

The number of 4-cycles and the cyclomatic number of a finite simple graph

TAKAYUKI HIBI AKI MORI

*Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University, Japan
hibi@math.sci.osaka-u.ac.jp akimk1214@gmail.com*

HIDEFUMI OHSUGI

*Department of Mathematical Sciences, School of Science
Kwansei Gakuin University
Sanda, Japan
ohsugi@kwansei.ac.jp*

Abstract

Let G be a finite connected simple graph with n vertices and m edges. We show that, when G is not bipartite, the number of 4-cycles contained in G is at most $\binom{m-n+1}{2}$. We further provide a short combinatorial proof of the bound $\binom{m-n+2}{2}$ which holds for bipartite graphs.

1 Introduction

In enumerative combinatorics on finite graphs, counting the number of prescribed subgraphs contained in a finite simple graph is one of the most traditional problems. One of the earliest paper on this topic is due to Erdős. Erdős [9] determined the maximum number of complete subgraphs K_t contained in K_r -free graphs. There are many papers in this field of study. Early references for general graphs are [3, 4]. Alon [3, 4] studied the number of subgraphs of prescribed type of graphs with m edges. As examples of the results on particular graphs, we give some references on the number of paths in graphs. Ahlswede and Katona [1] studied the maximum number of 2-edge paths in graphs with n vertices and m edges. Bollobás and Sarkar [6] determined the maximum number of 4-edge paths in graphs with m edges. Nagy [15] studied the minimum/maximum number of 4-edge paths in graphs with given edge density.

Throughout this paper, we assume that every graph is finite and simple. Let G be a connected graph with n vertices and m edges. Let $c(G)$ denote the number

of cycles contained in G . There are a large literature on the number of prescribed subgraphs contained in a graph even if we restrict a prescribed subgraph to a cycle. For example, Ahrens [2] proved that

$$m - n + 1 \leq c(G) \leq 2^{m-n+1} - 1,$$

where $m - n + 1$ is called the *cyclomatic number* or the *first Betti number* of G . Let $c_4(G)$ denote the number of 4-cycles in G . Alon [5, Special case of Corollary 2.1] showed that, for every fixed $\varepsilon > 0$, and for any graph with n vertices and at least εn^2 edges, $(1/2 + o(1))\binom{n}{2}^2(2\varepsilon)^4 \leq c_4(G)$, where the $o(1)$ terms tend to 0 as n tends to infinity. On the other hand, Hakimi and Schmeichel [10, Theorem 2] showed that, if G is a planar graph with $n \geq 4$ vertices, then $c_4(G) \leq (n^2 + 3n - 22)/2$.

In the present paper, we study upper bounds for $c_4(G)$ for a graph G in terms of the cyclomatic number $m - n + 1$ of G . Applying a result on commutative algebra given by Herzog et al. [12, Corollary 2.6] to the *edge ring* ([11, Chapter 5]) of G , it follows that

$$c_4(G) \leq \begin{cases} \binom{m-n+2}{2} & \text{if } G \text{ is bipartite,} \\ \binom{m-n+1}{2} + k_4(G) & \text{otherwise,} \end{cases} \tag{1}$$

where $k_4(G)$ is the number of complete graphs K_4 in G . We give the details in the Appendix for readers who are interested in the bridges between the algebraic result and the combinatorial statement.

The first main purpose of the present paper is to give a purely combinatorial proof of (1) for the bipartite case, and characterise the extremal graphs.

Proposition 1.1 *Let G be a connected bipartite graph with n vertices and m edges. Then*

$$c_4(G) \leq \binom{m - n + 2}{2}. \tag{2}$$

When G has no vertices of degree 1, equality holds if and only if G is the complete bipartite graph $K_{2,n-2}$.

The second main purpose of the present paper is to give an improvement of (1) for the non-bipartite case.

Theorem 1.2 *Let G be a connected graph with n vertices and m edges. Suppose that G has at least one odd cycle. Then*

$$c_4(G) \leq \binom{m - n + 1}{2}. \tag{3}$$

Note that equality holds in (3) if G is the complete graph K_n on $n \geq 3$ vertices. In fact,

$$c_4(K_n) = 3 \cdot \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{8} = \binom{\binom{n}{2} - n + 1}{2}. \tag{4}$$

In addition, if G is a graph obtained by gluing C_4 and an odd cycle along a vertex, then $m - n = 1$ and $c_4(G) = 1 = \binom{m-n+1}{2}$.

The present paper is organized as follows. In Section 2, after we introduce some results on lattice polytopes arising from graphs, a purely combinatorial proof of Proposition 1.1 will be achieved. Finally, in Section 3, we will show Theorem 1.2 for nonbipartite graphs.

2 Upper bounds for bipartite graphs

In the present section, we give a purely combinatorial proof of Proposition 1.1 for bipartite graphs. First, we introduce some results on lattice polytopes associated with graphs. Given a finite set of vectors $X = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subset \mathbb{R}^n$, the set

$$\text{conv}(X) := \left\{ \sum_{i=1}^m \lambda_i \mathbf{a}_i \in \mathbb{R}^n \mid 0 \leq \lambda_i \in \mathbb{R}, \sum_{i=1}^m \lambda_i = 1 \right\} \tag{5}$$

is called the *convex hull* of X . A subset $P \subset \mathbb{R}^n$ is called a *polytope* if there exists a finite set $X \subset \mathbb{R}^n$ such that $P = \text{conv}(X)$. The *dimension* of a polytope $\text{conv}(X)$ in (5) is the dimension of a vector space that is a translate of

$$\left\{ \sum_{i=1}^m \lambda_i \mathbf{a}_i \in \mathbb{R}^n \mid \lambda_i \in \mathbb{R}, \sum_{i=1}^m \lambda_i = 1 \right\}.$$

A subset F of a polytope $P \subset \mathbb{R}^n$ is called a *face* of P if there exists a vector $\mathbf{w} \in \mathbb{R}^n$ such that $F = \{\mathbf{x} \in P \mid \mathbf{w} \cdot \mathbf{y} \leq \mathbf{w} \cdot \mathbf{x} \text{ for any } \mathbf{y} \in P\}$. It is known that each face of a polytope is again a polytope. In addition, there are only finitely many faces of a polytope. The graph $G(P)$ of a polytope $P \subset \mathbb{R}^n$ is a graph whose vertex set consists of 0-dimensional faces of P and whose edge set consists of 1-dimensional faces of P . If α is a vertex of $G(P)$ whose neighbours are $\alpha_1, \dots, \alpha_s$ in $G(P)$, then it is not difficult to see that

$$P \subset \left\{ \alpha + \sum_{i=1}^s \lambda_i (\alpha_i - \alpha) \in \mathbb{R}^n \mid 0 \leq \lambda_i \in \mathbb{R} \right\}.$$

From this fact, $\dim P$ is equal to the dimension of the vector space spanned by $\alpha_1 - \alpha, \dots, \alpha_s - \alpha$. Hence we have a fundamental fact on $G(P)$.

