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123-Forcing matrices

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Abstract

A permutation σ of $\{1, 2, \ldots, n\}$ contains a 123-pattern provided it contains an increasing subsequence of length 3 and, otherwise, is 123-avoiding. In terms of the $n \times n$ permutation matrix P corresponding to σ, P contains a 123-pattern provided the 3×3 identity matrix I_3 is a submatrix of P. If A is an $n \times n$ $(0, 1)$-matrix, then A is 123-forcing provided every permutation matrix $P \leq A$ contains a 123-pattern. The main purpose of this paper is to characterize such matrices A with the minimum number of 0's.

1 Introduction

Let n be a positive integer and let $\mathcal{P}_n$ be the set of $n \times n$ permutation matrices corresponding to the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$. A permutation σ of $\{1, 2, \ldots, n\}$ *contains a* 123-*pattern* provided it contains an increasing subsequence of length 3 and, otherwise, is 123-*avoiding*. In terms of the $n \times n$ permutation matrix P corresponding to σ, P contains a 123-pattern provided the 3×3 identity matrix I_3 is a submatrix of P. If A is an $n \times n$ $(0, 1)$-matrix, then A is 123-*forcing* provided every permutation matrix $P \leq A$ (pointwise order) contains a 123-pattern; the matrix A thus *blocks* all 123-avoiding permutations in that every 123-avoiding permutation matrix has at least one 1 in a position of a 0 of A. The number of $n \times n$ 123-avoiding permutation matrices is the Catalan number

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

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In fact, this is the same number for any of the six permutations of $\{1, 2, 3\}$, see e.g. [1]. The ideas of forcing and blocking can be extended to other patterns [3].

The main purpose of this paper is to characterize 123-forcing matrices (equivalently, blockers of 123-avoiding matrices) A with the minimum number of 0's. Such matrices have been previously investigated in [3] where it was shown that the minimum possible number of 0's is n. The following example illustrates these concepts.

Example 1.1 Let $n = 6$ and let

$$A = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

Then every permutation matrix $P \leq A$ contains one of the two 1's from row 1, one of the three 1's from column 6, and then necessarily one of the 1's from the 2×3 submatrix formed by rows 2 and 3, and columns 3, 4, and 5, thereby resulting in a 123-pattern. Thus A is a 123-forcing matrix; equivalently, A blocks all 6×6 123-avoiding permutation matrices. Another example of a 123-forcing matrix with 6 0's that is readily checked is

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

Of course, an $n \times n$ matrix A with a row or column of all 0's is 123-forcing, since there are no permutation matrices $P \leq A$. □

2 Characterization of minimum 123-forcing matrices

In [3] the *cyclic-Hankel decomposition* of the $n \times n$ matrix J_n of all 1's into n permutation matrices was defined by starting with row 1 and cyclically permuting it as for circulant matrices, but in a right-to-left fashion, obtaining n disjoint permutation matrices. This is illustrated below for $n = 6$ using letters a, b, c, d, e, f below to designate the resulting permutation matrices:

$$\begin{bmatrix} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e \end{bmatrix}.$$

The cyclic-Hankel decomposition gives a decomposition of J_n into permutation matrices each of which avoids a 123-pattern, since each permutation in the decomposition corresponds to a decreasing subsequence followed by another decreasing subsequence (empty in one case). Such a decomposition was shown to be unique in [3]. The resulting permutation matrices $H_1, H_2, \ldots, H_n$ (our notation is such that the 1 in row 1 of H_i is in column i) are the $n \times n$ *cyclic-Hankel permutation matrices*, with H_n also called the *Hankel diagonal*. So with $n = 5$ we have

$$H_1 = \left[\begin{array}{c|c|c|c|c} 1 & & & & \\ \hline & & & & 1 \\ \hline & & & 1 & \\ \hline & & 1 & & \\ \hline & 1 & & & \end{array}\right], \quad H_2 = \left[\begin{array}{c|c|c|c|c} & 1 & & & \\ \hline 1 & & & & \\ \hline & & & & 1 \\ \hline & & & 1 & \\ \hline & & 1 & & \end{array}\right], \quad H_3 = \left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & & & 1 \\ \hline & & & 1 & \end{array}\right],$$

$$H_4 = \left[\begin{array}{c|c|c|c|c} & & & 1 & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & & & 1 \end{array}\right], \quad H_5 = \left[\begin{array}{c|c|c|c|c} & & & & 1 \\ \hline & & & 1 & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & & & \end{array}\right].$$

