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Subgraphs of Gallai-colored complete graphs spanned by edges using at most two colors

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Abstract

In Gallai-Ramsey theory for graphs, one seeks to identify the exact number of vertices a complete graph must have to guarantee that every coloring of its edges, in which rainbow triangles are avoided, necessarily contains a certain monochromatic subgraph. In this paper, we consider the analogous problem when subgraphs are sought that use at most two colors. Our results highlight certain structural properties of Gallai colorings and we give some exact evaluations when the subgraphs are complete graphs or cycles.

1 Introduction

Gallai-Ramsey theory offers an interesting variation on Ramsey theory, in which one only considers colorings of complete graphs that avoid rainbow triangles. The purpose of this paper is to demonstrate the usefulness of common structural properties of Gallai colorings by determining the values of several weakened Gallai-Ramsey numbers. Here, the word "weakened" implies that instead of monochromatic subgraphs, the numbers in question guarantee the existence of subgraphs that use at most a specified number of colors.

In order to describe our results, we must begin with definitions. A t*-coloring* of a complete graph K is a map $c : E(K) \longrightarrow \{1, 2, \ldots, t\}$. In general, we do not assume that such a map is surjective. Given graphs $G_1, G_2, \ldots, G_t$, the *Ramsey number* $r(G_1, G_2, \ldots, G_t)$ is defined to be the least natural number p such that every t-coloring of K_p (the complete graph of order p) contains a monochromatic subgraph that is isomorphic to G_i in color i, for some $1 \leq i \leq t$. When $G_1 = G_2 = \cdots = G_t$, we shorten our notation to $r^t(G_1)$ for the corresponding t-color Ramsey number. For a current overview of results on Ramsey numbers, the reader is referred to Radziszowski's dynamic survey [15].

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A *Gallai t-coloring* of a complete graph K is a t-coloring $c : E(K) \longrightarrow \{1, 2, \dots, t\}$ that lacks rainbow triangles (i.e., no subset $\{x, y, z\} \subseteq V(K)$ exists in which $|\{c(xy), c(yz), c(xz)\}| = 3$). For any graph G, the *Gallai-Ramsey number* $gr^t(G)$ is the least natural number p such that every Gallai t-coloring of K_p contains a monochromatic subgraph that is isomorphic to G. The following structural result for Gallai colorings appears in [11] as a reinterpretation of a result of Gallai [10] on transitive orientations of graphs (see [16] for an English translation of [10] by Maffray and Preissmann).

Theorem 1.1. *Every Gallai-colored complete graph can be obtained by substituting Gallai-colored complete graphs into the vertices of a* 2*-colored complete graph of order at least* 2*.*

For a fixed Gallai coloring, the 2-colored complete graph whose vertices are replaced with Gallai-colored complete graphs is called the *base graph*, while the substituted complete graphs are called the *blocks*. We note that the base graph is not unique for a given Gallai coloring. One immediate consequence of this theorem is that only the blocks can contain a color not used in the base graph, forcing such a color to span a disconnected graph (see [11]). The result is the following corollary.

Corollary 1.2. *Every Gallai-colored complete graph using at least three colors has a color that spans a disconnected graph.*

The beauty of Theorem 1.1 is that it provides a means of partitioning the collection of all Gallai colorings of a complete graph into sets in which specific techniques of proof can then be used. Typically, this process is employed in the determination of upper bounds for various Gallai-Ramsey numbers. The equivalence classes of the resulting partition correspond with the possible cardinalities of the base graphs. The following lemma will assist us in simplifying the use of Theorem 1.1 for proving upper bounds for various weakened Gallai-Ramsey numbers. This result is equivalent to Lemma 3.1 in [14].

Lemma 1.3. *Given a Gallai colored complete graph G, if the base graph is chosen so that its order is minimal, then its order is not equal to* 3*.*

Another useful property of Gallai colorings is described in the next theorem (see Erdős, Simonovits, and Sós [7]).

Theorem 1.4. *Let $n \geq 2$. Then every Gallai coloring of K_n uses at most $n - 1$ colors.*

This theorem is easily proved by induction on n. The dynamic survey of Fujita, Magnant, and Ozeki [9] and the recent book by Magnant and Salehi Nowbandegani [14] offer thorough overviews of the history and known evaluations of Gallai-Ramsey numbers for various graphs.

