AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **84(3)** (2022), Pages 341–356

Type IV-II codes over $\mathbb{Z}_4$ constructed from generalized bent functions

Sara Ban* Sanja Rukavina

Faculty of Mathematics, University of Rijeka
Radmile Matejčić 2, 51000 Rijeka
Croatia
sban@math.uniri.hr sanjar@math.uniri.hr

Abstract

A Type IV-II $\mathbb{Z}_4$-code is a self-dual code over $\mathbb{Z}_4$ with the property that all Euclidean weights are divisible by eight and all codewords have even Hamming weight. In this paper we use generalized bent functions for a construction of self-orthogonal codes over $\mathbb{Z}_4$ of length 2^m, for m odd, $m \geq 3$, and prove that for $m \geq 5$ those codes can be extended to Type IV-II $\mathbb{Z}_4$-codes. From this family of Type IV-II $\mathbb{Z}_4$-codes, we construct a family of self-dual Type II binary codes by using the Gray map. We also consider the weight distributions of the obtained codes and the structure of the supports of the minimum weight codewords.

1 Introduction

The discovery of good nonlinear binary codes arising via the Gray map from $\mathbb{Z}_4$-linear codes motivated the study of codes over rings in general (see [16]). In particular, self-dual $\mathbb{Z}_4$-codes have attracted much interest because of their connection with unimodular lattices (see, for example, [15, 17]). Type II self-dual $\mathbb{Z}_4$-codes are connected with even unimodular lattices. A Type IV self-dual $\mathbb{Z}_4$-code is closely related to a class of binary doubly even self-complementary codes (see [11]). In general, self-dual codes are one of the most interesting classes of codes, since many of the best known codes are of this type and they have a rich mathematical theory (see [23]). Thus, it is of interest to construct self-dual codes and study their properties. Many methods for constructing self-dual $\mathbb{Z}_4$-codes are known (see, for example, [3, 8, 22, 29]). In this paper, we give a construction of Type IV-II $\mathbb{Z}_4$-codes from generalized bent functions. To the best of our knowledge, this construction yields a new family of Type IV-II $\mathbb{Z}_4$-codes.

* Corresponding author.

ISSN: 2202-3518

According to [6], Rothaus wrote the first paper in English on bent functions in 1966, but its final version was published ten years later in [24]. Since then, bent functions have been a subject of interest of many researchers (see [6]). Among other things, relationships between bent functions and codes have been intensively studied. For example, cyclic codes and their connection with hyper-bent and bent functions are explored in [5], a construction of linear codes with two or three weights from weakly regular bent functions is given in [27], and in [9] bent vectorial functions are used for a construction of a two-parameter family of binary linear codes that do not satisfy the conditions of the Assmus-Mattson theorem, but nevertheless hold 2-designs, and a new coding-theoretic characterization of bent vectorial functions is presented. Trace codes over $\mathbb{Z}_4$ based on Boolean functions and their supports are explored in [26], and three-weight codes are obtained from bent and semi-bent functions.

Generalized bent functions were introduced in [21], and in [25] Schmidt considered generalized bent functions for a construction of constant-amplitude codes over $\mathbb{Z}_4$ of length 2^m. In this paper we use generalized bent functions for a construction of self-dual $\mathbb{Z}_4$-codes. We start from a pair of bent functions and obtain a cyclic self-orthogonal $\mathbb{Z}_4$-code of length 2^m, for m odd, $m \geq 3$, with all Euclidean weights divisible by 8. Further, we show that our construction gives a family of Type IV-II codes over $\mathbb{Z}_4$ of length 2^m, for m odd, $m \geq 5$. Consequently, this yields a family of self-dual Type II binary codes of length 2^{m+1}, for m odd, $m \geq 5$, constructed via the Gray map. We also consider the weight distributions of the obtained codes and the supports of the minimum weight codewords.

This paper is organized as follows. Section 2 gives definitions and basic properties of codes over $\mathbb{Z}_4$ and generalized bent functions. In Section 3 the construction of Type II codes over $\mathbb{Z}_4$ of length 2^m, for m odd, $m \geq 3$, from generalized bent functions is introduced. We prove that for $m \geq 5$ the constructed $\mathbb{Z}_4$-codes are also of Type IV. We give the Euclidean weight distribution, the Lee weight distribution and the symmetrized weight enumerator for the constructed codes. By using the Gray map, we obtain a family of self-dual Type II binary codes of length 2^{m+1}, for m odd, $m \geq 5$. Further, we observe the minimum weight codewords and the structure of their supports. For the construction of examples we used Magma [2].

2 Preliminaries

We assume that the reader is familiar with the basic facts of coding theory. We refer the reader to [19] for any terms not defined in this paper.

Let $\mathbb{F}_q$ be the field of order q, where q is a prime power. A code C over $\mathbb{F}_q$ of length n is any subset of $\mathbb{F}_q^n$. A k-dimensional subspace of $\mathbb{F}_q^n$ is called an $[n, k]$ *q-ary linear code*. An element of a code is called a *codeword*. A *generator matrix* for an $[n, k]$ code C is any $k \times n$ matrix whose rows form a basis for C. A linear code C of length n is *cyclic* if for every codeword $(c_0, \ldots, c_{n-2}, c_{n-1})$ in C the codeword $(c_{n-1}, c_0, \ldots, c_{n-2})$ is also in C.

If $q = 2$, then the code is called *binary*. The *(Hamming) weight* of a codeword $x \in \mathbb{F}_2^n$ is the number of non-zero coordinates in x. If the minimum weight d of an $[n, k]$ binary linear code is known, then we refer to the code as an $[n, k, d]$ binary linear code. Binary linear codes for which all codewords have even weight are called *(singly) even* and those among them for which all codewords have weight divisible by four are called *doubly even*.
Let C be a binary linear code of length n. The *dual code* $C^\perp$ of C is defined as

$$C^\perp = \{x \in \mathbb{F}_2^n \mid \langle x, y\rangle = 0 \text{ for all } y \in C\},$$

where $\langle x, y\rangle = x_1y_1+x_2y_2+\cdots+x_ny_n$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. The code C is *self-orthogonal* if $C \subseteq C^\perp$, and it is *self-dual* if $C = C^\perp$. The dual code of a cyclic code is cyclic. A self-dual doubly even binary code is called a *Type II binary code*.