Lemma 2.1 *Let $P \subset \mathbb{R}^n$ be a d -dimensional polytope. Then every vertex of $G(P)$ has degree at least d .*

A d -dimensional polytope $P \subset \mathbb{R}^n$ is called *simple* if every vertex of the graph $G(P)$ of P has degree d . A d -dimensional polytope $P \subset \mathbb{R}^n$ is called a *simplex* if $G(P)$ has $d + 1$ vertices. Since the graph $G(P)$ of a d -dimensional simplex P is the complete graph K_{d+1} , any simplex is simple. See [18, Chapter 3] for details on graphs of polytopes.

Given a graph G on the vertex set $\{v_1, \dots, v_n\}$ with the edge set $E(G)$, let $\mathcal{P}_G \subset \mathbb{R}^n$ denote the convex hull of $\{\mathbf{e}_i + \mathbf{e}_j \mid \{v_i, v_j\} \in E(G)\}$, where \mathbf{e}_i is the i th unit coordinate vector in \mathbb{R}^n . The polytope \mathcal{P}_G is called the *edge polytope* of G . See [11, Chapter 5] for details on edge polytopes. A characterization of graphs whose edge polytope is a simplex is known [11, Lemma 5.5]. Simple edge polytopes are classified in Ohsugi and Hibi [16, Corollary 5.4].

Proposition 2.2 *Let G be a connected graph. Then,*

- (i) \mathcal{P}_G is simple if and only if either \mathcal{P}_G is a simplex or G is a complete bipartite graph;
- (ii) \mathcal{P}_G is a simplex if and only if either G is a tree or G contains exactly one odd cycle and it is a unique cycle of G .

Let $f_1(G)$ be the number of the 1-dimensional faces of \mathcal{P}_G , that is, the number of the edges of $G(\mathcal{P}_G)$. Several bounds for $f_1(G)$ are given in Hibi et al. [13] and Tran and Ziegler [17]. In particular, the following proposition appears in Tran and Ziegler [17, Proposition 9].

Proposition 2.3 *Let G be a graph with m edges. Then*

$$f_1(G) = \frac{m(m-1)}{2} - 2c_4(G) + 3k_4(G).$$

Note that Propositions 2.2 (i) and 2.3 are proved by graph theoretical arguments based on the following fact: two distinct vertices $\mathbf{e}_i + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_\ell$ of \mathcal{P}_G are adjacent in $G(\mathcal{P}_G)$ if and only if the induced subgraph of G on the vertex set $\{v_i, v_j, v_k, v_\ell\}$ has no C_4 . By using Propositions 2.2 and 2.3, we have the following proposition.

Proposition 2.4 *Let G be a connected graph with n vertices and m edges.*

- (i) *If G is a bipartite graph, then*

$$c_4(G) \leq \frac{m(m-n+1)}{4},$$

and equality holds if and only if G is a tree or a complete bipartite graph.

- (ii) *If G has at least one odd cycle and no K_4 , then*

$$c_4(G) \leq \frac{m(m-n)}{4},$$

and equality holds if and only if G contains exactly one cycle and the length of the cycle is odd.

(iii) If G has at least one K_4 , then

$$c_4(G) < \frac{m(m-n)}{4} + \frac{3}{2}k_4(G).$$

Proof: By Proposition 2.3, we have

$$c_4(G) = \frac{1}{2} \left(\frac{m(m-1)}{2} - f_1(G) + 3k_4(G) \right) = \frac{m(m-1)}{4} - \frac{1}{2}f_1(G) + \frac{3}{2}k_4(G). \quad (6)$$

It is known [11, Lemmas 5.2 and 5.4] that the dimension of \mathcal{P}_G is

$$\dim \mathcal{P}_G = \begin{cases} n-2 & \text{if } G \text{ is bipartite,} \\ n-1 & \text{otherwise,} \end{cases}$$

and that $\{\mathbf{e}_i + \mathbf{e}_j \mid \{v_i, v_j\} \in E(G)\}$ is the set of all vertices (0-dimensional faces) of \mathcal{P}_G . Hence, the graph $G(\mathcal{P}_G)$ of \mathcal{P}_G has m vertices. From Lemma 2.1, the degree of each vertex of $G(\mathcal{P}_G)$ is at least $\dim \mathcal{P}_G$. Therefore,

$$f_1(G) \geq \begin{cases} \frac{m(n-2)}{2} & \text{if } G \text{ is bipartite,} \\ \frac{m(n-1)}{2} & \text{otherwise.} \end{cases}$$

Thus by (6), we have

$$c_4(G) \leq \begin{cases} \frac{m(m-n+1)}{4} & \text{if } G \text{ is bipartite,} \\ \frac{m(m-n)}{4} + \frac{3}{2}k_4(G) & \text{otherwise,} \end{cases}$$

and equality holds if and only if the edge polytope of G is a simple polytope, which is characterized in Proposition 2.2. In particular, if the edge polytope of G is simple, then G has no K_4 . \square

Now we turn to proving Proposition 1.1, which we do inductively. First we set some notation. A vertex v of a connected graph G is called a *cut vertex* if the graph obtained by the removal of v from G is disconnected. An edge e of a connected graph G is called a *bridge* if the graph obtained by the removal of e from G is disconnected. Let

$$\varepsilon(G) = \begin{cases} 1 & \text{if } G \text{ is bipartite,} \\ 0 & \text{otherwise.} \end{cases}$$

By the following lemma, we may assume that a graph G has no cut vertices.

Lemma 2.5 *Suppose that connected graphs G_1 and G_2 have exactly one common vertex, and each G_i has n_i vertices and m_i edges. Let G be the graph $G_1 \cup G_2$ with $n = n_1 + n_2 - 1$ vertices and $m = m_1 + m_2$ edges. If*

$$c_4(G_i) \leq \binom{m_i - n_i + 1 + \varepsilon(G_i)}{2}$$

for $i = 1, 2$, then we have

$$c_4(G) \leq \binom{m - n + 1 + \varepsilon(G)}{2}, \tag{7}$$

and equality holds if and only if $c_4(G_i) = \binom{m_i - n_i + 1 + \varepsilon(G_i)}{2}$ for $i = 1, 2$ and either (i) at least one G_i is a tree or (ii) G_1 contains exactly one cycle and the length of the cycle is odd and G_2 is bipartite, or vice versa.

Proof: Note that $\varepsilon(G) = \varepsilon(G_1) \cdot \varepsilon(G_2)$. Since every 4-cycle is included in a block of G , we have $c_4(G) = c_4(G_1) + c_4(G_2)$.

Case 1 (G_2 is a tree). Then $c_4(G_2) = \binom{m_2 - n_2 + 1 + \varepsilon(G_2)}{2} = 0$, $c_4(G) = c_4(G_1)$ and $\varepsilon(G) = \varepsilon(G_1)$. Hence (7) holds. Moreover, equality holds if and only if $c_4(G_1) = \binom{m_1 - n_1 + 1 + \varepsilon(G_1)}{2}$.