In [1], the authors defined the concept of a *weakened Gallai-Ramsey number* $gr^t_s(G)$, defined to be the least natural number p such that every Gallai t-coloring of K_p contains a subgraph isomorphic to G that is spanned by edges using at most s colors, where $s < t$. Throughout this paper, when we say that a Gallai t-colored complete graph contains an s-colored copy of H, we mean that there exists a subgraph isomorphic to H that is spanned by edges using at most s colors. In particular, an s-colored subgraph may use fewer than s colors. Weakened Gallai-Ramsey numbers may be viewed as a conglomeration of Gallai-Ramsey numbers and weakened Ramsey numbers, which were first introduced in [4] and [5], and further developed in [12] and [13]. We note that the number $gr_1^t(G)$ is just the usual Gallai-Ramsey number $gr^t(G)$.

In Section 2, we focus on the evaluation of $gr_2^t(K_n)$. In particular, we prove that $gr_2^t(K_4) = t + 2$ and $gr_2^3(K_5) = 9$ in Theorems 2.1 and 2.3, respectively. Besides these exact evaluations, we also prove that $gr_2^t(K_5) \geq 2^t + 1$ and $gr_2^3(K_n) \geq 2n - 1$ in Theorems 2.2 and 2.4, respectively. In Section 3, we turn our attention to weakened Gallai-Ramsey numbers for cycles. In the case of the cycle C_4 of order 4, we prove that every Gallai colored complete graph of order at least 4 contains a 2-colored C_4 (Theorem 3.1). Additionally, we prove that $gr_2^3(C_5) = t + 3$ in Theorem 3.2 and finish the section with a proof that $gr_2^t(C_{n+1}) \leq gr^t(C_n)$. We conclude in Section 4 with a conjecture and some directions for future research.

2 Complete Subgraphs Spanned by Edges Using at Most Two Colors

In this section, we focus on the evaluation of $gr_2^t(K_n)$. Before considering these numbers for specific values of n, note that whenever a Gallai colored complete graph contains a monochromatic K_n, it necessarily contains a 2-colored K_{n+1}. This follows from the fact that with the addition of each new vertex in a Gallai coloring, at most one new color may be introduced (see Lemma 5 of [1]). It follows immediately that

$$gr_2^t(K_{n+1}) \leq gr^t(K_n), \tag{1}$$

although this bound is not particularly strong for most values of t and n.

In [1], it was shown that for all $t \geq 3$,

$$t + 2 \leq gr_2^t(K_4) \leq t(gr_2^{t-1}(K_4) - 1) + 2. \tag{2}$$

In the case $t = 3$, it was further shown that $gr_2^3(K_4) = 5$, while the range

$$6 \leq gr_2^4(K_4) \leq 18$$

demonstrates how imprecise these bounds become for larger values of t. In fact, by using Theorem 1.1, we can show that the lower bound in (2) is the exact value for this weakened Gallai-Ramsey Number.

Theorem 2.1. *For all $t \geq 3$, $gr_2^t(K_4) = t + 2$.*

Proof. The lower bound $gr_2^t(K_4) \geq t + 2$ was proved in Theorem 7 of [1] using induction on $t \geq 3$. To prove the upper bound $gr_2^t(K_4) \leq t+2$, we proceed by strong induction on $t \geq 3$. The base case $gr_2^3(K_4) \leq 5$ was proved in Theorem 6 of [1]. Suppose that

$$gr_2^\ell(K_4) \leq \ell + 2, \quad \text{for all } \ell < t.$$

and consider a Gallai t-coloring of K_{t+2}. Taking advantage of the structure described in Theorem 1.1, denote by $\mathcal{B}$ a base graph of minimal order. If this Gallai coloring lacks a K_4 spanned by edges using at most two colors, then $|V(\mathcal{B})| < 4$. By Lemma 1.3, it follows that $|V(\mathcal{B})| \neq 3$. Hence, $|V(\mathcal{B})| = 2$. Label the blocks A and B and without loss of generality, suppose that the edges connecting A and B are red. If a red edge exists in A or B, then the Gallai coloring being considered contains a red K_3. The vertices of this K_3 along with any other vertex necessarily forms a K_4 spanned by edges using at most two colors (see Lemma 5 of [1]). Also, if A and B have edges in a common color, then the endpoints of these edges form a K_4 that uses at most two colors. Hence, the other $t-1$ colors are divided between A and B. Suppose that A has order k_1 and uses ℓ_1 colors and that B has order k_2 and uses ℓ_2 colors. Then $k_1 + k_2 = t + 2$ and $\ell_1 + \ell_2 = t - 1$. By the inductive hypothesis,

$$gr_2^{\ell_1}(K_4) \leq \ell_1 + 2 \quad \text{and} \quad gr_2^{\ell_2}(K_4) \leq \ell_2 + 2. \tag{3}$$