Let $\mathbb{Z}_4$ denote the ring of integers modulo 4. A linear code C of length n over $\mathbb{Z}_4$ (i.e., a $\mathbb{Z}_4$-*code*) is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. Two $\mathbb{Z}_4$-codes are *(monomially) equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called *permutation equivalent*. *The permutation automorphism group* of a $\mathbb{Z}_4$-code C is the group of all coordinate permutations that fix C set-wise. The *support* of a codeword $x \in \mathbb{Z}_4^n$ is the set of non-zero positions in x. Denote the number of coordinates i (where $i = 0, 1, 2, 3$) in a codeword $x \in \mathbb{Z}_4^n$ by $n_i(x)$. The codeword $x \in \mathbb{Z}_4^n$ is *even* if $n_1(x) = n_3(x) = 0$. The *Hamming weight* of a codeword x is $wt_H(x) = n_1(x)+n_2(x)+n_3(x)$, the *Lee weight* of x is $wt_L(x) = n_1(x)+2n_2(x)+n_3(x)$, and the *Euclidean weight* of x is $wt_E(x) = n_1(x) + 4n_2(x) + n_3(x)$. It holds that $wt_E(x) \equiv x_1^2 + \cdots + x_n^2 \pmod 8$ for every $x \in \mathbb{Z}_4^n$. We will denote by $d_H(C)$, $d_L(C)$ and $d_E(C)$, the minimum Hamming weight, the minimum Lee weight and the minimum Euclidean weight of the code C, respectively. The *symmetrized weight enumerator* of a $\mathbb{Z}_4$-code C is defined as

$$swe_C(a, b, c) = \sum_{x\in C} a^{n_0(x)}b^{n_1(x)+n_3(x)}c^{n_2(x)}.$$

Let C be a $\mathbb{Z}_4$-code of length n. The *dual code* $C^\perp$ of the code C is defined as

$$C^\perp = \{x \in \mathbb{Z}_4^n \mid \langle x, y\rangle = 0 \text{ for all } y \in C\},$$

where $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \pmod 4$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. The code C is *self-orthogonal* when $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. If C is a self-orthogonal $\mathbb{Z}_4$-code, then $wt_L(c)$ is even for all $c \in C$. A self-dual $\mathbb{Z}_4$-code of length n contains exactly 2^n codewords. *Type II* $\mathbb{Z}_4$*-codes* are self-dual $\mathbb{Z}_4$-codes which have the property that all Euclidean weights are divisible by eight. *Type IV* $\mathbb{Z}_4$*-codes* are self-dual $\mathbb{Z}_4$-codes with all codewords of even Hamming weight (see [11]). A Type IV code that is also Type II is called a *Type IV-II* $\mathbb{Z}_4$*-code*.

Every $\mathbb{Z}_4$-code C contains a set of $k_1 + k_2$ codewords $\{c_1, c_2, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_{k_1+k_2}\}$ such that every codeword in C is uniquely expressible in the form

$$\sum_{i=1}^{k_1} a_i c_i + \sum_{i=k_1+1}^{k_1+k_2} a_i c_i,$$

where $a_i \in \mathbb{Z}_4$ and c_i has at least one coordinate equal to 1 or 3, for $1 \leq i \leq k_1$, $a_i \in \mathbb{Z}_2$ and c_i has all coordinates equal to 0 or 2, for $k_1 + 1 \leq i \leq k_1 + k_2$. We say that C is of *type* $4^{k_1}2^{k_2}$. The matrix whose rows are c_i, $1 \leq i \leq k_1 + k_2$, is called a *generator matrix* for C. A generator matrix G of a $\mathbb{Z}_4$-code C is in *standard form* if

$$G = \begin{bmatrix} I_{k_1} & A & B_1 + 2B_2 \\ O & 2I_{k_2} & 2D \end{bmatrix}, \tag{1}$$

where A, B_1, B_2 and D are matrices with entries from $\{0, 1\}$, O is the $k_2 \times k_1$ null matrix, and I_m denotes the identity matrix of order m. If C is a self-dual $\mathbb{Z}_4$-code of length n, then $2k_1 + k_2 = n$ and the matrix $B_1 + 2B_2$ in G is of order k_1. Any $\mathbb{Z}_4$-code is permutation equivalent to a code with generator matrix in standard form.

Since

$$wt_E(x + y) \equiv wt_E(x) + wt_E(y) + 2\langle x, y\rangle \pmod 8 \tag{2}$$

for all $x, y \in \mathbb{Z}_4^n$, every self-orthogonal $\mathbb{Z}_4$-code which has a generator matrix such that all rows have Euclidean weights divisible by 8 consists of codewords whose Euclidean weights are divisible by 8.

Let C be a $\mathbb{Z}_4$-code of length n. There are two binary linear codes of length n associated with C: the binary code $C^{(1)} = \{c \pmod 2 \mid c \in C\}$, which is called the *residue code* of C, and the binary code $C^{(2)} = \{c \in \mathbb{Z}_2^n \mid 2c \in C\}$, which is called the *torsion code* of C. If C is a $\mathbb{Z}_4$-code of type $4^{k_1}2^{k_2}$, then $C^{(1)}$ is a binary code of dimension k_1 generated by the matrix

$$\begin{bmatrix} I_{k_1} & A & B_1 \end{bmatrix}.$$

If C is a self-dual $\mathbb{Z}_4$-code, then $C^{(1)}$ is doubly even and $C^{(1)} = C^{(2)\perp}$ (see [8]).

According to [17], the following statement holds.