Remark 2.1 The famous Frobenius-König Theorem can be put in the context of our investigations. Consider the empty permutation σ_0. Then every permutation of $\{1, 2, \ldots, n\}$ contains the pattern σ_0. Thus every $n \times n$ $(0,1)$-matrix A is σ_0-forcing, and no permutation matrix is σ_0-avoiding. Thus the property that the $n \times n$ $(0,1)$-matrix A blocks all σ_0-avoiding permutation matrices is equivalent to the property that there does not exist a permutation matrix $P \leq A$. By the Frobenius-König Theorem, this holds if and only if A contains an $r \times (n + 1 - r)$ zero submatrix for some r with $1 \leq r \leq n$.

Lemma 2.2 *The number of* 0*'s in a* 123*-forcing* $n \times n$ $(0,1)$*-matrix is at least* n*. A* 123*-forcing* $n \times n$ $(0,1)$*-matrix with exactly* n 0*'s contains exactly one* 0 *from the positions of the* 1*'s of each cyclic-Hankel permutation matrix.*

Proof. The cyclic-Hankel decomposition of J_n consists of n mutually disjoint 123-avoiding permutation matrices. Hence a 123-forcing $n \times n$ matrix must have a 0 in a position of a 1 of each of them, and thus must contain at least n 0's. □

We characterize the 123-forcing $n \times n$ $(0,1)$-matrix with the minimum number n of 0's. Our characterization is based on the following construction generalizing the matrix A constructed in Example 1.1.

Let $k \leq n$ and let a and b be integers with $1 \leq a, b \leq n$ where $a + b = k + 1$. By $L_n^k(a, b)$ we denote the $n \times n$ $(0,1)$-matrix with exactly k 0's forming an L-shaped region whose last a positions in row 1 equal 0 and whose first b positions in column n equal to 0, giving a total of k 0's. In particular, there is a 0 in the corner position

$(1, n)$. (Sometimes we refer to the set of positions of the 0's of $L_n^k(a, b)$.) For example, we have

$$L_7^6(4,3) = \left[\begin{array}{c|c|c|c|c|c|c} & & & 0 & 0 & 0 & 0 \\ \hline & & & & & & 0 \\ \hline & & & & & & 0 \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \end{array}\right].$$

Lemma 2.3 *The $n \times n$ matrices $L_n^n(a, b)$ with $a + b = n + 1$ are* 123*-forcing* $(0, 1)$*-matrices with the minimum number n of* 0*'s.*

Proof. The number of 0's in $L_n^n(a, b)$ equals n. If a or b equals n, we have a row or column of all 0's and so (vacuously) a 123-forcing matrix. Now assume that neither a nor b equals 1. The matrix $L_n^n(a, b)$ contains an $n \times n$ matrix which is the direct sum of the following matrices of all 1's: $J_{1,n-a}$, $J_{b-1,a-1}$, and $J_{n-b,1}$. Every permutation matrix $P \leq L_n^n(a, b)$ contains a 1 from the $J_{1,n-a}$ and a 1 from the $J_{n-b,1}$. Since $b - 1 = n - a$, such a permutation matrix must also contain a 1 from the $J_{b-1,a-1}$ and hence has a 123-pattern. □

There is a similar construction and lemma with $L_n^n(a, b)$ replaced with the L-shaped region $V_n^k(a, b)$ with corner at position $(n, 1)$, the transpose of $L_n^k(a, b)$.

The following example illustrates the complexities involved in characterizing the 123-forcing $n \times n$ $(0, 1)$-matrix with the minimum number n of 0's.