Case 2 (G_i is not a tree for $i \in \{1, 2\}$). It then follows that

$$\begin{aligned} & \left(\binom{(m_1 + m_2) - (n_1 + n_2 - 1) + 1 + \varepsilon(G)}{2} \right) - c_4(G) \\ & \geq \left(\binom{(m_1 + m_2) - (n_1 + n_2) + 2 + \varepsilon(G)}{2} \right) \\ & \quad - \left(\binom{m_1 - n_1 + 1 + \varepsilon(G_1)}{2} \right) - \left(\binom{m_2 - n_2 + 1 + \varepsilon(G_2)}{2} \right) \\ & = \begin{cases} (m_1 - n_1 + 1)(m_2 - n_2 + 1) & \text{if } \varepsilon(G_1) = \varepsilon(G_2), \\ (m_1 - n_1)(m_2 - n_2 + 1) & \text{if } \varepsilon(G_1) = 0 \text{ and } \varepsilon(G_2) = 1, \\ (m_1 - n_1 + 1)(m_2 - n_2) & \text{if } \varepsilon(G_1) = 1 \text{ and } \varepsilon(G_2) = 0 \end{cases} \end{aligned}$$

is nonnegative. Thus we have equation (7).

Since G_i is not a tree for $i = 1, 2$, both $m_1 - n_1 + 1$ and $m_2 - n_2 + 1$ are positive. If $\varepsilon(G_1) = \varepsilon(G_2)$, then $(m_1 - n_1 + 1)(m_2 - n_2 + 1)$ is not zero. Suppose that $\varepsilon(G_1) = 0$ and $\varepsilon(G_2) = 1$. Then $(m_1 - n_1)(m_2 - n_2 + 1) = 0$ if and only if $m_1 = n_1$ if and only if G_1 contains exactly one cycle and the length of the cycle is odd and G_2 is bipartite. The case when $\varepsilon(G_1) = 1$ and $\varepsilon(G_2) = 0$ is similar. \square

We now give a purely combinatorial proof of Proposition 1.1 for bipartite graphs.

Proof of Proposition 1.1. We proceed by induction on m . Let G be a bipartite graph such that $c_4(G) > \binom{m-n+2}{2}$ with minimal m . By Lemma 2.5, we may assume that G has no cut vertices (and so no bridges). Let G' be a (connected) subgraph of G obtained by deleting an edge e_0 of G . Let $c_4(e_0)$ denote the number of 4-cycles of G containing e_0 . By the induction hypothesis, we have

$$c_4(G') \leq \binom{(m-1) - n + 2}{2} = \binom{m - n + 2}{2} - (m - n + 1).$$

Hence $m - n + 2 \leq c_4(e_0)$. Since all edges satisfy this condition,

$$c_4(G) \geq \frac{m(m - n + 2)}{4} > \frac{m(m - n + 1)}{4}.$$

This contradicts Proposition 2.4 and hence (2) holds.

On the other hand, we have

$$c_4(K_{2,n-2}) = \binom{n-2}{2} = \binom{2(n-2) - n + 2}{2}.$$

Conversely, let G be a bipartite graph with $c_4(G) = \binom{m-n+2}{2}$ having no vertices of degree 1. By Lemma 2.5, G has no cut vertices (and so no bridges) since G has no vertices of degree 1 and no odd cycles. Let G' be a (connected) subgraph of G obtained by deleting an edge e_0 of G . Since

$$c_4(G') \leq \binom{(m-1) - n + 2}{2} = c_4(G) - (m - n + 1)$$

holds, it follows that $m - n + 1 \leq c_4(e_0)$ for each edge $e_0 \in E(G)$. Thus $\frac{m(m-n+1)}{4} \leq c_4(G)$. By Proposition 2.4 (i), $c_4(G) = \frac{m(m-n+1)}{4}$ and hence G is a complete bipartite graph. Since G is not a tree, $m - n + 1 \neq 0$. Hence $\frac{(m-n+1)(m-n+2)}{2} = \frac{m(m-n+1)}{4}$ if and only if $m = 2(n - 2)$. It then follows that G is the complete bipartite graph $K_{2,n-2}$. \square

3 Upper bound for nonbipartite graphs

In this section, we prove the main theorem (Theorem 1.2) of the present paper. Theorem 1.2 will be proved by induction on the number of edges, and Propositions 3.5 and 3.6 will play important roles in the proof. In order to show these propositions, we use the following theorem, lemma, and propositions for nonbipartite graphs.

- Proposition 3.1 states that $c_4(G) \leq \binom{m-n+1}{2}$ if $k_4(G) \leq 1$. The proof of this proposition is similar to the proof of Proposition 1.1. In addition, the argument in the proof will be used for the proof of Proposition 3.6.
- Lemma 3.2 states that $c_4(G) \leq \binom{m-n+1}{2}$ if K_{n-1} is a subgraph of G .
- Theorem 3.3 is Motzkin–Straus Theorem for the maximum value of the function $\sum_{\{v_i, v_j\} \in E(G)} x_i x_j$. We will give a sketch of the proof for the readers.
- Proposition 3.4 gives an upper bound for $c_4(G)$ in terms of m , n , the minimum degree, the clique number, and the sum Σ_2 of the squares of the degrees for G . We will give a proof of Proposition 3.4 by using Motzkin–Straus Theorem. This will be useful because upper bounds (8) and (9) for Σ_2 in terms of m , n , the minimum degree, and the maximum degree are known.

- Proposition 3.5 states that $c_4(G) \leq \binom{m-n+1}{2}$ if the minimum degree, and the maximum degree of a graph satisfy some conditions. The main tool for the proof is Proposition 3.4.
- Proposition 3.6 states that $c_4(G) \leq \binom{m-n+1}{2}$ if G satisfies some conditions on the minimum degree, the maximum degree, and any nonbipartite subgraph $G \setminus v$ or $G \setminus e$ with $v \in V(G)$, $e \in E(G)$ satisfies such an inequality. Again, the main tool for the proof is Proposition 3.4.

First, we show that, if G has at most one K_4 , then $c_4(G) \leq \binom{m-n+1}{2}$. Given a graph G , a *block* of G is a maximal connected subgraph of G with no cut vertices.

Proposition 3.1 *Let G be a connected graph having at least one odd cycle. If $k_4(G) \leq 1$, then we have*

$$c_4(G) \leq \binom{m-n+1}{2}.$$

When G has no vertices of degree 1 and $k_4(G) = 0$, equality holds if and only if G is one of the following:

- (a) an odd cycle;
- (b) the union of $K_{2,n'}$ and a path P where common vertices of $K_{2,n'}$ and P are end vertices of P ;
- (c) a graph whose set of blocks consists of one $K_{2,n'}$, one odd cycle and some bridges.

Proof: We proceed by induction on m . Let G be a connected graph having at least one odd cycle such that $k_4(G) \leq 1$ and $c_4(G) > \binom{m-n+1}{2}$ with minimal m . By Lemma 2.5, we may assume that G has no cut vertices. Let G' be a (connected) subgraph of G obtained by deleting an edge e_0 of G . Then we claim the following:

Claim 1. G' is not bipartite.

If G' is bipartite, then e_0 joins two vertices in the same part in G' since G is not bipartite. Hence there exists no C_4 that contains e_0 . Thus

$$c_4(G) = c_4(G') \leq \binom{(m-1)-n+2}{2} = \binom{m-n+1}{2}$$

which is a contradiction. Therefore G' is not bipartite.

