, c_2, and c_3 are determined by equations (17), . . . , (20), h_3 is determined by equation (15), and h_4 by equation (16). It is easy to then verify that equations (11), (12), and (13) are satisfied. In what follows we will specify k, h_5, k_1, k_2, and k_3 instead. Note that each value of k determines three values of (h_1,h_2). Thus, as in the proof of Theorem 3.1, to determine whether these solutions yield orthomorphisms or not, we need to check that no row or column of M_θ contains more than one 1. In particular, we obtain six normalized orthomorphisms in $C(1)$ when $c_0=c_1=c_2=c_3=0$: no orthomorphisms if, for some $i\in\{0,1,2,3\}$, $c_i=c_{i+1}\neq 0$, subscripts added modulo

4; and three normalized orthomorphisms in $C(1)$ for any other solution. We will determine $|C(1)|$ case by case. There are three cases, $k = 0$, $k = 1$, and $k = 2$.

Case 1: $k = 0$.

Subcase 1i: $h_5 = 1$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 0$, $c_1 = 1 + k_1$, $c_2 = k_1$, and $c_3 = 2 + k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (0, 2, 1, 0)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (0, 0, 2, 1)$. This yields six normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 0$, $c_1 = 1$, $c_2 = 0$, and $c_3 = 2$. This yields three normalized orthomorphisms in $C(1)$ for each choice of k_1.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = -k_1$, $c_1 = 1$, $c_2 = k_1$, and $c_3 = 2$. This yields no normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = -k_1$, $c_1 = 1 - k_1$, $c_2 = 0$, and $c_3 = 2 - k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (2, 0, 0, 1)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (1, 2, 0, 0)$ yielding six normalized orthomorphisms in $C(1)$.

Subcase 1i yields eighteen normalized orthomorphisms in $C(1)$.

Subcase 1ii: $h_5 = 2$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 1$, $c_1 = k_1$, $c_2 = 2 + k_1$, and $c_3 = k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (1, 1, 0, 1)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (1, 2, 1, 2)$. This yields three normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 1$, $c_1 = 0$, $c_2 = 2$, and $c_3 = 0$. This yields three normalized orthomorphisms in $C(1)$ for each choice of k_1.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = 1 - k_1$, $c_1 = 0$, $c_2 = 2 + k_1$, and $c_3 = 0$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (0, 0, 0, 0)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (2, 0, 1, 0)$. This yields nine normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = 1 - k_1$, $c_1 = -k_1$, $c_2 = 2$, and $c_3 = -k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (0, 2, 2, 2)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (2, 1, 2, 1)$ yielding three normalized orthomorphisms in $C(1)$.

Subcase 1ii yields twenty one normalized orthomorphisms in $C(1)$.

Case 2: $k = 1$.

Subcase 2i: $h_5 = 1$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 2$, $c_1 = 2 + k_1$, $c_2 = 2 + k_1$, and $c_3 = k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (2, 0, 0, 1)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (2, 1, 1, 2)$. This yields three normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 2$, $c_1 = 2$, $c_2 = 2$, and $c_3 = 0$. This yields no normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = 2 - k_1$, $c_1 = 2$, $c_2 = 2 + k_1$, and $c_3 = 0$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (1, 2, 0, 0)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (0, 2, 1, 0)$ yielding six normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = 2 - k_1$, $c_1 = 2 - k_1$, $c_2 = 2$, and $c_3 = -k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (1, 1, 2, 2)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (0, 0, 2, 1)$ yielding three normalized orthomorphisms in $C(1)$.

Subcase 2i yields twelve normalized orthomorphisms in $C(1)$.

Subcase 2ii: $h_5 = 2$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 0$, $c_1 = 1 + k_1$, $c_2 = 1 + k_1$, and $c_3 = 1 + k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (0, 2, 2, 2)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (0, 0, 0, 0)$. This yields six normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 0$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 1$. This yields no normalized orthomorphisms in $C(1)$.

0If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = -k_1$, $c_1 = 1$, $c_2 = 1 + k_1$, and $c_3 = 1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (2, 1, 2, 1)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (1, 1, 0, 1)$. This yields three normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = -k_1$, $c_1 = 1 - k_1$, $c_2 = 1$, and $c_3 = 1 - k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (2, 0, 1, 0)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (1, 2, 1, 2)$ yielding six normalized orthomorphisms in $C(1)$.