Example 2.4 Consider $n = 10$ and the labeling of the positions of a 10×10 matrix with $a, b, c, \ldots$, where all the positions on the same cyclic-Hankel permutation matrix are labeled the same. We start with the 123-forcing matrix $L_{10}^{10}(5, 6)$. We move its first two 0's in row 1 (the positions labeled f and g there), down their cyclic-Hankel permutation matrices to the positions $z_1 = (4, 3)$ (on H_6) and $z_2 = (6, 2)$ (on H_7); these are colored, respectively, red and green in (1). The remaining positions of $L_{10}^{10}(5, 6)$, now forming a $L_{10}^8(3, 6)$, are colored yellow. This results in a set of 10 positions (the colored positions).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline j & a & b & c & d & e & f & g & h & i \end{array}\right]. \tag{1}$$

The resulting matrix of 10 0's is not 123-forcing as shown by the 123-avoiding permutation matrix colored blue in (2) that does not intersect it.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline j & a & b & c & d & e & f & g & h & i \end{array}\right]. \tag{2}$$

If, instead, we move the positions labeled f and g in row 1 to the positions colored red in (3), we obtain a 123-forcing matrix:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline j & a & b & c & d & e & f & g & h & i \end{array}\right] \rightarrow \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} & & & & & & & 0 & 0 & 0 \\ \hline & & & & & & & & & 0 \\ \hline & & & & & & & & & 0 \\ \hline & & & 0 & & & & & & 0 \\ \hline & & & & & & & & & 0 \\ \hline 0 & & & & & & & & & 0 \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \end{array}\right] \tag{3}$$

We argue this as follows referring to the labels of the positions. Suppose that the matrix in (3) (illustrated there as a $(0,1)$-matrix with its 1's in the empty positions) contains a 123-avoiding permutation matrix Q. In row 1, either position f or g must contain a 1 of Q since, by Lemma 2.3, $L_{10}^{10}(5,6)$ is a 123-forcing matrix. Suppose first that the f in row 1 is a 1 of Q. Then, since Q is a 123-avoiding permutation matrix, the submatrix determined rows $2,3,4,5,6$ and columns $7,8,9$ cannot contain a 1 of Q implying that Q has to have only 1's on the Hankel diagonal of the 6×6 submatrix determined by rows and columns $1,2,3,4,5,6$; but the 0 in position f precludes that.

Now suppose that the position of g in row 1 is a 1 of Q. Then the submatrix of Q determined by rows $2,3,4,5,6$ and columns $7,8,9,10$ cannot contain a 1 of Q and the submatrix of Q determined by rows $8,9,10$ and columns $8,9,10$ must contain a 1 of Q. Since the Hankel diagonal of the 7×7 submatrix determined by rows and columns $1,2,\ldots,7$ contains a 0 in position g, it now follows that the matrix in (3) is a 123-forcing matrix. $\square$

The following result, Theorem 2.10 from [3], is important in characterizing the $n\times n$ 123-forcing $(0,1)$-matrices with the minimum number n of 0's.

Theorem 2.5 *Let $n \geq 3$. If an $n \times n$ 123-forcing $(0,1)$-matrix contains the minimum number n of 0's, then it must contain one of the positions $(1,n)$ and $(n,1)$; if it contains a 0 in position $(1,n)$ (respectively, position $(n,1)$), then it also contains a 0 in either the position $(1,n-1)$ or position $(2,n)$ (respectively, position $(n,2)$ or position $(n-1,1)$).*

In view of Theorem 2.5, by symmetry it is enough to consider minimum 123-forcing $(0,1)$-matrices that contain a 0 in the positions $(1,n)$ and $(1,n-1)$, and we assume this throughout.

We now label the positions in an $n \times n$ matrix A with the integers $1, 2, \ldots, n$ where the positions in row 1 are labeled, in order, $1, 2, \ldots, n$ and the positions on the corresponding cyclic-Hankel permutation matrices have the same labels. We call this the *standard labeling*. For example, with $n = 5$, the standard labeling is

$$\left[\begin{array}{c|c|c|c|c} 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 & 5 & 1 \\ \hline 3 & 4 & 5 & 1 & 2 \\ \hline 4 & 5 & 1 & 2 & 3 \\ \hline 5 & 1 & 2 & 3 & 4 \end{array}\right].$$