Claim 2. $m-n+1 \leq c_4(e_0)$.

By the induction hypothesis, we have

$$c_4(G') \leq \binom{(m-1)-n+1}{2} = \binom{m-n+1}{2} - (m-n).$$

Hence $m-n+1 \leq c_4(e_0)$.

Since every edge e_0 of G satisfies $m - n + 1 \leq c_4(e_0)$,

$$c_4(G) \geq \frac{m(m - n + 1)}{4}.$$

However, by Proposition 2.4,

$$c_4(G) \leq \frac{m(m - n)}{4} < \frac{m(m - n + 1)}{4},$$

if G has no K_4 , and

$$c_4(G) < \frac{m(m - n)}{4} + \frac{3}{2} \leq \frac{m(m - n)}{4} + \frac{m}{4} = \frac{m(m - n + 1)}{4}$$

if G has one K_4 (and hence $m \geq 6$). This is a contradiction and hence $c_4(G) \leq \binom{m-n+1}{2}$.

On the other hand, let G be a connected graph having at least one odd cycle such that $c_4(G) = \binom{m-n+1}{2}$, $k_4(G) = 0$, and G has no vertices of degree 1.

Case 1. (G has a cut vertex.) Since G is not bipartite, $\varepsilon(G) = 0$. Since G has no vertices of degree 1, by Lemma 2.5, $G = G_1 \cup G_2$ where

- G_1 and G_2 have exactly one common vertex,
- G_1 contains exactly one cycle and the length of the cycle is odd, and
- G_2 is a bipartite graph with n' vertices and m' edges such that $c_4(G_2) = \binom{m'-n'+1}{2}$.

Moreover, by Proposition 1.1, G_2 is a graph whose set of blocks consists of one $K_{2,\ell}$ and some bridges. Thus G satisfies (c).

Case 2. (G has no cut vertices.) Suppose that there exists an edge e_0 of G such that the subgraph G' of G obtained by deleting an edge e_0 of G is bipartite. Then there exists no C_4 that contains e_0 . Thus we have

$$c_4(G') = c_4(G) = \binom{m - n + 1}{2} = \binom{(m - 1) - n + 2}{2}.$$

Since G' is a bipartite graph having n vertices and $m - 1$ edges, by Proposition 1.1, G' is either a tree or the complete bipartite graph $K_{2,n'}$ (by removing vertices of degree 1). Therefore G satisfies either (b) or (c). Suppose that for any edge e_0 of G , the subgraph G' of G obtained by deleting e_0 has at least one odd cycle. Since $k_4(G') = 0$, we have

$$c_4(G') \leq \binom{(m - 1) - n + 1}{2} = c_4(G) - (m - n).$$

Hence $m - n \leq c_4(G) - c_4(G') = c_4(e_0)$ for each edge e_0 of G . Thus $c_4(G) \geq \frac{m(m-n)}{4}$. Since $k_4(G) = 0$, by Proposition 2.4 (ii), $c_4(G) \leq \frac{m(m-n)}{4}$ and hence $c_4(G) = \frac{m(m-n)}{4}$. From Proposition 2.4 (ii), G contains exactly one cycle and the length of the cycle is odd. Since G has no vertices of degree 1, G is an odd cycle. This contradicts the hypothesis that G' has at least one odd cycle. \square

A *clique* of a graph G is a subgraph of G that is a complete graph. The *clique number* $\omega(G)$ of a graph G is the number of vertices in a maximum clique of G . If G has at least one odd cycle and $\omega(G)$ is at most 3, then G has no K_4 and hence $c_4(G) \leq \binom{m-n+1}{2}$. On the other hand, if $\omega(G)$ is very large, then $c_4(G) \leq \binom{m-n+1}{2}$ also holds for G .

Lemma 3.2 *Let G be a connected graph with $n \geq 4$ vertices and $\omega(G) \geq n - 1$. Then $c_4(G) \leq \binom{m-n+1}{2}$.*

Proof: If $\omega(G) = n$, then $G = K_n$. This case is proved in (4) of Section 1. If $\omega(G) = n - 1$, then K_{n-1} is a subgraph of G . Suppose that $m = \binom{n-1}{2} + k$ where $1 \leq k \leq n - 2$. Then

$$\begin{aligned} & \binom{m-n+1}{2} - c_4(G) \\ &= \frac{((\binom{n-1}{2} + k - n)(\binom{n-1}{2} + k - n + 1))}{2} - 3\binom{n-1}{4} - (n-3)\binom{k}{2} \\ &= \frac{(n-4)(k-1)(n-1-k)}{2} \geq 0. \end{aligned}$$

Thus $c_4(G) \leq \binom{m-n+1}{2}$. □

Next, we give an upper bound for $c_4(G)$ in terms of several parameters on G (Proposition 3.4). The following theorem will play an important role for the proof of Proposition 3.4.

Theorem 3.3 (Motzkin–Straus [14]) *Let G be a graph on the vertex set $\{v_1, \dots, v_n\}$. Then*

$$\max \left\{ \sum_{\{v_i, v_j\} \in E(G)} x_i x_j \mid x_i \geq 0, \sum_{i=1}^n x_i = 1 \right\} = \frac{1}{2} \left(1 - \frac{1}{\omega(G)} \right).$$

Sketch of Proof. Suppose that (x_1, \dots, x_n) with $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$ attains the maximum for the function $\sum_{\{v_i, v_j\} \in E(G)} x_i x_j$. If $x_i, x_j > 0$ ($i \neq j$) and $\{v_i, v_j\}$ is not an edge of G , then, by replacing (x_i, x_j) with either $(x_i + x_j, 0)$ or $(0, x_i + x_j)$, we obtain a new (x_1, \dots, x_n) such that the value of the function does not decrease. Thus we may assume that $S = \{v_i \mid x_i > 0\}$ is the vertex set of a clique of G . Then

$$\sum_{\{v_i, v_j\} \in E(G)} x_i x_j = \frac{1}{2} \left(\left(\sum_{v_i \in S} x_i \right)^2 - \sum_{v_i \in S} x_i^2 \right) = \frac{1}{2} \left(1 - \sum_{v_i \in S} x_i^2 \right) \leq \frac{1}{2} \left(1 - \frac{1}{|S|} \right),$$

and the equality holds when $x_i = 1/|S|$ for all $v_i \in S$. □

Given a graph G on the vertex set $\{v_1, \dots, v_n\}$, let $\delta(G)$ denote the *minimum degree* of G , and let $\Delta(G)$ denote the *maximum degree* of G . That is,

$$\delta(G) = \min\{\deg_G(v_i) \mid 1 \leq i \leq n\}, \quad \Delta(G) = \max\{\deg_G(v_i) \mid 1 \leq i \leq n\}.$$

Let

$$\Sigma_2 = \sum_{i=1}^n \deg_G(v_i)^2 \left(= \sum_{\{v_i, v_j\} \in E(G)} (\deg_G(v_i) + \deg_G(v_j)) \right).$$