Theorem 2.1. *Let C be a Type II $\mathbb{Z}_4$-code of length n. Then $C^{(2)}$ is an even binary code, $C^{(1)}$ contains the all-ones binary vector, and $n \equiv 0 \pmod 8$.*

A *Boolean function* on n variables is a mapping $f : \mathbb{F}_2^n \to \mathbb{F}_2$. Its *truth table* is the $(0, 1)$ sequence $(f((0, \ldots, 0)), f((0, \ldots, 0, 1)), \ldots, f((1, \ldots, 1)))$. The *Walsh-Hadamard transformation* of f is

$$W_f(v) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+\langle v,x\rangle}.$$

A *bent function* is a Boolean function f such that $W_f(v) = \pm 2^{\frac{n}{2}}$, for every $v \in \mathbb{F}_2^n$. If f is bent, then the number of its variables is an even number. It was proven by Rothaus in the 1960s ([24]) that the number of zeros of a bent function equals

$$2^{n-1}\left(\pm\frac{1}{2^{\frac{n}{2}}}+1\right).$$

A *generalized Boolean function* on n variables is a mapping $f : \mathbb{F}_2^n \to \mathbb{Z}_{2^h}$. The *generalized Walsh-Hadamard transformation* of f is

$$\tilde{f}(v) = \sum_{x\in\mathbb{F}_2^n} \omega^{f(x)}(-1)^{\langle v,x\rangle},$$

where $\omega = e^{\frac{2\pi i}{2^h}}$. A *generalized bent function (gbent function)* is a generalized Boolean function f such that $|\tilde{f}(v)| = 2^{\frac{n}{2}}$, for every $v \in \mathbb{F}_2^n$. In this paper we will consider generalized bent functions from $\mathbb{F}_2^n$ into $\mathbb{Z}_4$.

3 Codes constructed from gbent functions

According to [25], the following theorem holds.

Theorem 3.1. *Let $m \geq 3$ be odd, and let $a, b : \mathbb{F}_2^{m-1} \to \mathbb{F}_2$ be bent functions. Then $f : \mathbb{F}_2^m \to \mathbb{Z}_4$ given by*

$$f(x, y) = 2a(x)(1 + y) + 2b(x)y + y, \quad x \in \mathbb{F}_2^{m-1},\ y \in \mathbb{F}_2,$$

is a gbent function.

Lemma 3.2. *Let $m \geq 3$ be odd and let $f : \mathbb{F}_2^m \to \mathbb{Z}_4$ be a gbent function constructed from bent functions a and b as in Theorem 3.1. Let c_f be a codeword*

$$(f((0,\ldots,0)), f((0,\ldots,0,1)),\ldots, f((1,\ldots,1))) \in \mathbb{Z}_4^{2^m}.$$

Then $wt_E(c_f) \equiv 0 \pmod 8$ and $\langle c_f, c_f\rangle = 0$.

Proof. By the construction, c_f has 2^{m-1} even and 2^{m-1} odd coordinates. The number of zeros in c_f is equal to the number of zeros in the bent function a. This number is equal to $2^{m-2}\left(\pm\frac{1}{2^{\frac{m-1}{2}}}+1\right)$. It follows that $wt_E(c_f) = 2^m + 2^{m-1} \pm 2^{\frac{m+1}{2}}$. So $wt_E(c_f)$ is divisible by 8. Since $wt_E(x) \equiv x_1^2 + \cdots + x_n^2 \pmod 8$ for every $x \in \mathbb{Z}_4^n$, we have $\langle c_f, c_f\rangle = 0$. □

Remark 3.3. Note that for $c_f = (f((0,\ldots,0)), f((0,\ldots,0,1)),\ldots, f((1,\ldots,1)))$ and $m = 3$, Euclidean weight $wt_E(c_f)$ takes value 8 or 16. If $m \geq 5$, then $wt_E(c_f) \geq \frac{2^m}{3} + 8$.

3.1 Codes over $\mathbb{Z}_4$

An $n \times n$ *circulant matrix* is a matrix of the form

$$\begin{bmatrix} x_0 & x_{n-1} & \dots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & \dots & x_2 \\ \vdots & & & & \vdots \\ x_{n-1} & \dots & \dots & x_1 & x_0 \end{bmatrix}.$$

Proposition 3.4. *Let $m \geq 3$ be odd, and let $a, b : \mathbb{F}_2^{m-1} \to \mathbb{F}_2$ be bent functions. Let $f : \mathbb{F}_2^m \to \mathbb{Z}_4$ be a gbent function given by $f(x, y) = 2a(x)(1 + y) + 2b(x)y + y$, $x \in \mathbb{F}_2^{m-1}$, $y \in \mathbb{F}_2$, and let c_f be a codeword*

$$(f((0, \dots, 0)), f((0, \dots, 0, 1)), \dots, f((1, \dots, 1))) \in \mathbb{Z}_4^{2^m}.$$

Let C_f be a $\mathbb{Z}_4$-code generated by the $2^m \times 2^m$ circulant matrix whose first row is the codeword c_f. Then C_f is a self-orthogonal $\mathbb{Z}_4$-code of length 2^m, all its codewords have Euclidean weights divisible by 8 and the residue code $C_f^{(1)}$ has dimension 2.

Proof. By Lemma 3.2, $\langle c_f, c_f \rangle = 0$. Note that even and odd coordinates alternate in the codeword c_f.

Let c_i and c_j, $i \neq j$, be the i-th and the j-th row of the circulant generator matrix of C_f. Since c_i and c_j are rows of a circulant matrix, $\langle c_i, c_j \rangle$ depends only on $j - i$.

If $j - i$ is odd, then

$$\langle c_i, c_j \rangle \equiv 0 \cdot s_1 + 2 \cdot s_2 \pmod 4,$$

where s_1 is the sum of $2n_0(c_f)$ ones and threes, and s_2 is the sum of $2n_2(c_f)$ ones and threes. So s_2 is an even number. It follows that $\langle c_i, c_j \rangle = 0$.

If $j - i$ is even, then

$$\langle c_i, c_j \rangle \equiv \alpha_1 0 \cdot 0 + \alpha_2 0 \cdot 2 + \alpha_3 2 \cdot 0 + \alpha_4 2 \cdot 2 + \alpha_5 1 \cdot 1 + \alpha_6 1 \cdot 3 + \alpha_7 3 \cdot 1 + \alpha_8 3 \cdot 3 \pmod 4,$$

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 2^{m-1}$. By considering elementary properties of the cyclic shift we obtain $\alpha_6 = \alpha_7$. It follows that

$$\langle c_i, c_j \rangle \equiv \alpha_5 - 2\alpha_6 + \alpha_8 \pmod 4 \equiv 2^{m-1} - 4\alpha_6 \pmod 4.$$

So c_i and c_j are orthogonal codewords in this case as well. Therefore C_f is a self-orthogonal $\mathbb{Z}_4$-code.

By Lemma 3.2, c_f has Euclidean weight divisible by 8. Since C_f is a self-orthogonal code, it follows from (2) that all codewords of C_f have Euclidean weights divisible by 8.