In what follows, A is an $n \times n$ matrix with the standard labeling and exactly n 0's. We start with $A = L_n^n(a,b)$ where $a + b = n + 1$ so that A contains a total of n 0's. Let a' and b' be integers with $0 \leq a' < a$ and $0 \leq b' < b$. The L-shaped matrix $L_n^k(a - a', b - b')$ is obtained from $L_n^n(a,b)$ by removing the 0's in the first a' positions in row 1 and the last b' positions in column n leaving $k = n - a' - b'$ 0's. In order that we have a 123-blocking matrix with exactly n 0's, the $(a' + b')$ 0's that are removed from $L_n^n(a,b)$ need to be shifted to new positions on their corresponding cyclic-Hankel permutation matrices. This is what was done in Example 2.4 in two cases one of which gave a 123-forcing matrix and one of which did not. We refer to matrices obtained in this way from an $L_n^n(a,b)$ as *L-cyclic matrices*.

We now set out to characterize the 123-forcing $(0,1)$-matrices with the minimum number n of 0's. The $k \times k$ *leading Hankel principal submatrix* of an $n \times n$ matrix A is the $k \times k$ submatrix of A determined by the first k rows and last k columns of A.

Lemma 2.6 *Let A be an $n \times n$ 123-forcing $(0,1)$-matrix containing exactly n 0's with a 0 in position $(1,n)$ and let $2 \leq k \leq n$. Then the $k \times k$ leading Hankel principal submatrix A_k of A is a $k \times k$ 123-forcing matrix.*

Proof. If there is a 123-avoiding permutation matrix $P \leq A_k$, then since there is a 0 in position $(1,n)$ (and thus no more 0's on the Hankel diagonal of A by Lemma 2.2), with the Hankel diagonal of the complementary $(n-k) \times (n-k)$ matrix, we obtain a 123-avoiding permutation matrix in A, a contradiction. □

Corollary 2.7 *Let A be a 123-forcing matrix containing exactly n 0's with a 0 in position $(1,n)$. If A contains two 0's in row n, then A is not a 123-forcing matrix.*

Proof. If there are two 0's of A in row n, then the leading $(n-1) \times (n-1)$ Hankel principal submatrix contains at most $(n-2)$ 0's, and then with Lemma 2.2 this gives a contradiction of Lemma 2.6 with $k = n-1$. $\square$

Lemma 2.8 *Let A an $n \times n$ $(0,1)$-matrix with exactly n 0's including 0's in the positions $(1,1)$ and $(1,n)$, but not the position $(1,2)$. Then A is not a 123-forcing matrix.*

Proof. Suppose that A is a 123-forcing matrix. We illustrate the argument with $n = 10$. The positions $(1,1)$ and $(1,n)$ are in red below; the position $(1,2)$ with a b is in green. None of the other positions labeled a or j can be 0 by Lemma 2.2. Then the n positions colored green in (4) give a 123-avoiding permutation matrix which cannot contain a 0 of A, no matter what the other positions of the 0's in A.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{4}$$

$\square$

The following theorem is crucial for our characterization of the $n \times n$ 123-forcing matrices with the minimum number n of 0's. It implies, assuming (as we know we can) that position $(1,n)$ has a 0 in a 123-forcing matrix, that every $n \times n$ 123-forcing matrix with the minimum number n of 0's is obtained from an $L_n(a,b)$ with $a+b = n+1$ by shifting, along the corresponding cyclic-Hankel permutation matrices, x initial zeros in row 1 of $L_n(a,b)$ and y terminal zeros in column n of $L_(a,b)$ where $0 \le x \le a-1$ and $0 \le y \le b-1$. The problem then becomes how these should be shifted in order to obtain a 123-forcing matrix. It may be useful here to recall our Example 2.4.