Several upper bounds for Σ_2 are known. For example, if $\delta \leq \delta(G) \leq \Delta(G) \leq \Delta$, then

$$\Sigma_2 \leq m \left(\frac{2m}{n-1} + n - 2 \right), \tag{8}$$

$$\Sigma_2 \leq 2m(\Delta + \delta) - n\Delta\delta. \tag{9}$$

Here (8) is given by de Caen [8, Theorem 1], and (9) is given by Das [7, Theorem 4.3]:

$$\begin{aligned} \Sigma_2 &= \sum_{i=1}^n (\deg_G(v_i)(\deg_G(v_i) - \delta) + \deg_G(v_i)\delta) \\ &\leq \sum_{i=1}^n (\Delta(\deg_G(v_i) - \delta) + \deg_G(v_i)\delta) = 2m(\Delta + \delta) - n\Delta\delta. \end{aligned}$$

Proposition 3.4 *Let G be a connected graph with $\delta(G) \geq 2$. Then, for any $2 \leq \alpha \leq \delta(G)$ and $\beta \geq \omega(G)$, we have*

$$c_4(G) \leq \frac{(2m - \alpha n)^2}{8} \left(1 - \frac{1}{\beta} \right) + \frac{m(n - \alpha^2 + 1)}{4} + \frac{\alpha - 2}{4} \Sigma_2. \tag{10}$$

In addition,

$$c_4(G) = \frac{(2m - \delta(G)n)^2}{8} \left(1 - \frac{1}{\omega(G)} \right) + \frac{m(n - \delta(G)^2 + 1)}{4} + \frac{\delta(G) - 2}{4} \Sigma_2 \tag{11}$$

holds if and only if G is either K_n or $K_{\ell, \ell}$ with $n = 2\ell$.

Proof: We will prove (10) by applying Motzkin–Straus Theorem. First we have to rewrite the expression so that we can apply Motzkin–Straus Theorem.

Let $c_t(i, j)$ denote the number of t -cycles of G containing $\{v_i, v_j\} \in E(G)$. Then

$$\begin{aligned} c_4(i, j) &\leq (\deg_G(v_i) - 1)(\deg_G(v_j) - 1) - c_3(i, j) \\ &\leq (\deg_G(v_i) - 1)(\deg_G(v_j) - 1) - (\deg_G(v_i) + \deg_G(v_j) - n) \\ &= (\deg_G(v_i) - \alpha)(\deg_G(v_j) - \alpha) + (\alpha - 2)(\deg_G(v_i) + \deg_G(v_j)) + n - \alpha^2 + 1. \end{aligned}$$

Since $\sum_{\{v_i, v_j\} \in E(G)} c_4(i, j) = 4c_4(G)$ and $\sum_{\{v_i, v_j\} \in E(G)} (\deg_G(v_i) + \deg_G(v_j)) = \Sigma_2$, we have

$$c_4(G) \leq \frac{1}{4} \sum_{\{v_i, v_j\} \in E(G)} (\deg_G(v_i) - \alpha)(\deg_G(v_j) - \alpha) + \frac{m(n - \alpha^2 + 1)}{4} + \frac{\alpha - 2}{4} \Sigma_2.$$

Since $\alpha n \leq \delta(G)n \leq 2m$, we have $2m - \alpha n \geq 0$. Equality holds if and only if $\alpha = \delta(G)$ and G is a regular graph. In this case, $\sum_{\{v_i, v_j\} \in E(G)} (\deg_G(v_i) - \alpha)(\deg_G(v_j) - \alpha) = 0$. If $2m - \alpha n > 0$, then let

$$x_i = \frac{\deg_G(v_i) - \alpha}{2m - \alpha n} \geq 0$$

for each vertex v_i . Since $\sum_{i=1}^n (\deg_G(v_i) - \alpha) = 2m - \alpha n$, we have $\sum_{i=1}^n x_i = 1$. By Motzkin–Straus Theorem,

$$\sum_{\{v_i, v_j\} \in E(G)} x_i x_j \leq \frac{1}{2} \left(1 - \frac{1}{\omega(G)} \right).$$

It follows that

$$\begin{aligned} c_4(G) &\leq \frac{(2m - \alpha n)^2}{8} \left(1 - \frac{1}{\omega(G)} \right) + \frac{m(n - \alpha^2 + 1)}{4} + \frac{\alpha - 2}{4} \Sigma_2 \\ &\leq \frac{(2m - \alpha n)^2}{8} \left(1 - \frac{1}{\beta} \right) + \frac{m(n - \alpha^2 + 1)}{4} + \frac{\alpha - 2}{4} \Sigma_2 \end{aligned}$$

for any $\beta \geq \omega(G)$.

Next we will classify graphs G satisfying (11). The complete graph K_n has $m = n(n - 1)/2$ edges and satisfies $\delta(K_n) = n - 1$ and $\Sigma_2 = n(n - 1)^2$. By substituting these values into the right-hand side of (11), we have

$$\begin{aligned} &\frac{(2m - \delta(K_n)n)^2}{8} \left(1 - \frac{1}{\omega(K_n)} \right) + \frac{m(n - \delta(K_n)^2 + 1)}{4} + \frac{\delta(K_n) - 2}{4} \Sigma_2 \\ &= n(n - 1)(n - 2)(n - 3)/8 \\ &= c_4(K_n). \end{aligned}$$

On the other hand, if $G = K_{\ell, \ell}$, then $(n, m) = (2\ell, \ell^2)$, $\delta(G) = \ell$, and $\Sigma_2 = 2\ell^3$. By substituting these values into the right-hand side of (11), we have

$$\begin{aligned} &\frac{(2m - \delta(K_{\ell, \ell})n)^2}{8} \left(1 - \frac{1}{\omega(K_{\ell, \ell})} \right) + \frac{m(n - \delta(K_{\ell, \ell})^2 + 1)}{4} + \frac{\delta(K_{\ell, \ell}) - 2}{4} \Sigma_2 \\ &= \frac{\ell^2(\ell - 1)^2}{4} = \binom{\ell}{2}^2 = c_4(K_{\ell, \ell}). \end{aligned}$$

Conversely, suppose that $G \notin \{K_n, K_{\ell, \ell}\}$ satisfies (11). It then follows that

$$c_4(i, j) = (\deg_G(v_i) - 1)(\deg_G(v_j) - 1) - c_3(i, j) \tag{12}$$

$$c_3(i, j) = \deg_G(v_i) + \deg_G(v_j) - n \tag{13}$$

for any edge $\{v_i, v_j\}$ of G . From (13), we have

$$(\deg_G(v_i) - 1) + (\deg_G(v_j) - 1) - c_3(i, j) = n - 2.$$

Hence any vertex of G is adjacent to either v_i or v_j (or both). Moreover, from (12), if $\{v_i, v_k\}$ and $\{v_j, v_\ell\}$ are edges of G with $k \neq j$, $\ell \neq i$, $k \neq \ell$, then $\{v_k, v_\ell\}$ is an edge of G . Let K_r ($r < n$) be a maximum clique of G .

Case 1 ($r \geq 3$). Since G is connected, there exists an edge $\{v_i, v_j\}$ of G such that v_i is a vertex of K_r and v_j is not a vertex of K_r . Then the above claim guarantees that v_j is adjacent to all vertices of K_r . This contradicts the hypothesis that K_r is a maximum clique.