If $k_1 \geq \ell_1 + 2$, then the first Gallai-Ramsey number in (3) implies that there exists a K_4 that uses at most two colors. Otherwise, $k_1 \leq \ell_1 + 1$ and we find that

$$k_2 = t - k_1 + 2 \geq t - \ell_1 + 1 = (t - \ell_1 - 1) + 2 = \ell_2 + 2.$$

It follows from the second inequality in (3) that B contains a K_4 that uses at most two colors. We have proved that every Gallai t-colored K_{t+2} contains a 2-colored K_4, implying that $gr_2^t(K_4) \leq t + 2$, from which the theorem follows. □

While $gr_2^t(K_4)$ grows linearly with respect to t, in the next theorem, we prove that $gr_2^t(K_5)$ is at least exponential with respect to t.

Theorem 2.2. *For all $t \geq 3$, $gr_2^t(K_5) \geq 2^t + 1$.*

Proof. We proceed by induction on $t \geq 3$, to prove that there exists a Gallai t-colored K_{2^t} that lacks a K_5-subgraph spanned by edges using at most two colors and in which every K_3-subgraph is spanned by edges using exactly two colors. For the base case, consider the Gallai 3-colored K_8 formed by taking two copies of K_4-subgraphs in which a cycle of length four receives color 1 (red) and the other pair of disjoint edges receive color 2 (blue), then all edges interconnecting the two K_4-subgraphs receive color 3 (green). See Figure 1 for the resulting K_8 and observe that every K_3-subgraph is spanned by edges using exactly two colors. It is also easily confirmed that every K_5-subgraph includes edges in all three colors. Now assume that for $k \geq 3$, there exists a Gallai k-colored K_{2^k} that lacks a K_5-subgraph spanned by edges using at most two colors and in which every K_3-subgraph is spanned by edges using exactly two colors. Form a Gallai $(k+1)$-colored $K_{2^{k+1}}$ by taking a copy of K_2 with edge in color $k+1$ as the base graph and substituting the above K_{2^k} for both blocks. Every K_3-subgraph in the resulting $K_{2^{k+1}}$ is spanned by edges using exactly two colors. No

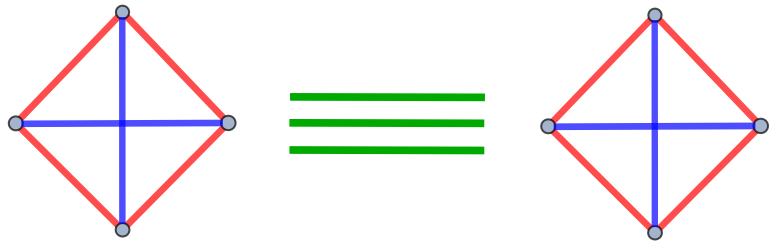

Figure 1: A Gallai 3-coloring of K_8 that lacks a K_5-subgraph spanned by edges using at most 2 colors.

K_5-subgraph contained entirely within one of the blocks is spanned by edges using at most two colors by the inductive hypothesis. For K_5-subgraphs that include vertices from both blocks, there are two cases. If a K_5-subgraph includes two vertices from one block and three from the other, then the K_3-subgraph uses exactly two colors, with the interconnecting edges adding a third color. If the K_5-subgraph includes one vertex from one block and four vertices from the other, then the K_4-subgraph uses at least two colors while the interconnecting edges add a third color. Thus, we have proved that for all $t \geq 3$, there exists a Gallai t-colored K_{2^t} that lacks a K_5-subgraph spanned by edges using at most two colors. □

When $t = 3$, the following theorem shows that the lower bound in Theorem 2.2 is the value of the corresponding weakened Gallai-Ramsey number.

Theorem 2.3. $gr_2^3(K_5) = 9$.