Theorem 2.9 *Let $A = [a_{ij}]$ be an $n \times n$ 123-forcing $(0,1)$-matrix with exactly n 0's not all in a row or a column. Without loss of generality, assume that there is a 0 in position $(1,n)$. Then the 0's of A in the first row are consecutive and the 0's of A in the last column are consecutive.*

Proof. Since the 123-forcing property is preserved by reflecting with respect to the Hankel diagonal, we only need to show that the statement is true for the 0's of A in the first row. Thus we need to show that there does not exist k with $1 < k < n-1$

such that $a_{1,k-1} = 0$ while $a_{1,k} \neq 0$. We prove the result by assuming that we have such a k and obtain a contradiction. Note that Lemma 2.8 shows the theorem is true for $k = 2$. So we just need to show the theorem is true for $2 < k \leq n-1$. If $n = 3$, then there is nothing more to prove. We now proceed by induction on n using a 10×10 matrix to elucidate the general proof.

Referring to (5), suppose that the position $(1, k-1)$ (the d in row 1) contains a 0 but the positions $(1, k)$ and e.g., $(1, k+1)$ (the e and f in row 1) contain 1's, with the positions $(1, k+2), \ldots, (1, n)$ also containing 0's (those labeled g, h, i, j below). These zero positions are colored red in (5) below. (There could be more than just two positions e and f with 1's but the argument will be the same.)

There are two cases to consider.

(I) The position $(n, k-1)$ (the green c) does not contain a 0. Then we can construct a 123-avoiding permutation matrix as shown in (5) in color green since the positions of the green d's cannot contain 0's as we already have a 0 in the position labeled d in row 1.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right] \to \left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{5}$$

(II) The position $(n, k-1)$ (now the red c in (6)) contains a 0.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right] \tag{6}$$

We now consider the $(n-1) \times (n-1)$ submatrix of (6) obtained by deleting the first column and last row. Since the position of the red c in the last row contains a 0, this submatrix contains at most $n-1$ 0's. By induction this submatrix contains an $(n-1) \times (n-1)$ 123-avoiding permutation matrix which with the green j (which cannot contain a 0 since the position of the red j contains a 0) gives an $n \times n$ 123-avoiding permutation matrix.

Thus the theorem holds by induction. $\square$

Lemma 2.10 *Let A be an $n \times n$ $(0,1)$-matrix with exactly n 0's having a 0 in position $(1,n)$. Assume that positions $z_1 = (i,k)$ and $z_2 = (j,l)$ above the Hankel diagonal with $i < j$ and $l \leq k$ contain 0's. Then A is not a 123-forcing matrix.*

Proof. Let $\mathcal{Z}$ be the set of positions of A with a 0. Since both z_1 and z_2 are above the Hankel diagonal, then in our standard labeling, z_1 has label $(i+k-1)$ and z_2 has label $(j+l-1)$. If $i+k < j+l-1$, we can always do a comparison using positions having consecutive labels between $i+k$ and $j+l-2$ inclusively with z_1 and z_2. Thus without loss of generality, we assume that $i+k = j+l+1$ meaning that the labels of z_1 and z_2 are consecutive integers and the cyclic-Hankel permutation matrix corresponding to z_1 immediately precedes the cyclic-Hankel permutation matrix corresponding to z_2.

For ease of understanding, we argue with a 12×12 matrix and two specific positions but the argument is easily seen to hold in general. Suppose that the positions of the red 8 and 9 in (7) contain a 0. There are two possibilities to consider: (i) z_1 and z_2 are not in the same column, and (ii) z_1 and z_2 are in the same column.

(i) z_1 and z_2 are not in the same column, like the red positions in (7).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{7}$$

(a) If the yellow 4 in (7) is not in $\mathcal{Z}$, then we can construct a 123-avoiding permutation matrix as in (8) using the fact that we already have a 0 in a position labeled with an 8,

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{8}$$

(b) If the yellow 6 in (7) is not in $\mathcal{Z}$, then we can construct a 123-avoiding permutation matrix as in (9).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{9}$$

(c) If both the positions of the yellow 4 and yellow 6 in (7) are in $\mathcal{Z}$, then that $\mathcal{Z}$ does not give a blocking follows directly from Corollary 2.7.

(ii) z_1 and z_2 are in the same column as in (10).

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}. \tag{10}$$

(a) If the position of the yellow 5 in (10) is not in $\mathcal{Z}$, then we can construct a 123-avoiding permutation matrix as shown in green in (11).