Case 2 ($r = 2$). Let $\{v_i, v_j\}$ be an edge of G . Since G has no triangles, any vertex of G is adjacent to exactly one of v_i and v_j . Moreover, since

$$c_4(i, j) = (\deg_G(v_i) - 1)(\deg_G(v_j) - 1),$$

G is a complete bipartite graph K_{ℓ_1, ℓ_2} , where $\ell_1 = \deg_G(v_i)$, $\ell_2 = \deg_G(v_j)$. We may assume that $\ell_2 \leq \ell_1$. Then K_{ℓ_1, ℓ_2} has $\ell_1 + \ell_2$ vertices, $\ell_1 \ell_2$ edges, and satisfies

$$\begin{aligned} \delta(K_{\ell_1, \ell_2}) &= \ell_2, \\ \omega(K_{\ell_1, \ell_2}) &= 2, \\ \Sigma_2 &= \ell_1 \ell_2^2 + \ell_2 \ell_1^2 = \ell_1 \ell_2 (\ell_1 + \ell_2), \\ c_4(K_{\ell_1, \ell_2}) &= \binom{\ell_1}{2} \binom{\ell_2}{2}. \end{aligned}$$

Hence the right-hand side of (11) is

$$\begin{aligned} & \frac{(2\ell_1 \ell_2 - \ell_2(\ell_1 + \ell_2))^2}{16} + \frac{\ell_1 \ell_2 (\ell_1 + \ell_2 - \ell_2^2 + 1)}{4} + \frac{(\ell_2 - 2)\ell_1 \ell_2 (\ell_1 + \ell_2)}{4} \\ &= \frac{\ell_2^2 (\ell_1 - \ell_2)^2}{16} + c_4(K_{\ell_1, \ell_2}). \end{aligned}$$

Thus equality (11) holds if and only if $\ell_1 = \ell_2$. □

As an application, we have the following propositions that will play an important role for a proof of the main theorem.

Proposition 3.5 *Let G be a connected graph with n vertices having at least one odd cycle. If G satisfies at least one of the following conditions, then $c_4(G) \leq \binom{m-n+1}{2}$.*

- (i) $\delta(G) \geq 4$ and $\Delta(G) \leq \frac{3n+1}{4}$;
- (ii) $\delta(G) \geq 12$;
- (iii) $5 \leq \delta(G) \leq \Delta(G) \leq n - 2$ and $n \leq 27$.

Proof: By Lemma 3.2, we may assume that $\omega(G) \leq n - 2$. If $n \leq 5$, then G has no K_4 and hence the inequality holds from Proposition 3.1. We may assume that $n \geq 6$. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$. We will use Proposition 3.4 together with (8) and (9) for the proof.

(i) By (9), we have $\Sigma_2 \leq 2m(\Delta + 4) - 4n\Delta$ since $\delta \geq 4$. By this and Proposition 3.4, it follows that

$$c_4(G) \leq \frac{(m - 2n)^2}{2} \left(1 - \frac{1}{\omega(G)}\right) + \frac{m(n - 15)}{4} + m(\Delta + 4) - 2n\Delta.$$

Hence

$$\begin{aligned} & \binom{m - n + 1}{2} - c_4(G) \\ & \geq \binom{m - n + 1}{2} - \left(\frac{(m - 2n)^2}{2} \left(1 - \frac{1}{\omega(G)}\right) + \frac{m(n - 15)}{4} + m(\Delta + 4) - 2n\Delta \right) \\ & = \frac{(m - 2n)((m - 2n) - \frac{\omega(G)}{2}(4\Delta - 3n - 1))}{2\omega(G)}. \end{aligned}$$

Since $\delta \geq 4$, we have $m \geq 2n$. Thus $\binom{m - n + 1}{2} - c_4(G) \geq 0$ if $\Delta \leq \frac{3n + 1}{4}$ holds.

(ii) Let $\delta(G) \geq 12$. Then $n \geq 13$ and $m \geq 6n$. By substituting $\alpha = 4$ and $\beta = n - 2$ in the inequality in Proposition 3.4 and (8), we have

$$\begin{aligned} & \binom{m - n + 1}{2} - c_4(G) \\ & \geq \binom{m - n + 1}{2} - \left(\frac{(m - 2n)^2}{2} \left(1 - \frac{1}{n - 2}\right) + \frac{m(n - 15)}{4} + \frac{1}{2}m \left(\frac{2m}{n - 1} + n - 2 \right) \right) \\ & = \frac{1}{2(n - 2)} \left(-\frac{n - 3}{n - 1}m^2 + \frac{(n - 3)(n + 14)}{2}m - n(3n^2 - 9n - 2) \right). \end{aligned}$$

Let

$$\varphi(x) = -\frac{n - 3}{n - 1}x^2 + \frac{(n - 3)(n + 14)}{2}x - n(3n^2 - 9n - 2).$$

Since $n \geq 13$,

$$\begin{aligned} \varphi(6n) &= \frac{2n(3n^2 - 29n + 62)}{n - 1} > 0 \\ \varphi\left(\binom{n}{2}\right) &= \frac{n(n - 5)^2}{2} > 0. \end{aligned}$$

Thus $\varphi(x) > 0$ for all $6n \leq x \leq \binom{n}{2}$. Since $6n \leq m \leq \binom{n}{2}$, it follows that $c_4(G) \leq \binom{m - n + 1}{2}$.

(iii) Note that $n \geq 7$ since $5 \leq n - 2$. Since $5 \leq \delta \leq \Delta \leq n - 2$, by substituting $\alpha = 5$ and $\beta = n - 2$ in the inequality in Proposition 3.4 and (9), we have

$$\begin{aligned} & \binom{m-n+1}{2} - c_4(G) \\ \geq & \binom{m-n+1}{2} - \left(\frac{(m-\frac{5n}{2})^2}{2} \left(1 - \frac{1}{n-2} \right) + \frac{m(n-24)}{4} + \frac{3}{4} (2m(n+3) - 5n(n-2)) \right) \\ = & \frac{1}{2(n-2)} \left(m^2 - \frac{1}{2} (n^2 + 16) m + \frac{1}{4} n (9n^2 - 57n + 128) \right). \end{aligned}$$

The minimum value of this function is

$$\frac{1}{32} (-n^3 + 34n^2 - 192n + 128) = \frac{1}{32} ((27 - n)(n^2 - 7n + 3) + 47)$$

when $m = \frac{1}{4}(n^2 + 16)$ and this minimum value is positive for $n = 7, 8, \dots, 27$. \square

Proposition 3.6 *Let G be a 2-connected graph with $n \geq 6$ vertices and m edges that has at least one odd cycle. Let $\delta = \delta(G) \geq 2$ and $\Delta = \Delta(G)$. Suppose that $c_4(H) \leq \binom{m'-n'+1}{2}$ for every graph H with n' vertices and m' edges having at least one odd cycle obtained by deleting either a vertex or an edge of G . If G satisfies at least one of the following conditions, then $c_4(G) \leq \binom{m-n+1}{2}$:*

- (i) $\Delta = n - 1$;
- (ii) $\delta \geq 3$ and $m < \delta(n - \delta)$;
- (iii) $m \geq \frac{(\delta+2)n - (3\delta+1)}{2} + \frac{1}{2(\delta-1)}$;
- (iv) $\delta \leq 3$.