Proof. The inequality $gr_2^3(K_5) \geq 9$ follows from Theorem 2.2. It remains to be shown that every Gallai 3-coloring of K_9 contains a K_5-subgraph spanned by edges using at most two colors. Consider a Gallai 3-colored K_9 and denote its base graph by $\mathcal{B}$, which is assumed to be of minimal order. Since the base graph is 2-colored (by Theorem 1.1), if we select a single vertex from each block, we trivially obtain a K_5-subgraph spanned by edges using at most two colors when $|V(\mathcal{B})| \geq 5$. By Lemma 1.3, $|V(\mathcal{B})| \neq 3$, so it remains for us to handle the cases $|V(\mathcal{B})| = 2$ and $|V(\mathcal{B})| = 4$ separately.

Case 1 Suppose that $|V(\mathcal{B})| = 2$. Without loss of generality, assume the edges connecting the two blocks are red and the other two colors (which only appear in the blocks) are blue and green.

Subcase 1.1 Suppose that each block contains red edges. Selecting the endpoints of a single red edge in each block forms a red K_4. Adding in any other vertex necessarily produces a 2-colored K_5 (by Lemma 5 of [1]).

Subcase 1.2 Suppose that neither block contains a red edge. By the pigeonhole principle, one block contains at least five vertices, which form a K_5-subgraph that avoids red edges.

Subcase 1.3 Suppose that exactly one block contains red edges and the other block has order at least 2. The blue/green block contains a 2-colored K_5 if it has order at least 5, so assume this block has an order of at most 4. Without loss of generality, suppose the blue/green block contains a blue edge uv. Since this block contains at most four vertices, the block containing a red edge has an order of at least 5. Let xy be a red edge in this block and label three other vertices in this block a, b, and c (see the image in Figure 2). If the subgraph induced by $\{a, x, y\}$ is a red/blue K_3, then the

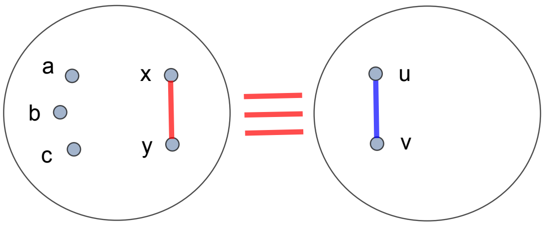

Figure 2: Subcase 1.3 in the proof of Theorem 2.3.

subgraph induced by $\{a, x, y, u, v\}$ forms a red/blue K_5. The same is true for $\{b, x, y\}$ and $\{c, x, y\}$. Therefore, each of a, b, and c must have a green edge connecting to xy, and any other edges connecting $\{a, b, c\}$ to xy must be red. If ab is red or green, then the subgraph induced by $\{a, b, x, y, u\}$ is a red/green K_5. A similar argument can be made if either ac or bc is red or green. The only remaining possibility is that the subgraph induced by $\{a, b, c\}$ is a blue K_3, and this in combination with uv results in a red/blue K_5.

Subcase 1.4 Suppose that one block contains a red edge and the other block consists of a single vertex u. Label a red edge in the first block xy and the other vertices a, b, c, d, e, and f (see the first image in Figure 3). If any of a, b, c, d, e, f form

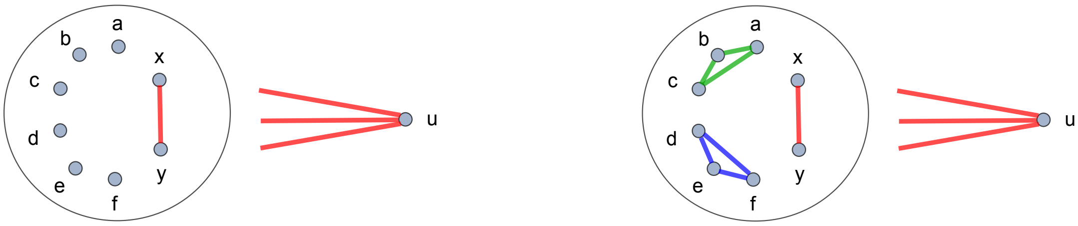

Figure 3: Subcase 1.4 in the proof of Theorem 2.3.