Even and odd coordinates alternate in the codeword c_f. Consequently, the residue code $C_f^{(1)}$ has dimension 2 and C_f has $4^2 2^{k_2}$ codewords for some $k_2 \leq 2^m - 4$.

□

Example 3.5. There are exactly eight bent functions on two variables. We constructed gbent functions f from all pairs (a, b) of bent functions $a, b : \mathbb{F}_2^2 \to \mathbb{F}_2$, as given in Theorem 3.1. In that way, 64 codewords $c_f \in \mathbb{Z}_4^8$ were obtained (see Lemma 3.2). Among associated codes C_f, constructed as in Proposition 3.4, there are two inequivalent codes. One of them is the code $C_{f_{2^3\_1}}$ generated by the codeword $c_{f_{2^3\_1}} = (0, 1, 0, 1, 0, 3, 2, 1)$, which is obtained from the pair $(x_1x_2, x_1 + x_1x_2)$. The other code arises from the pair (x_1x_2, x_1x_2), i.e., from the codeword $c_{f_{2^3\_2}} = (0, 1, 0, 1, 0, 1, 2, 3)$. Both codes are self-orthogonal codes of type $4^2 2^3$ and their permutation automorphism groups have order 64. The codes obtained from the remaining 62 gbent functions on two variables are equal to $C_{f_{2^3\_1}}$ or to $C_{f_{2^3\_2}}$.

Remark 3.6. According to [12], cyclic codes over rings are an important class of codes from both theoretical and practical points of view. In [12], the structure of cyclic codes over $\mathbb{Z}_4$ of even length is determined.

3.1.1 Type II codes over $\mathbb{Z}_4$

In the next theorem, we use the self-orthogonal cyclic $\mathbb{Z}_4$-code C_f constructed as in Proposition 3.4 to construct a Type II $\mathbb{Z}_4$-code $\widetilde{C_f}$. Moreover, for $m \geq 5$ the code $\widetilde{C_f}$ is of Type IV-II. To the best of our knowledge, this construction was not previously known and leads to a new family of Type IV-II $\mathbb{Z}_4$-codes.

Theorem 3.7. *Let $m \geq 3$ be odd, and let $a, b : \mathbb{F}_2^{m-1} \to \mathbb{F}_2$ be bent functions. Let $f : \mathbb{F}_2^m \to \mathbb{Z}_4$ be a gbent function given by $f(x, y) = 2a(x)(1 + y) + 2b(x)y + y$, $x \in \mathbb{F}_2^{m-1}$, $y \in \mathbb{F}_2$, and let c_f be a codeword*

$$(f((0, \ldots, 0)), f((0, \ldots, 0, 1)), \ldots, f((1, \ldots, 1))) \in \mathbb{Z}_4^{2^m}.$$

Let C_f be a cyclic $\mathbb{Z}_4$-code of type $4^2 2^{k_2}$ generated by c_f. Let G be a generator matrix of C_f in standard form. Let $k_3 = 2^m - 2^2 - k_2$ and let

$$\widetilde{D} = \begin{bmatrix} O & 2I_{k_3} & H \end{bmatrix}$$

be a $k_3 \times 2^m$ matrix, where O is the $k_3 \times (k_2 + 2)$ null matrix and H is a $k_3 \times 2$ matrix whose rows $h_i, 1 \leq i \leq k_3$ are defined as follows.

If k_2 is odd, then

$$h_i = \begin{cases} (0, 2), & \text{if } i \text{ is odd,} \\ (2, 0), & \text{if } i \text{ is even.} \end{cases}$$

If k_2 is even, then

$$h_i = \begin{cases} (2, 0), & \text{if } i \text{ is odd,} \\ (0, 2), & \text{if } i \text{ is even.} \end{cases}$$

(i) The code $\widetilde{C_f}$ generated by the matrix $\widetilde{G} = \begin{bmatrix} G \\ \widetilde{D} \end{bmatrix}$ is a Type II $\mathbb{Z}_4$-code of length 2^m.

(ii) If $m \geq 5$, then $\widetilde{C_f}$ is a Type IV $\mathbb{Z}_4$-code.

(iii) Up to equivalence, $\widetilde{C_f}$ does not depend on the choice of bent functions a and b.

Proof. (i) By Proposition 3.4, C_f is a self-orthogonal cyclic $\mathbb{Z}_4$-code generated by c_f. It is of type $4^2 2^{k_2}$ and length 2^m, and all its codewords have Euclidean weights divisible by 8.

The first and the second row of the matrix G, namely g_1 and g_2, are the only non-even rows in G.

Let $\widetilde{d}_i$ be the i-th row of the matrix $\widetilde{D}$. Then $\left\langle g_1, \widetilde{d}_i \right\rangle = 0$ and $\left\langle g_2, \widetilde{d}_i \right\rangle = 0$, for all $i = 1, \ldots, k_3$. Therefore, $\widetilde{C_f}$ is a self-orthogonal $\mathbb{Z}_4$-code of type $4^2 2^{2^m - 2^2}$, i.e., $\widetilde{C_f}$ is a self-dual $\mathbb{Z}_4$-code.

Moreover, all rows in $\widetilde{D}$ have Euclidean weight 8. It follows that Euclidean weights of all codewords in $\widetilde{C_f}$ are divisible by 8. We conclude that $\widetilde{C_f}$ is a Type II $\mathbb{Z}_4$-code of length 2^m.

(ii) Let $m \geq 5$ and let $c \in \widetilde{C_f}$. We have $n_1(c) + n_3(c) \in \{0, 2^{m-1}, 2^m\}$. The Euclidean weight of c is divisible by 8. So $n_2(c)$ is an even number. It follows that c has even Hamming weight. Therefore $\widetilde{C_f}$ is a Type IV $\mathbb{Z}_4$-code for $m \geq 5$.