Proof: First, we explain the reason why we may assume that any subgraph H of G obtained by removing either a vertex or an edge from G satisfies $c_4(H) \leq \binom{m'-n'+1}{2}$. Let H be an induced subgraph of G obtained by removing a vertex $v \in V(G)$ from G . If H has no triangles, then G has no K_4 , and hence Proposition 3.1 guarantees $c_4(G) \leq \binom{m-n+1}{2}$. Thus we may assume that such an induced subgraph H has a triangle, and hence H satisfies $c_4(H) \leq \binom{m'-n'+1}{2}$ by the hypothesis. On the other hand, let H be a subgraph of G obtained by removing an edge $e_0 \in E(G)$ from G . By the same argument in Claim 1 in the proof of Proposition 3.1, we may assume that H is not bipartite. By the hypothesis, H satisfies $c_4(H) \leq \binom{m'-n'+1}{2}$.

In addition, by the same argument in Claim 2 in the proof of Proposition 3.1, we may assume that any edge e_0 is contained in at least $(m - n + 1)$ 4-cycles in G . This fact will play an important role for (ii) and (iv).

(i) Let v be a vertex of G of degree $n - 1$ and let G' an induced subgraph of G obtained by removing v from G . Then G' is connected, not bipartite, and has $n - 1$

vertices and $m - n + 1$ edges. Then G' satisfies $c_4(G') \leq \binom{(m-n+1)-(n-1)+1}{2}$, and by applying (8) to $\sum_{v' \in V(G')} \deg_{G'}(v')^2$, it follows that

$$\begin{aligned} c_4(G) &= c_4(G') + \sum_{v' \in V(G')} \binom{\deg_{G'}(v')}{2} \\ &\leq \binom{m - 2n + 3}{2} + \frac{1}{2} \sum_{v' \in V(G')} (\deg_{G'}(v')^2 - \deg_{G'}(v')) \\ &\leq \binom{m - 2n + 3}{2} + \frac{1}{2}(m - n + 1) \left(\frac{2(m - n + 1)}{n - 2} + n - 3 \right) - (m - n + 1) \\ &= \binom{m - n + 1}{2} - \frac{\left(\binom{n}{2} - m \right) (m - 2n + 3)}{n - 2}. \end{aligned}$$

Since G' is a connected graph with $n - 1$ vertices and $m - n + 1$ edges, it follows that $(n - 1) - 1 \leq m - n + 1$. Hence $2n - 3 \leq m \leq \binom{n}{2}$. Thus $c_4(G) \leq \binom{m-n+1}{2}$.

(ii) Suppose that G satisfies $c_4(G) > \binom{m-n+1}{2}$, $\delta \geq 3$ and $m < \delta(n - \delta)$. Let v be a vertex of G such that the degree of a vertex v is δ . Suppose that v_1, \dots, v_δ are incident with v in G . Let $\deg_G(v_i) = \delta + \alpha_i$ for $i = 1, 2, \dots, \delta$ and $\alpha = \min\{\alpha_1, \dots, \alpha_\delta\}$. Then

$$\frac{1}{2}\delta(n + \alpha) \leq \frac{1}{2} \left(\sum_{i=1}^{\delta} (\delta + \alpha) + \sum_{i=\delta+1}^n \delta \right) \leq \frac{1}{2} \sum_{i=1}^n \deg_G(v_i) = m. \tag{14}$$

Suppose that $\alpha = \alpha_j$. Let $c_t(v, v_j)$ denote the number of t -cycles of G containing $\{v, v_j\}$. Then we have

$$m - n + 1 \leq c_4(v, v_j).$$

In addition, as in the proof of Proposition 3.4, it follows that

$$c_4(v, v_j) \leq (\deg_G(v) - 1)(\deg_G(v_j) - 1) - c_3(v, v_j) \leq (\deg_G(v) - 1)(\deg_G(v_j) - 1).$$

Thus

$$m - n + 1 \leq c_4(v, v_j) \leq (\deg_G(v) - 1)(\deg_G(v_j) - 1) = (\delta - 1)(\delta + \alpha - 1)$$

and hence

$$m \leq (\delta - 1)(\delta + \alpha - 1) + n - 1. \tag{15}$$

From (14) and (15),

$$\frac{1}{2}\delta(n + \alpha) \leq m \leq (\delta - 1)(\delta + \alpha - 1) + n - 1.$$

This inequality simplifies to $(\delta - 2)(n - 2\delta) \leq (\delta - 2)\alpha$. Since $\delta \geq 3$ by hypothesis, canceling implies $n - 2\delta \leq \alpha$. Thus $2(n - \delta) \leq n + \alpha$ and hence

$$\delta(n - \delta) \leq \frac{1}{2}\delta(n + \alpha) \leq m,$$

which contradicts the hypothesis $m < \delta(n - \delta)$.

(iii) We may assume that $\Delta \leq n - 2$ by (i). Let v be a vertex of G of degree δ that is adjacent to v_1, \dots, v_δ and let G' an induced subgraph of G obtained by deleting v . Note that the number of copies of C_4 that contains v is equal to $\sum_{1 \leq i < j \leq \delta} s_{ij}$, where

$$s_{ij} = |\{v' \mid \{v_i, v'\}, \{v_j, v'\} \in E(G), v' \neq v\}|.$$

It is trivial that $s_{ij} \leq n - 3$. On the other hand, if $s_{ij} = s_{ik} = n - 3$ ($j \neq k$), then the degree of v_i is $n - 1$. Hence the number of i, j such that $s_{ij} = n - 3$ is at most $\lfloor \delta/2 \rfloor$. Thus the number of copies of C_4 that contains v is at most $(n - 4) \binom{\delta}{2} + \lfloor \frac{\delta}{2} \rfloor$. Therefore

$$c_4(G) \leq c_4(G') + (n - 4) \binom{\delta}{2} + \left\lfloor \frac{\delta}{2} \right\rfloor.$$

Since G' satisfies $c_4(G') \leq \binom{(m-\delta)-(n-1)+1}{2}$, we have

$$\begin{aligned} & \binom{m - n + 1}{2} - c_4(G) \\ \geq & \binom{m - n + 1}{2} - \binom{(m - \delta) - (n - 1) + 1}{2} - (n - 4) \binom{\delta}{2} - \left\lfloor \frac{\delta}{2} \right\rfloor \\ = & (\delta - 1) \left(m - \frac{(\delta + 2)n - (3\delta + 1)}{2} \right) - \frac{1}{2} + \frac{\delta}{2} - \left\lfloor \frac{\delta}{2} \right\rfloor. \end{aligned}$$

This is nonnegative if

$$m \geq \begin{cases} \frac{(\delta+2)n-(3\delta+1)}{2} + \frac{1}{2(\delta-1)} & \delta \text{ is even,} \\ \frac{(\delta+2)n-(3\delta+1)}{2} & \delta \text{ is odd.} \end{cases}$$

(iv) Let G be a 2-connected graph with minimum degree $\delta \in \{2, 3\}$ having at least one odd cycle such that

$$c_4(G) > \binom{m - n + 1}{2}.$$

Suppose that $\delta = 2$, $\deg(v_1) = 2$, and v_1 is contained in k 4-cycles of G . Then $0 \leq k \leq n - 3$. Since each edge of G is contained in at least $(m - n + 1)$ 4-cycles of G , we have $m - n + 1 \leq k$ and hence $m \leq n + k - 1$. The union of k C_4 's is a complete bipartite graph $K_{2,k+1}$ with $2(k + 1)$ edges. If $k = n - 3$, then G has at least $2(k + 1) + 1 = 2k + 3 = n + k$ edges since G has at least one odd cycle. If $k < n - 3$, then G has at least $2(k + 1) + (n - k - 3) + 1 = n + k$ edges since remaining $n - k - 3$ vertices are of degree ≥ 2 and G is 2-connected. Thus we have $n + k \leq m \leq n + k - 1$, a contradiction.