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}. \tag{11}$$

(b) If the yellow 5 (now colored red in (12)) is in $\mathcal{Z}$, then we take the $(n, 1)$ position and then consider the $(n-1) \times (n-1)$ submatrix, obtained by removing the first column and the last row, which contains at most $n-1$

positions in $\mathcal{Z}$.

$$\left[\begin{array}{c||c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline\hline
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{12}$$

The matrix in (13) is this $(n-1)\times(n-1)$ matrix relabeled using our standard labeling.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}\right]. \tag{13}$$

We now have to consider several possibilities.

(i) If the position of the yellow 4 in (13) does not contain a 0, we can construct an $(n-1)\times(n-1)$ 123-avoiding permutation matrix as in the following matrix (14).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}\right]. \tag{14}$$

(ii) If the position of the yellow 3 in (15) contains a 0, then we consider the $(n-2) \times (n-2)$ submatrix obtained by removing the first two columns and bottom two rows as in (15).

$$\left[\begin{array}{c|c||c|c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline\hline
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{15}$$

(iii) We can repeat this process if the position of the yellow 3 in (15) contains a 0 and continue until we arrive at the situation displayed in (16).

$$\left[\begin{array}{c|c|c|c|c||c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\ \hline
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\ \hline
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\ \hline
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline\hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{16}$$

The position of the yellow 12 in (16) is not in $\mathcal{Z}$, since the $(1, n)$ position is in $\mathcal{Z}$. We then obtain a 123-avoiding permutation matrix as shown in green in (17).

$$\left[\begin{array}{ccccc||ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 \\
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline\hline
8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}\right]. \tag{17}$$

This completes the proof. □

An analogous lemma holds for positions below the Hankel diagonal by reflection with respect to the Hankel diagonal with $i \le j$ and $k < l$.

Lemma 2.11 *Let A be an $n \times n$ $(0,1)$-matrix with exactly n 0's having a 0 in position $(1,n)$. Assume that positions $z_1 = (i,k)$ and $z_2 = (j,l)$ below the Hankel diagonal with $i \le j$ and $k < l$ contain 0's. Then A is not a 123-forcing matrix.* □

Before formulating the next lemma, we consider a revealing example.

Example 2.12 Consider $n = 8$ and an 8×8 123-forcing $(0,1)$-matrix A with 8 0's with some of our standard labeling shown in (18). There are 0's assumed in the positions labeled 5,6,7,8,1 2 as in $L_8^6(4,3)$. The positions 4 in row 1 and position 3 in column 8 are assumed not to contain 0's. Suppose the position 4 in red contains a 0.

$$\left[\begin{array}{cccc||cccc}
 & & & 4 & 5 & 6 & 7 & 8 \\
 & & & & & & & 1 \\
 & & & & & & & 2 \\
\hline\hline
4 & & & & & & & 3 \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}\right]. \tag{18}$$

Then none of the other positions labeled with a 4 can contain a 0 and, as demonstrated in (19), the positions colored yellow give a 123-avoiding permutation matrix. Hence the position of the red 4 in the lower left submatrix of (18) cannot contain a

0 in a 123-forcing matrix with $n = 8$ 0's including those 0's in an $L_8^6(4, 3)$.

$$\left[\begin{array}{cccc||cccc}
 & & & 4 & 5 & 6 & 7 & 8 \\
 & & 4 & & & & & 1 \\
 & 4 & & & & & & 2 \\
\hline\hline
4 & & & & & & & 3 \\
5 & & & & & & & \\
 & & & & & & 4 & \\
 & & & & & 4 & & \\
 & & & & 4 & & &
\end{array}\right] \tag{19}$$

. ∎

The preceding example illustrates the following lemma.