a red K_3 with xy then the inclusion of u yields a red K_4, and the addition of any other vertex produces a 2-colored K_5 (by Lemma 5 of [1]). This means that each of a, b, c, d, e, f must have a blue or green edge connecting to xy (but not both). Consider the case where at least four of these vertices connect to xy via the same color (from among blue and green). Without loss of generality, assume that a, b, c, and d each connect to xy using at least one blue edge. If there exists any red or blue edge in the subgraph induced by $\{a, b, c, d\}$, then we obtain a 2-colored K_5 (e.g., if ab is red or blue, then the subgraph induced by $\{a, b, x, y, u\}$ is a red/blue K_5). The only remaining possibility is that the subgraph induced by $\{a, b, c, d\}$ forms a green K_4 and including any other vertex in the graph creates a 2-colored K_5 (by Lemma 5 of [1]). Finally, consider the case where exactly three of a, b, c, d, e, and f connect to xy with at least one blue edge and the other three connect to xy with at least one green edge. Without loss of generality, suppose a, b, and c each connect to xy via a blue edge and d, e, and f each connect to xy via a green edge. If ab is red or blue, then the subgraph induced by $\{a, b, x, y, u\}$ is a red/blue K_5. The same can be said of ac and bc. The only other possibility is that the subgraph induced by $\{a, b, c\}$ is a green K_3. Similarly, If de is red or green, then the subgraph induced by $\{d, e, x, y, u\}$ is a red/green K_5. This is also true for df and ef. The only other possibility is that the subgraph induced by $\{d, e, f\}$ is a blue K_3. Now consider these two disjoint K_3-subgraphs, one blue, and one green (see the second image in Figure 3). If there exists a red edge between the two K_3 subgraphs, then a 2-colored K_5 must be formed. For example, if ad is red, then ae and af must be red or blue, which makes the subgraph induced by $\{a, d, e, f, u\}$ a red/blue K_5. Otherwise, there exists no red edge between the two K_3-subgraphs and the subgraph induced by $\{a, b, c, d, e, f\}$ is a blue/green K_6.

Case 2 Suppose that $|V(\mathcal{B})| = 4$. If any vertex in $\mathcal{B}$ is only incident with edges in one color, then the other vertices in $\mathcal{B}$ can be unioned together to form a block in a base graph isomorphic to K_2, contradicting the assumption that the base graph was chosen with minimal order. So, without loss of generality, suppose that every vertex in $\mathcal{B}$ is incident with edges in colors red and blue. If any red or blue edge exists within a block, then choosing its endpoints, along with a single vertex from each of the other blocks forms a 2-colored K_5. So, assume all the blocks contain only green edges. If any block contains four or more vertices, then selecting four such vertices along with any other vertex creates a 2-colored K_5. Otherwise, each block contains at most three vertices. Note that some block must contain at least three vertices by the pigeonhole principle. Also, some block must contain at least two vertices and the five resulting vertices selected in this way yield a 2-colored K_5. □

We have now seen that the lower bound given in Theorem 2.2 is the exact value of the corresponding weakened Gallai-Ramsey numbers when $t = 3$, but it is not clear if this bound is exact when $t > 3$. One can try to prove that this is the case using a similar approach to the proof of Theorem 2.3. Unfortunately, the Subcases 1.3 and 1.4 in the above proof do not easily extend to larger values of t.

In the following theorem, we make use of the lower bound proved by Chvátal and Harary [6] for 2-color Ramsey numbers:

$$r(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + 1,$$

where $c(G_1)$ is the order of the largest connected component in G_1 and $\chi(G_2)$ is the chromatic number of G_2. In particular, applying this bound to complete graphs, we have that

$$r(K_m, K_n) \geq (m-1)(n-1)+1. \tag{4}$$

Having already proved that $gr_2^3(K_4) = 5$ and $gr_2^3(K_5) = 9$, we now turn to proving a general lower bound for $gr_2^3(K_n)$ when $n \geq 6$.

Theorem 2.4. *For all $n \geq 6$, $gr_2^3(K_n) \geq 2n-1$.*

Proof. We prove this lower bound by constructing a Gallai 3-colored $K_{2(n-1)}$ that avoids a 2-colored K_n. Our construction involves using a base graph isomorphic to K_2 with a red edge, then replacing each vertex in the K_2 with blue/green K_{n-1}-subgraphs. We must argue that no 2-colored K_n is formed in our construction, and it is necessary to handle the cases where n is even or odd separately.