Suppose that $\delta = 3$. By (ii) and (iii), we may assume that

$$3(n - 3) \leq m < \frac{(\delta + 2)n - (3\delta + 1)}{2} = \frac{5}{2}n - 5. \tag{16}$$

Since $3n - 9 < \frac{5}{2}n - 5$, we have $n < 8$ and hence $n = 6, 7$. If $n = 6$, then $m = 9$ by (16). Then G is a connected 3-regular graph with 6 vertices. It is easy to see that G has no K_4 , a contradiction. If $n = 7$, then $m = 12$ by (16). Let $\{v_1, \dots, v_7\}$ be the vertex set of G with $\deg_G(v_1) = 3$ and $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\} \in E(G)$. Since $m = \delta(n - \delta)$, by the argument in the proof of (ii), the degree of v_i ($i = 2, 3, 4$) is $\delta + \alpha = \delta + n - 2\delta = 4$. Then

$$24 = 2m = \sum_{i=1}^7 \deg_G(v_i) = 15 + \deg_G(v_5) + \deg_G(v_6) + \deg_G(v_7).$$

Since $\deg_G(v_i) \geq 3$, we have $\deg_G(v_i) = 3$ for $i = 5, 6, 7$. Note that the degree of a vertex incident with one of the vertices v_1, v_5, v_6, v_7 of degree 3 should be 4. Thus G is the complete bipartite graph $K_{3,4}$ on the vertex set $\{v_2, v_3, v_4\} \cup \{v_1, v_5, v_6, v_7\}$, a contradiction. \square

We are now in a position to prove the main theorem of the present paper.

Proof of Theorem 1.2. Let G be a 2-connected graph with $n \geq 6$ vertices having at least one odd cycle with

$$c_4(G) > \binom{m - n + 1}{2}.$$

Suppose that the assertion holds for any connected graph G' having at least one odd cycle obtained by deleting either a vertex or an edge of G . For $n \in \{6, 7, 8, 9\}$, we have $\lfloor (3n + 1)/4 \rfloor = n - 2$. By Proposition 3.5 (i) and Proposition 3.6 (i) and (iv), it follows that $n \geq 10$. In addition, by Proposition 3.5 (ii) and Proposition 3.6, we may assume that G is a 2-connected graph with $4 \leq \delta \leq 11$, $\Delta \leq n - 2$, and

$$\delta(n - \delta) \leq m < \frac{(\delta + 2)n - (3\delta + 1)}{2} + \frac{1}{2(\delta - 1)}.$$

In particular,

$$\delta(n - \delta) < \frac{(\delta + 2)n - (3\delta + 1)}{2} + \frac{1}{2(\delta - 1)}$$

holds. Since $\delta \geq 4$, we have

$$n < \frac{2\delta^3 - 5\delta^2 + 2\delta + 2}{(\delta - 2)(\delta - 1)} = 2\delta + 1 + \frac{\delta}{(\delta - 2)(\delta - 1)} < 2\delta + 2.$$

If $\delta = 4$, then $n < 2\delta + 2 = 10$, a contradiction. Thus $\delta \geq 5$. By Proposition 3.5 (iii), we have $n \geq 28$. However, since $\delta \leq 11$, we have $n < 2\delta + 2 \leq 24$. This is a contradiction. \square

It would, of course, be of interest to classify all connected graphs G which satisfy the equality $c_4(G) = \binom{m - n + 1}{2}$.

Appendix: Algebraic proof for equation (1)

In this appendix, we give an algebraic proof of equation (1) in Section 1. Let G be a connected graph on the vertex set $\{v_1, \dots, v_n\}$ whose edge set is $\{e_1, \dots, e_m\}$. Then the toric ideal I_G of the edge ring $K[G]$ of G is defined as follows. Let $K[x_1, \dots, x_m]$ and $K[t_1, \dots, t_n]$ be polynomial rings over a field K . Define the ring homomorphism $\pi_G : K[x_1, \dots, x_m] \rightarrow K[t_1, \dots, t_n]$ by $\pi_G(x_i) = t_j t_k \in K[t_1, \dots, t_n]$ where $e_i = \{v_j, v_k\}$ for each $1 \leq i \leq m$. Then the toric ring $K[G]$ is the image of π_G , and the toric ideal I_G of $K[G]$ is the kernel of π_G . See [11, Chapter 5] for details. It is known [11, Lemma 5.9] that the toric ideal I_G is generated by homogeneous binomials of the form

$$f_\Gamma = \prod_{k=1}^{\ell} x_{i_{2k-1}} - \prod_{k=1}^{\ell} x_{i_{2k}},$$

where $\Gamma = (e_{i_1}, \dots, e_{i_{2\ell}})$ is a closed walk of even length in G . In particular, $f_\Gamma \in I_G$ is quadratic if and only if Γ is a 4-cycle contained in G .

Applying a result by Herzog et al. [12, Corollary 2.6] to the edge ring of G , it follows that the number of quadratic binomials in a minimal set of generators of I_G is less than or equal to $\binom{m - \dim K[G] + 1}{2}$, where

$$\dim K[G] = \begin{cases} n - 1 & \text{if } G \text{ is bipartite,} \\ n & \text{otherwise.} \end{cases}$$

If $f_\Gamma \in I_G$ is quadratic, then f_Γ is generated by other (quadratic) binomials of I_G if and only if the induced subgraph G' of G on the vertex set $V(\Gamma)$ is a complete graph K_4 . More precisely, if G' is K_4 , then G' has three 4-cycles $\Gamma, \Gamma_1, \Gamma_2$ and $f_\Gamma = f_{\Gamma_1} + f_{\Gamma_2}$. Thus the number of quadratic binomials in a minimal set of generators of I_G is $c_4(G) - k_4(G)$. If G is bipartite, then $k_4(G) = 0$. Hence

$$c_4(G) \leq \begin{cases} \binom{m-n+2}{2} & \text{if } G \text{ is bipartite,} \\ \binom{m-n+1}{2} + k_4(G) & \text{otherwise.} \end{cases}$$

Acknowledgments

The authors would like to thank the referees for their careful reading and important advice that improved the writing of the present paper.

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(Received 18 Aug 2021; revised 2 Oct 2022)