Lemma 2.13 *Let A be an $n \times n$ 123-forcing $(0, 1)$-matrix with exactly n 0's where the 0's in row 1 and column n are precisely the 0's of an $L_n^k(a, b)$ where $a + b \leq n + 1$ and $k = a + b$. Let X be the $(n - b) \times (n - a)$ submatrix of A formed by rows $b + 1, b + 2, \ldots, n$ and columns $1, 2, \ldots, n - a$. Then A does not contain any 0's in X.*

Proof. We assume the standard labeling of the positions of A. If $a + b = n + 1$, there is nothing to prove and so we assume that $a + b \leq n$. We prove the lemma by induction starting with the position labeled $n - a$ in the first row. Suppose the blocker uses a position α with label $n - a$ in X. Then we choose those positions labeled $n - a$ starting from row 1 down to, but not including, that position α. We then choose below α the positions on the cyclic Hankel-permutation matrix labeled $(n - a + 1)$ down to column 1, say in row p. We also choose the position in column n in the same row as α. We complete with the positions on the Hankel diagonal containing $(n - a)$ starting with row $p + 1$ down to the last row to obtain a 123-avoiding permutation matrix. This is illustrated with $n = 10, a = 3, b = 3$ in (20).

$$\left[\begin{array}{ccccccc||ccc}
 & & & & & & 7 & 8 & 9 & 10 \\
 & & & & & 7 & & & & 1 \\
 & & & & 7 & & & & & 2 \\
\hline\hline
 & & & 7 & & & & & & \\
 & & 7 & & & & & & & \\
 & 7 & & & & & & & & 5 \\
7 & 8 & & & & & & & & \\
8 & & & & & & & & & \\
 & & & & & & & & 7 & \\
 & & & & & & & 7 & &
\end{array}\right]. \tag{20}$$

We now proceed by induction.

Suppose the lemma holds for positions with labels $q+1, q+2, \ldots, n-a$, and we consider the position in row 1 with label q. Suppose the blocker uses a position α labeled q in X. We then choose the positions labeled q on the cyclic-Hankel diagonal with labels q starting from row 1 down to, but not including, position α. Below position α we choose the positions on the cyclic-Hankel permutation matrix labeled $q+1$ down to column 1, say in row r. We also choose the position in column n in the same row as α. Finally, we choose the positions on the cyclic Hankel diagonal labeled q in the lower $(r+1) \times (r+1)$ submatrix. Using the induction hypothesis, we obtain a 123-avoiding permutation matrix without any 0's. This is illustrated in Example 2.14. ☐

Example 2.14 To illustrate Lemma 2.13, let $n = 12$ and consider a 123-forcing $(0,1)$-matrix A given in (21) whose 0's in row 1 and column 12 are those where $L_{12}^6(4,3)$ has 0's (colored yellow). Suppose we know that A does not contain a 0 in positions labeled 7 within the lower left 9×8 matrix X, and consider the positions labeled 6. If A has a 0 in a position in X containing a 6 as shown in color green, we then choose the positions colored red as shown .

$$\left[\begin{array}{c|c|c|c|c|c|c|c||c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
 & & & & 6 & & & & & & & 1 \\ \hline
 & & & 6 & & & & & & & & 2 \\ \hline\hline
 & & 6 & & & & & & & & & 3 \\ \hline
 & 6 & & & & & & & & & & 4 \\ \hline
 & 7 & & & & & & & & & & 5 \\ \hline
7 & & & & & & & & & & & 6 \\ \hline
 & & & & & & & & & & 6 & 7 \\ \hline
 & & & & & & & & & 6 & & 8 \\ \hline
 & & & & & & & & 6 & & & 9 \\ \hline
 & & & & & & & 6 & & & & 9 \\ \hline
 & & & & & & 6 & & & & & 10
\end{array}\right]. \tag{21}$$

Since the position labeled 4 in the last column is not a 0 in the $L_{12}^6(4,3)$, and so is not a 0 in A, we get a 123-avoiding permutation matrix. ☐

We now show that the properties given in Lemmas 2.10, 2.11, and 2.13 characterize the 123-forcing $(0,1)$-matrices with the minimum number n of 0's.