(iii) Let $\begin{bmatrix} F & \tilde{I}_2 \end{bmatrix}$, where $\tilde{I}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be the generator matrix of the residue code of $\widetilde{C_f}$. The number of Type II $\mathbb{Z}_4$-codes of type $4^2 2^{2^m - 2^2}$ with the same residue code $\widetilde{C_f}^{(1)}$ is 2^2 (see [14], [22]). The generator matrices of these four codes could be given in the form (see [22, Theorem 3])

$$\begin{bmatrix} F & \tilde{I}_2 + 2B \\ 2H & O \end{bmatrix},$$

where the possibilities for B are $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, if $m = 3$. If $m \geq 5$, then the possibilities for B are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore those four codes are equivalent for every $m \geq 3$. □

Example 3.8. The construction described in Theorem 3.7, when applied to the codes $C_{f_{2^3\_1}}$ and $C_{f_{2^3\_2}}$ from Example 3.5, yields a code equivalent to $\mathscr{K}_8'$, a unique Type II $\mathbb{Z}_4$-code of length 8 and type $4^2 2^4$, whose permutation automorphism group has size 1152 (see [8], [18]). According to [8], it was introduced by Klemm in [20]. Let W_i^E and W_i^L denote the number of codewords of Euclidean weight i and Lee weight i in a $\mathbb{Z}_4$-code, respectively. The code $\mathscr{K}_8'$ has Euclidean weight distribution

$$(W_0^E, W_8^E, W_{16}^E, W_{24}^E, W_{32}^E) = (1, 140, 102, 12, 1).$$

Its Lee weight distribution is

$$(W_0^L, W_4^L, W_6^L, W_8^L, W_{10}^L, W_{12}^L, W_{16}^L) = (1, 12, 64, 102, 64, 12, 1).$$

In the sequel we will consider weight distributions for codes $\widetilde{C_f}$ of length 2^m for odd $m, m \geq 3$. By A_i we denote the number of codewords of weight i in a binary code. It follows from the MacWilliams identity (see, for example, [19], p. 252.) that the weight distribution $(A_0, \ldots, A_n)$ of a binary linear $[n, k]$ code and the weight distribution $(A'_0, \ldots, A'_n)$ of its dual code are connected by the equations

$$A'_j = \frac{1}{2^k} \sum_{i=0}^{n} A_i \sum_{l=0}^{j} (-1)^l \binom{i}{l} \binom{n-i}{j-l}, \ j = 0, \ldots, n. \tag{3}$$

Lemma 3.9. *Let $\widetilde{C_f}$ be a Type II $\mathbb{Z}_4$-code of length 2^m for odd $m, m \geq 3$, constructed as in Theorem 3.7. Then the weight distribution of its torsion code $\widetilde{C_f}^{(2)}$ is $(A'_0, \ldots, A'_{2^m})$, where*

$$A'_j = \frac{1}{2} \left(\binom{2^m}{j} + \sum_{l=0}^{j} (-1)^l \binom{2^{m-1}}{l} \binom{2^{m-1}}{j-l} \right)$$

for even j and $A'_j = 0$ for odd j, $j = 0, \ldots, 2^m$.

Proof. The $\mathbb{Z}_4$-code $\widetilde{C_f}$ constructed as in Theorem 3.7 is a self-dual $\mathbb{Z}_4$-code. Therefore the torsion code $\widetilde{C_f}^{(2)}$ is the dual code of $\widetilde{C_f}^{(1)}$. Further, $\widetilde{C_f}$ is a Type II $\mathbb{Z}_4$-code. So $\widetilde{C_f}^{(2)}$ is an even binary code. By the construction, the residue code $\widetilde{C_f}^{(1)}$ contains codewords of weights $0, 2^{m-1}$ and 2^m with $A_0 = 1, A_{2^{m-1}} = 2, A_{2^m} = 1$. The statement of the lemma follows from the expression (3). □

Theorem 3.10. *Let $\widetilde{C_f}$ be a Type II $\mathbb{Z}_4$-code of length 2^m for odd $m, m \geq 3$, constructed as in Theorem 3.7, and let $(A'_0, \ldots, A'_{2^m})$ be the weight distribution of its torsion code $\widetilde{C_f}^{(2)}$. Then:*

(i) $\widetilde{C_f}$ has Euclidean weight distribution $(W_0^E, \ldots, W_{2^{m+2}}^E)$ with $W_i^E = 0$ for $i \not\equiv 0 \pmod 8$ and, for i divisible by 8, we have

$$W_i^E = A'_{\frac{i}{4}} + s_i + t_i,$$

(ii) the symmetrized weight enumerator of the code $\widetilde{C_f}$ is

$$swe_{\widetilde{C_f}}(a, b, c) = s_{2^m} b^{2^m} + \sum_{i=0}^{2^m} \left(A'_i a^{2^m - i} c^i + t_{4i} a^{5 \cdot 2^{m-3} - i} b^{2^{m-1}} c^{i - 2^{m-3}} \right),$$

(iii) if $m \geq 5$, then $\widetilde{C_f}$ has Lee weight distribution $(W_0^L, \ldots, W_{2^{m+1}}^L)$ with $W_i^L = 0$ for $i \not\equiv 0 \pmod 4$ and, for i divisible by 4, we have

$$W_i^L = A'_{\frac{i}{2}} + s_i + u_i,$$

where

$$A'_j = \frac{1}{2}\left(\binom{2^m}{j} + \sum_{l=0}^{j}(-1)^l \binom{2^{m-1}}{l}\binom{2^{m-1}}{j-l}\right)$$

for even j *and* $A'_j = 0$ *for odd* j, $j = 0, \ldots, 2^m$, *and*

$$s_i = \begin{cases} 2^{2^m-2} & \text{if } i = 2^m, \\ 0 & \text{otherwise;} \end{cases}$$

$$t_i = \begin{cases} 2^{2^{m-1}}\binom{2^{m-1}}{(2i-2^m)/8} & \text{if } 2^{m-1} \le i \le 5\cdot 2^{m-1}, \\ 0 & \text{otherwise;} \end{cases}$$

$$u_i = \begin{cases} 2^{2^{m-1}}\binom{2^{m-1}}{(2i-2^m)/4} & \text{if } 2^{m-1} \le i \le 3\cdot 2^{m-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The expression for $(A'_0, \ldots, A'_{2^m})$ is determined by Lemma 3.9.

By Theorem 3.7, $\widetilde{C_f}$ is a Type II $\mathbb{Z}_4$-code, i.e., all Euclidean weights are divisible by 8. If $m \ge 5$, then all Lee weights in $\widetilde{C_f}$ are divisible by four.