Case 1 Assume that n is even. For the blocks in our construction, we must color the two K_{n-1}-subgraphs in blue/green while avoiding a monochromatic $K_{n/2}$ in either color (so that the vertices of a 2-colored K_n cannot be divided between the two blocks). From Inequality (4), we have that

$$r(K_{n/2}, K_{n/2}) \geq \frac{n^2}{4} - n + 2,$$

which is greater than $n-1$ whenever $n > 6$. For the case $n = 6$, it is well-known that $r(K_3, K_3) = 6$, which is certainly greater than 5. In all cases, we have shown that it is possible to color K_{n-1} blue and green without producing a monochromatic $K_{n/2}$. Thus, no 2-colored K_n exists.

Case 2 Assume that n is odd. For the blocks in our construction, we must color the two K_{n-1}-subgraphs in blue/green while avoiding a monochromatic $K_{(n+1)/2}$ in either color (so that the vertices of a 2-colored K_n cannot be divided between the two blocks). From Inequality (4), we have that

$$r(K_{(n+1)/2}, K_{(n+1)/2}) \geq \frac{n^2}{4} - \frac{n}{2} + \frac{5}{4},$$

which is greater than $n-1$ whenever $n > 3$. We have shown that it is possible to color K_{n-1} blue and green without producing a monochromatic $K_{(n+1)/2}$. Thus, no 2-colored K_n exists. □

In general, the lower bounds given in Theorem 2.4 are unlikely to be exact as Inequality (4) is weak for large m and n. A stronger way of expressing the bounds described in Theorem 2.4 is

$$gr_2^3(K_n) \geq \begin{cases} 2r(K_{n/2}, K_{n/2}) - 1 & \text{if } n \text{ is even} \\ 2r(K_{(n+1)/2}, K_{(n+1)/2}) - 1 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \geq 6$.

3 Cycles Spanned by Edges Using at Most Two Colors

In this section, we turn our attention to weakened Gallai-Ramsey numbers for cycles. We use the usual notation C_n to denote a cycle of order n. In the case $n = 4$, we find that every Gallai colored complete graph of order at least 4 contains a 2-colored C_4.

Theorem 3.1. *For $n \geq 4$, every Gallai coloring of K_n contains a C_4 spanned by edges using at most two colors.*

Proof. Consider a Gallai colored K_n with $n \geq 4$ and select four distinct vertices a, b, c, and d. If the cycle $abcda$ uses only one or two colors, we are done. So suppose it uses at least three colors. By Theorem 1.4, it uses exactly three colors, exactly one of which occurs on two edges in the cycle. We must consider cases based on whether the two edges in the cycle that share a common color are adjacent or not. Without loss of generality, suppose the color that is repeated is red and we have the cases given in Figure 4.

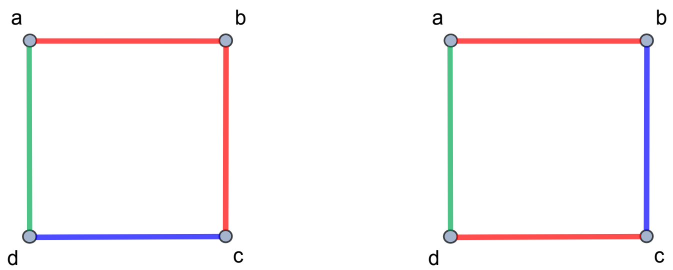

Figure 4: Two cases of 3-colored cycles of order 4.

Case 1 In the first image in Figure 4, avoiding a rainbow triangle forces edge bd to be red and edge ac to be either blue or green. If ac is blue, then $abdca$ is a red/blue C_4. If ac is green, then $acbda$ is a red/green C_4. Either way, we obtain a C_4 spanned by edges using two colors.

Case 2 In the second image in Figure 4, avoiding a rainbow triangle forces edges ac and bd to be red. Then $acdba$ is a red C_4.
In both cases, we have shown that there exists a C_4 spanned by edges using at most two colors. □

Of course, if every Gallai colored complete graph contains a 2-colored C_4, then it also contains a 2-colored P_4 (a path of order 4). In the proof of the following theorem, we will need to use the fact that $gr^t(P_4) = t + 3$, which was proved in Theorem 7 of [8].