Theorem 2.15 *Let A be an $n \times n$ $(0,1)$ with exactly n 0's with one 0 on each cyclic-Hankel permutation matrix where the position $(1,n)$ contains a 0. Let a and b be maximum such that $k = a+b \leq n+1$ and A has 0's where $L_n^k(a,b)$ has 0's. Then A is a 123-forcing matrix if and only if the following conditions hold:*

(a) *No other positions of A in row 1 and column n contain a 0.*

(b) *A does not have 0's in two positions $z_1 = (i,k)$ and $z_2 = (j,l)$ above the Hankel diagonal with $i < j$ and $k \leq l$.*

(c) *A does not have 0's in two positions $z_1 = (i, k)$ and $z_2 = (j, l)$ below the Hankel diagonal with $i < j$ and $k \leq l$.*

(d) *The $(n - b) \times (n - a)$ submatrix of A formed by rows $b + 1, b + 2, \ldots, n$ and columns $1, 2, \ldots, n - a$ does not contain any 0's.*

Proof. The assumption that the position $(1, n)$ contains a 0 is without loss of generality. The necessity follows from previous lemmas, and we now prove these properties are sufficient to guarantee that every permutation matrix $P \leq A$ contains a 123-pattern.

We partition the $(0, 1)$ matrix A as

$$\left[\begin{array}{c||c} A_1 & A_2 \\ \hline\hline A_3 & A_4 \end{array}\right]$$

where A_2 is $b \times a$ and so A_1 is $b \times (n - a)$, $A_3 = J_{n-b,n-a}$, and A_4 is $(n - b) \times a$. Since A_2 contains 0's in the positions of the 0's of $L_n^k(a, b)$, it follows from (a) that A_2 has 0's only in positions of its first row and last column with those 0's in $(a + b - 1)$ cyclic-Hankel permutation matrices.

Since there is exactly one 0 in each cyclic-Hankel permutation matrix, A_1 contains $p \leq n - (a + b - 1)$ positions with a 0. Thus there are $q = (n - (a + b - 1) - p)$ columns of A_1 not containing a 0 in A and thus this many positions with a 0 in A_3. Since the first row of A_1 and the last column of A_2 each contain a 1 of every permutation matrix $P \leq A$, a 123-avoiding permutation matrix $P \leq A$ cannot use a 1 in A_2. Thus to get a 123-avoiding permutation matrix $P \leq A$, P must contain a strictly decreasing sequence (subpermutation) of b 1's in A_1 and a strictly decreasing sequence (subpermutation) of a 1's in A_4. An example of this situation is given in (22) with $n = 15$, $a = 6$, $b = 4$, and $p = 3$, where $x = 0$ denotes 0's of A_1 (shifted from the red squares in row 1) and 0's in A_4 (shifted from the red squares in A_4).

$$\left[\begin{array}{ccccccccc||cccccc}
 & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 1 & x & & & & & & & 0 \\
 & & & & 1 & x & & & & & & & & & 0 \\
 & & 1 & x & & & & & & & & & & & 0 \\ \hline\hline
 & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & x & \\
 & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & x & & \\
 & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & x & & & & \\
 & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & &
\end{array}\right]. \tag{22}$$

In (22) we need to have a decreasing subpermutation of size 4 in the upper left 4×9 (a submatrix equal to a Hankel diagonal matrix H_k, $k = 4$ in (22)) and a decreasing subpermutation of size 6 in the lower right 11×6 (so a submatrix equal to a Hankel diagonal matrix H_l, $l = 6$ in (22)). We show examples of these in (22).

With the $x = 0$'s on different cyclic-Hankel diagonals, it follows that the 1 (colored yellow) in the H_k in the last row of the upper left submatrix is in column $(n - b - 1)$ or earlier (it is in column 3 in the example), and the 1 (also colored yellow) in row H_l is the first column of the lower right submatrix (it is in row 13 in the example).

Now consider the submatrix A' determined by the rows and columns not yet containing a 1 (the 5×5 submatrix in two shades of blue in (22)). The positions in the submatrix of A_3 determined by the columns of the $x = 0$'s in A_1 and the rows of the $x = 0$'s in A_4 (colored dark blue in the example) must contain only 0's, otherwise with the two yellow 1's we get a 123 pattern. This gives an $l \times l$ zero submatrix of A' ($l = 3$ in the example) which violates the easy part of the Frobenius-König Theorem, and hence we cannot complete the 1's to a permutation matrix. Hence there does not exist a 123-avoiding permutation matrix $P \leq A$, completing the proof. $\square$

There is an analogous theorem where in Theorem 2.15 we assume the position $(n, 1)$ contains a 0, thereby taking care of all possibilities.

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