Let $\widetilde{G}_s$ be the generator matrix of $\widetilde{C_f}$ in standard form. Denote by r_i the i-th row of $\widetilde{G}_s$, $i = 1, \ldots, 2^m - 2$. Note that the matrix $B_1 + 2B_2$ in (1) is of order 2. Further, for $m = 3$, each of the rows r_1 and r_2 contains the number 2 exactly once, and for $m \ge 5$, there are no 2s in r_1 and r_2.

Let $c \in \widetilde{C_f}$. Then $n_1(c) + n_3(c) \in \{0, 2^{m-1}, 2^m\}$ and

$$c = a_1 r_1 + a_2 r_2 + \sum_{i=3}^{2^m-2} a_i r_i,$$

where $a_1, a_2 \in \mathbb{Z}_4$ and $a_i \in \mathbb{Z}_2$ for $i = 3, \ldots, 2^m - 2$. The code $\widetilde{C_f}$ contains exactly 2^{2^m-2} codewords c with $n_1(c) + n_3(c) = 0$. These are the codewords with even a_1 and a_2. Furthermore, $\widetilde{C_f}$ contains exactly 2^{2^m-2} codewords c with $n_1(c) + n_3(c) = 2^m$. These are the codewords with odd a_1 and a_2. Finally, $\widetilde{C_f}$ contains exactly 2^{2^m-1} codewords c with $n_1(c) + n_3(c) = 2^{m-1}$. These are the codewords where one of the elements in $\{a_1, a_2\}$ is odd and the other is even.

The codewords c with $n_1(c) + n_3(c) = 2^m$ have Euclidean and Lee weight equal to 2^m.

Let $c \in \widetilde{C_f}$ be a codeword with $n_1(c) + n_3(c) = 2^{m-1}$. If $m = 3$, half of these codewords have $n_2(c) = 1$ and half of them have $n_2(c) = 3$. If $m \ge 5$, then $n_2(c)$ is an even number, $n_2(c) \in \{0, \ldots, 2^{m-1}\}$, and there are exactly

$$2 \cdot 2^{2^{m-1}-1}\binom{2^{m-1}}{n_2(c)}$$

codewords with Euclidean weight $2^{m-1} + 4n_2(c)$ and Lee weight $2^{m-1} + 2n_2(c)$, for each of these numbers $n_2(c)$. If a_1 is odd, then the even codeword $a_2 r_2 + \sum_{i=3}^{2^m-2} a_i r_i$ has 2s

on exactly $n_2(c)$ even coordinate positions and the remaining even number of 2s are on odd coordinate positions. If a_2 is odd, then the even codeword $a_1r_1 + \sum_{i=3}^{2^m-2} a_ir_i$ has 2s on exactly $n_2(c)$ odd coordinate positions and the remaining even number of 2s are on even coordinate positions.

From these observations the weight distributions in (i) and (iii) are obtained.

For the coefficients of the symmetrized weight enumerator of $\widetilde{C_f}$, we count the codewords c with $n_1(c) + n_3(c) = 2^m$, the even codewords and the codewords c with $n_1(c) + n_3(c) = 2^{m-1}$ in $\widetilde{C_f}$. □

3.2 Binary Type II self-dual codes

Self-dual codes over $\mathbb{Z}_4$ and their images under the Gray map are examined in [10]. According to [10], it is of interest to understand what the Gray map of a self-dual code is and, especially, when its image is a self-dual code.

The *Gray map* $\phi : \mathbb{Z}_4^n \to \mathbb{F}_2^{2n}$ is the componentwise extension of the map $\psi : \mathbb{Z}_4 \to \mathbb{F}_2^2$ defined by $\psi(0) = (0,0)$, $\psi(1) = (0,1)$, $\psi(2) = (1,1)$, $\psi(3) = (1,0)$. Note that a $\mathbb{Z}_4$-code C and the corresponding binary code $\phi(C)$ have the same size and that the Lee weight of a codeword $x \in \mathbb{Z}_4^n$ is equal to the (Hamming) weight of its Gray image $\phi(x)$.

If C is a $\mathbb{Z}_4$-code of length n, its Gray image $\phi(C)$ is a binary code of length $2n$, which is in general nonlinear. However, the following theorem holds (see [4, Theorem 8]).

Theorem 3.11. *If C is a self-dual $\mathbb{Z}_4$-code with all Lee weights divisible by 4, then the binary image of C under the Gray map is linear.*

Moreover, according to [11, Proposition 2.6], the following statement holds.

Theorem 3.12. *If C is a Type IV $\mathbb{Z}_4$-code then all the Lee weights of C are divisible by 4 and its Gray image is a self-dual doubly even binary code.*

If $x, y \in \mathbb{Z}_4^n$, we define xy as the componentwise product $(x_1y_1, \ldots, x_ny_n)$. According to [13], the following statement holds.

Lemma 3.13. *Let C be a $\mathbb{Z}_4$-code of type $4^{k_1}2^{k_2}$. Let G be a generator matrix of C in standard form and let g_i, $i \in \{1, 2, \ldots, k_1 + k_2\}$, be its i-th row. The binary code $\phi(C)$ is linear if and only if for all $i, j \in \{1, \ldots, k_1\}$ we have $2g_ig_j \in C$.*

As a consequence of our previous observations we have the following corollaries.

Corollary 3.14. *Let $\widetilde{C_f}$ be a Type II $\mathbb{Z}_4$-code of length 2^m for odd m, $m \geq 3$, constructed as in Theorem 3.7. Then:*

(i) The Gray image $\phi(\widetilde{C_f})$ is a self-dual binary code of length 2^{m+1}. If $m \geq 5$, then $\phi(\widetilde{C_f})$ is doubly even.

(ii) The Gray image $\phi(C_f)$ is a self-orthogonal linear binary code of length 2^{m+1}. If $m \geq 5$, then $\phi(C_f)$ is doubly even.

Proof. (i) According to Theorem 3.7, $\widetilde{C_f}$ is a Type II $\mathbb{Z}_4$-code of length 2^m and for $m \geq 5$, $\widetilde{C_f}$ is a Type IV $\mathbb{Z}_4$-code. Moreover, for $m = 3$, a construction yields the code $\mathscr{K}_8'$ (see Examples 3.5 and 3.8). Its Gray image is an even $[16, 8, 4]$ binary code. Together with Theorem 3.12 this concludes the proof.