Subcase 2ii yields fifteen normalized orthomorphisms in $C(1)$.

Case 3: $k = 2$.

Subcase 3i: $h_5 = 1$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 1$, $c_1 = k_1$, $c_2 = 1 + k_1$, and $c_3 = 1 + k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (1, 1, 2, 2)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (1, 2, 0, 0)$. This yields three normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 1$, $c_1 = 0$, $c_2 = 1$, and $c_3 = 1$. This yields three no orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = 1 - k_1$, $c_1 = 0$, $c_2 = 1 + k_1$, and $c_3 = 1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (0, 0, 2, 1)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (2, 0, 0, 1)$ yielding six normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = 1 - k_1$, $c_1 = -k_1$, $c_2 = 1$, and $c_3 = 1 - k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (0, 2, 1, 0)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (2, 1, 1, 2)$ yielding three normalized orthomorphisms in $C(1)$.

Subcase 3i yields twelve normalized orthomorphisms in $C(1)$.

Subcase 3ii: $h_5 = 2$.

If $(k_2, k_3) = (k_1, k_1)$, $c_0 = 2$, $c_1 = 2 + k_1$, $c_2 = k_1$, and $c_3 = 2 + k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (2, 0, 1, 0)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (2, 1, 2, 1)$. This yields six normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (k_1, -k_1)$, $c_0 = 2$, $c_1 = 2$, $c_2 = 0$, and $c_3 = 2$. This yields no normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, k_1)$, $c_0 = 2 - k_1$, $c_1 = 2$, $c_2 = k_1$, and $c_3 = 2$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (1, 2, 1, 2)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (0, 2, 2, 2)$. This yields three normalized orthomorphisms in $C(1)$.

If $(k_2, k_3) = (-k_1, -k_1)$, $c_0 = 2 - k_1$, $c_1 = 2 - k_1$, $c_2 = 0$, and $c_3 = 2 - k_1$. If $k_1 = 1$, then $(c_0, c_1, c_2, c_3) = (1, 1, 0, 1)$, and if $k_1 = 2$, then $(c_0, c_1, c_2, c_3) = (0, 0, 0, 0)$ yielding six normalized orthomorphisms in $C(1)$.

Subcase 3ii yields fifteen normalized orthomorphisms in $C(1)$.

Thus $|C(1)| = 18 + 21 + 12 + 15 + 12 + 15 = 93$. It follows that the number of normalized orthomorphisms of $GF(9)$ is $|300| + 2 \times |210| + 2 \times |120| + 2 \times |C(1)| = 9 + 2 \times 18 + 2 \times 9 + 2 \times 93 = 249$. □

The classification and methods of Sections 2 and 3 can, in principle, be applied to any group of order $3n$ with a normal subgroup of index 3. Each normalized orthomorphism of such a group is in a class abc, for some $a, b, c \geq 0$, $a \neq 0$. The number of such classes is $\binom{n+1}{2}$, the coefficient of x^n in $(x + x^2 + x^3 + \dots)(1 + x + x^2 + \dots)^2$.

The next smallest groups, with normal subgroups of index 3, for which the number of normalized orthomorphisms has not been explained theoretically are of orders 12 and 15. There are two groups of order 12 that have non-cyclic Sylow 2-subgroups and normal subgroups of index 3, $\mathbb{Z}_2 \times \mathbb{Z}_6$ and A_4, and there is one group of order 15, $\mathbb{Z}_{15}$. In [3] , using the “method of exhaustion”, A_4 was found to have 3,776 normalized orthomorphisms, whereas subsequent computer searches found the number to be 3,840: see [6], [12], and [17]. Computer searches reported in [10], [11], [16], and [17] found $\mathbb{Z}_2 \times \mathbb{Z}_6$ to have 16,512 normalized orthomorphisms, and computer searches reported in [12], [15], and [16] found $\mathbb{Z}_{15}$ to have 2,424,195 normalized orthomorphisms.

Acknowledgements

I would like to thank the referees for their comments that have improved this paper.

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(Received 5 Oct 2022; revised 24 Feb 2023)