Theorem 3.2. *For all $t \geq 3$, $gr^t_2(C_5) = t + 3$.*

Proof. First, we construct a Gallai t-colored K_{t+2} that avoids a 2-colored C_5. Begin with a K_3 in color 1. Add a new vertex, call it x_1, and color all edges connecting

x_1 to the existing K_3 using color 2. Next, add in a vertex x_2 and color all edges connecting x_2 to the existing graph using color 3. At this point, we obtain a Gallai 3-colored K_5. If such a coloring includes a 2-colored C_5, then the two colors used must be colors 2 and 3 since these are the only colors used on edges that are incident with x_1. A 2-colored C_5 would also require the use of exactly two edges in color 3 as every such edge is incident with vertex x_2. All other edges in the cycle would then have to use color 2, which is not possible since no monochromatic P_4 exists in our construction. Thus, we have produced a Gallai 3-colored K_5 that lacks a 2-colored C_5, from which it follows that $gr_2^3(C_5) \geq 6$. We use this coloring as the basis for our general construction. Next, add in vertex x_3, and color all edges connecting x_3 to the existing graph using color 4. Continue in this manner, adding in one vertex at a time, coloring all edges connecting the new vertex to the existing graph using a new color each time. Label the added vertices $x_3, x_4, \ldots, x_{t-1}$. Observe that the subgraphs spanned by colors $2, 3, \ldots, t$ in the resulting Gallai t-colored K_{t+2} are all stars and the subgraph spanned by color 1 is a K_3 (see Figure 5). Any C_5-subgraph uses at

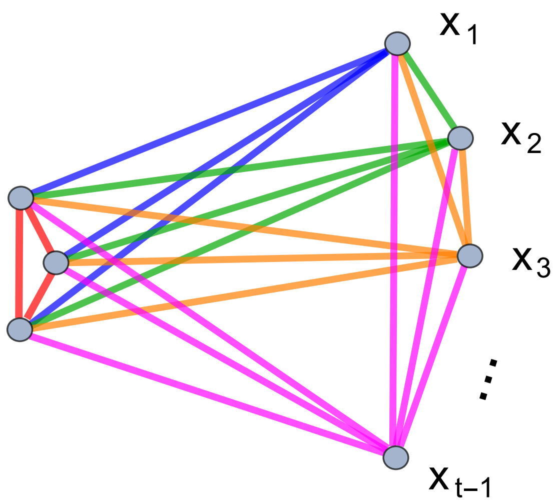

Figure 5: A Gallai t-colored K_{t+2} that lacks a 2-colored C_5.

most two edges in any given color, and hence, uses at least three colors in total. It follows that $gr_2^t(C_5) \geq t+3$. To prove the upper bound, consider a Gallai t-colored K_{t+3} and let $\mathcal{B}$ be its base graph, chosen to have minimal order. If $|V(\mathcal{B})| \geq 5$, then selecting a single vertex from each block results in a 2-colored $K_{V(\mathcal{B})}$, which necessarily contains a 2-colored C_5. Thus, we only have the cases $2 \leq |V(\mathcal{B})| \leq 4$ left to consider and Lemma 1.3 implies that $|V(\mathcal{B})| \neq 3$. So we are left with the two cases $|V(\mathcal{B})| = 2$ and $|V(\mathcal{B})| = 4$.

<u>Case 1</u> Assume that $|V(\mathcal{B})| = 2$ and the edges connecting the two blocks are all red. We divide this case into three subcases.

<u>Subcase 1.1</u> Assume that each block contains at least two vertices. Since $t \geq 3$, our

graph contains at least 6 vertices, so one block must contain at least 3 vertices. Label three vertices a, b, and c in one block and two vertices d and e in the other block. Then $adbec$ is a monochromatic P_5, and regardless of what color edge ac receives, the cycle $adbeca$ is 2-colored.

Subcase 1.2 Assume that one block consists of a single vertex (call it u), the other block contains $t + 2$ vertices, and the larger block is assumed to use fewer than t colors. The larger block is a $(t - 1)$-colored K_{t+2}, which contains a monochromatic P_4 since $gr^{t-1}(P_4) = t + 2$. This P_4, along with vertex u, forms a 2-colored C_5.