(ii) The code $\phi(C_f)$ is a subcode of the code $\phi(\widetilde{C_f})$. Let G be a generator matrix of C_f in standard form. Codewords g_1 and g_2 have alternating odd and even coordinates. So, $g_1 g_2$ is an even codeword. Then $2g_1 g_2$ is a codeword with all coordinates equal to 0. It follows from Lemma 3.13 that $\phi(C_f)$ is linear. Therefore the statement holds. □

Corollary 3.15. *Let $\widetilde{C_f}$ be a Type IV-II $\mathbb{Z}_4$-code of length 2^m for odd m, $m \geq 5$, constructed as in Theorem 3.7, and let $(A'_0, \ldots, A'_{2^m})$ be the weight distribution of its torsion code $\widetilde{C_f}^{(2)}$. Then the Gray image $\phi(\widetilde{C_f})$ has weight distribution $(W_0, \ldots, W_{2^{m+1}})$ with $W_i = 0$ for $i \not\equiv 0 \pmod 4$ and, for i divisible by 4, we have*

$$W_i = A'_{\frac{i}{2}} + s_i + u_i,$$

where

$$s_i = \begin{cases} 2^{2^m-2} & \text{if } i = 2^m, \\ 0 & \text{otherwise} \end{cases},$$

$$u_i = \begin{cases} 2^{2^{m-1}} \binom{2^{m-1}}{(2i-2^m)/4} & \text{if } 2^{m-1} \leq i \leq 3 \cdot 2^{m-1}, \\ 0 & \text{otherwise.} \end{cases}.$$

Proof. This follows directly from Theorem 3.10 (iii). □

3.3 On minimum weight codewords

In the study of the properties of codes, one of the desirable properties is the possible connection to various combinatorial objects, which is sometimes realized in such a way that supports of the minimum weight codewords of the code correspond to the blocks of a design (see [28]). An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) *design*, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. We assume that the reader is familiar with the basic facts of design theory (see, for example, [1], [7]).

In previous sections we constructed codes C_f, $\widetilde{C_f}$ and $\phi(\widetilde{C_f})$ for $m = 3$. In this section we give an example for $m = 5$. We also observe the minimum weight codewords and their relation with combinatorial designs.

Example 3.16. Let $(a, b) = (x_1x_2 + x_1x_3 + x_2x_4, x_1x_2 + x_3x_4)$ be a pair of bent functions $a, b : \mathbb{F}_2^4 \to \mathbb{F}_2$. From this pair, we constructed the gbent function f_{2^5} as described in Theorem 3.1. Further, the codeword

$$c_{f_{2^5}} = (0, 1, 0, 1, 0, 1, 0, 3, 0, 1, 2, 1, 0, 1, 2, 3, 0, 1, 0, 1, 2, 1, 2, 3, 2, 3, 0, 3, 0, 3, 2, 1)$$

is constructed as in Lemma 3.2. The self-orthogonal $\mathbb{Z}_4$-code $C_{f_{2^5}}$, constructed by Proposition 3.4, is of type $4^2 2^{21}$, its dual $C_{f_{2^5}}^{\perp}$ is of type $4^9 2^{21}$ and its Gray image is a doubly even $[64, 25, 4]$ binary code. The permutation automorphism group of $C_{f_{2^5}}$ is of order 9 663 676 416.

The $\mathbb{Z}_4$-code $C_{f_{2^5}}$ has $d_H(C_{f_{2^5}}) = 2$, $d_L(C_{f_{2^5}}) = 4$ and $d_E(C_{f_{2^5}}) = 8$, and the sets of minimum weight codewords are the same for all three weights. The supports of these codewords form a resolvable 1-$(32, 2, 1)$ design with 16 blocks and block intersection number 0. So the minimum weight codewords of its Gray image yield a resolvable 1-$(64, 4, 1)$ design.

For $C_{f_{2^5}}^{\perp}$, we have $d_H(C_{f_{2^5}}^{\perp}) = 2$, $d_L(C_{f_{2^5}}^{\perp}) = 4$ and $d_E(C_{f_{2^5}}^{\perp}) = 8$. The sets of minimum weight codewords are the same for Hamming and Lee weight. The codewords of minimum Euclidean weight have Lee weight equal to 4, 6 or 8. The supports of the codewords with Euclidean weight and Lee weight equal to 8 form a 1-$(32, 8, 7)$ design with 28 blocks and block intersection numbers 0 and 4. This design is a $(4, 7; 2)$-net, i.e., an affine resolvable 1-design. Its block intersection graph G_0 is a strongly regular graph with parameters $(28, 15, 6, 10)$. Affine resolvable 1-designs are important due to their connection with orthogonal arrays and partial geometries (see [7]).

Remark 3.17. In a similar way to Example 3.16, for $m = 3$ one can obtain that the supports of the codewords of minimum Euclidean weight form a 1-$(8, 4, 2)$ design with 4 blocks and block intersection numbers 0 and 2. This is a $(2, 2; 2)$-net, i.e., an affine resolvable 1-design. Because of the computational complexity, it is out of our reach to analyze minimum weight codewords in the case $m \geq 7$. However, for $m = 7$ and $(a, b) = (x_1x_4 + x_2x_5 + x_3x_6, x_1x_4 + x_2x_5 + x_3x_6 + x_1x_2x_3)$, we calculated $d_E(C_{f_{2^7}}^{\perp}) = 8$ from the minimum weight of the corresponding residue and torsion code. So the supports of the codewords with minimum Euclidean weight in $C_{f_{2^7}}^{\perp}$ cannot form a net.

Observing the minimum weight codewords in the torsion code $\widetilde{C_f}^{(2)}$, for odd m, $m \geq 3$, we obtain that the supports of the minimum weight codewords split into 2-sets which forms a resolvable 1-design, as given in the following remark.