Subcase 1.3 Assume that one block consists of a single vertex (call it u), the other block contains $t + 2$ vertices, and the larger block has edges in all t colors. In particular, there exists a red edge in the larger block, which we label xy. Label the other vertices in this block a_1, a_2, …, a_t. Suppose that some a_i connects to xy via a red edge. Without loss of generality, suppose that a_1x is red. Then $a_1xyua_2a_1$ is a 2-colored C_5. It remains for us to consider the case where no a_i connects to xy via red edges. Note that for each a_i, the edges a_ix and a_iy must receive the same color and the colors for each such pair of edges are chosen from among $t - 1$ colors (any color other than red). By the pigeonhole principle, there exists some a_i and a_j, with $i \neq j$, such that a_ix, a_jx, a_iy, and a_jy all receive the same color (call it blue). Then $a_ixa_jyua_i$ is a red/blue C_5. o

Case 2 Suppose that $|V(\mathcal{B})| = 4$ and $\mathcal{B}$ has its edges colored red and blue. If any vertex in the base graph is only incident with edges in a single color, then unioning the other vertices together produces a base graph of order 2, contradicting the minimal order of $\mathcal{B}$. So, every vertex in $\mathcal{B}$ is incident with both red and blue edges. We have two subcases to consider.

Subcase 2.1 Suppose that one block (call it A) has order at least 3 and label three of its vertices a, b, and c. Since there are three other blocks, at least two of them must be connected to A via the same color (call it red). Select a single vertex from each of these two blocks and call them x and y. Then $axbyca$ is a 2-colored C_5.

Subcase 2.2 Suppose that each block has order at most 2. Since there are only four blocks, at least two blocks (call them A and B) have order 2. Assume that all edges connecting A and B are red. If one of A or B is adjacent in $\mathcal{B}$ to another block C via a red edge, then let a and b be in A, w and z be in B, and x be in C. It follows that $abzxwa$ is a 2-color C_5. Otherwise, A and B connect to the other blocks via only blue edges. Denote the other blocks by C and D and let $x \in C$ and $y \in D$. It follows that $abxwya$ is a 2-colored C_5.

In all cases, we find that every Gallai t-coloring of K_{t+3} contains a 2-colored C_5, completing the proof of the theorem. □

We conclude with a general result on 2-colored cycles, similar to Inequality (1), which held for complete graphs.

Theorem 3.3. *For all $t \geq 3$ and $n \geq 4$, $gr_2^t(C_{n+1}) \leq gr^t(C_n)$.*

Proof. Let $p = gr^t(C_n)$ and consider a Gallai t-colored K_p. It follows that there exists a monochromatic C_n, and we label the vertices in this cycle by $x_1x_2\cdots x_nx_1$ and assume that all of the edges in the cycle are given color 1. Let y be any other vertex. Then the edges yx_1 and yx_n are given color 1 or some other color, say color 2, but they cannot be two distinct colors other than color 1 (as the subgraph induced by $\{y, x_1, x_n\}$ would then be a rainbow triangle). It follows that $x_1x_2\cdots x_nyx_1$ is a 2-colored C_{n+1}, from which we find that $gr_2^t(C_{n+1}) \leq p$. □

4 Conclusion

Besides computing the values of the weakened Gallai-Ramsey numbers $gr_2^t(K_n)$ and $gr_2^t(C_n)$ that we have not evaluated here, there are several other ways in which this work can be generalized/varied. First, we conjecture that the lower bound given in Theorem 2.2 is exact.

Conjecture 4.1. *For all $t \geq 3$, $gr_2^t(K_5) = 2^t + 1$.*

Note that for the cases where $t \geq 4$, one can use the same approach that was used in Theorem 2.3, but when considering the parts of the proof that correspond to Subcases 1.3 and 1.4, similar arguments no longer suffice.

Besides this conjecture, we offer a brief description of some additional topics for future research.

1. Consider $gr_s^t(G)$, where $s > 2$, and G is complete or a cycle. A good starting point would be the case $s = 3$ and $t \geq 4$. The techniques used for the upper bounds of such numbers may resemble the techniques used in this paper. Lower bounds may present more of an obstacle.

2. Consider $gr_2^t(G)$ for graphs other than complete graphs or cycles. In particular, fans, paths, stars, and other trees have not yet been studied.

3. Consider $gr_2^t(H)$ where H is an r-uniform hypergraph (see [2] for the definition of a Gallai-Ramsey hypergraph number and [3] for the definition of a weakened Ramsey hypergraph number). This topic may be more challenging as there is currently no analogue of Theorem 1.1 for Gallai colorings of r-uniform hypergraphs.

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