Remark 3.18. Let $\widetilde{C_f}$ be a Type II $\mathbb{Z}_4$-code of length 2^m for odd m, $m \geq 3$, constructed as in Theorem 3.7. Let $\{i, j\}$, $i < j$, be a support of a minimum weight codeword in $\widetilde{C_f}^{(2)}$. Then $j - i$ has to be an even number because $\widetilde{C_f}^{(2)}$ is the dual code of $\widetilde{C_f}^{(1)}$. So the supports $\{i, j\} \subseteq \{1, \ldots, 2^m\}$, $i < j$, of the minimum weight codewords in $\widetilde{C_f}^{(2)}$ are divided into $\frac{2^m}{4} = 2^{m-2}$ classes:

$$\mathcal{M}(k) = \{\{i, j\} \ : \ j - i = k \text{ or } j - i = 2^m - k\},$$

where $k \in \{2, 4, \ldots, 2^{m-1} - 2\}$, and

$$\mathcal{M}\left(2^{m-1}\right) = \left\{\{i, j\} \ : \ j - i = 2^{m-1}\right\}.$$

Now $\widetilde{C_f}^{(2)}$ is cyclic. So the class $\mathcal{M}(k)$, $k \in \{2, 4, \ldots, 2^{m-1} - 2\}$, consists of 2^m supports and every coordinate position occurs in exactly two supports in the class. The class $\mathcal{M}(2^{m-1})$ consists of 2^{m-1} supports and every coordinate position occurs in exactly one of the supports. So the set

$$\mathcal{M}(2) \cup \cdots \cup \mathcal{M}\left(2^{m-1} - 2\right) \cup \mathcal{M}\left(2^{m-1}\right)$$

is the set of the blocks of a 1-$(2^m, 2, 2^{m-1} - 1)$ design with $2^{m-1}(2^{m-1} - 1)$ blocks.

Acknowledgements

This work has been supported by Croatian Science Foundation under the project 6732 and by the University of Rijeka under the project uniri-prirod-18-45. The authors would like to thank the anonymous referees for helpful comments that improved the presentation of the paper.

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, 2nd ed., Cambridge University Press, Cambridge, 1999.

[2] W. Bosma and J. Cannon, *Handbook of Magma Functions*, University of Sydney, 1994, available at: `http://magma.maths.usyd.edu.au/magma`.

[3] S. Bouyuklieva and M. Harada, On Type IV self-dual codes over $\mathbb{Z}_4$, *Discrete Math.* **247** (2002), 25–50.

[4] A. R. Calderbank and N. J. A. Sloane, Double circulant codes over $\mathbb{Z}_4$ and even unimodular lattices, *J. Algebr. Combin.* **6** (1997), 119–131.

[5] C. Carlet and P. Gaborit, Hyper-bent functions and cyclic codes, *J. Combin. Theory Ser. A* **113**(3) (2006), 466–482.

[6] C. Carlet and S. Mesnager, Four decades of research on bent functions, *Des. Codes Crypto.* **78** (2016), 5–50.

[7] C. J. Colbourn and J. H. Dinitz (eds.), *Handbook of Combinatorial Designs*, 2nd ed., Chapman & Hall/CRC Press, Boca Raton, 2007.

[8] J. H. Conway and N. J. A. Sloane, Self-Dual Codes over the Integers Modulo 4, *J. Combin. Theory Ser. A* **62** (1993), 30–45.

[9] C. Ding, A. Munemasa and V. D. Tonchev, Bent Vectorial Functions, Codes and Designs, *IEEE Trans. Inform. Theory* **65**(11) (2019), 7533–7541.

[10] S. T. Dougherty and C. Fernández-Córdoba, Codes over $\mathbb{Z}_{2^k}$, Gray map and Self-Dual Codes, *Adv. Math. Commun.* **5**(4) (2011) 571–588.

[11] S. T. Dougherty, P. Gaborit, M. Harada, A. Munemasa and P. Solé, Type IV self-dual codes over rings, *IEEE Trans. Inform. Theory* **45**(7) (1999) 2345–2360.

[12] S. T. Dougherty and S. Ling, Cyclic codes over $\mathbb{Z}_4$ of even length, *Des. Codes Crypto.* **39**(2) (2006) 127–153.

[13] C. Fernández-Córdoba, J. Pujol and M. Villanueva, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: rank and kernel, *Des. Codes Crypto.* **56** (2010), 43–59.

[14] P. Gaborit, Mass formulas for self-dual codes over $\mathbb{Z}_4$ and $\mathbb{F}+u\mathbb{F}_q$ rings, *IEEE Trans. Inform. Theory* **42**(4) (1996), 1222–1228.

[15] T. A. Gulliver and M. Harada, An Optimal Unimodular Lattice in Dimension 39, *J. Combin. Theory Ser. A* **88** (1999) 158–161.

[16] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Trans. Inform. Theory* **40** (1994), 301–319.

[17] M. Harada, P. Solé and P. Gaborit, Self-dual codes over $\mathbb{Z}_4$ and unimodular lattices: a survey, in: *Algebra and Combinatorics: an International Congress, ICAC'97, Hong Kong*, (Eds.: K.-P. Shum, E.J. Taft and Z.-X. Wan), Springer, Singapore, 1999, 255–275.

[18] W. C. Huffman, On the classification and enumeration of self-dual codes, *Finite Fields Appl.* **11** (2005), 451–490.

[19] W. C. Huffman and V. Pless, *Fundamentals of Error-Correcting Codes*, Cambridge University Press, Cambridge, 2003.

[20] M. Klemm, Selbstduale Codes über dem Ring der ganzen Zahlen modulo 4, *Arch. Math.* **53** (1989), 201–207.

[21] P. V. Kumar, R. A. Scholtz and L. R. Welch, Generalized bent functions and their properties, *J. Combin. Theory Ser. A* **40** (1985), 90–107.

[22] V. Pless, J. Leon and J. Fields, All $\mathbb{Z}_4$ codes of Type II and length 16 are known, *J. Combin. Theory Ser. A* **78** (1997), 32–50.

[23] E. Rains and N. J. A. Sloane, Self-dual codes, in: *Handbook of Coding Theory*, (Eds.: V.S. Pless and W.C. Huffman), Elsevier, Amsterdam, 1998, 177–294.

[24] O. S. Rothaus, On "Bent" Functions, *J. Combin. Theory Ser. A* **20** (1976), 300–305.

[25] K. U. Schmidt, Quaternary constant-amplitude codes for multicode CDMA, *IEEE Trans. Inform. Theory* **55** (2009), 1824–1832.

[26] M. Shi, Y. Liu, H. Randriambololona, L. Sok and P. Solé, Trace codes over $\mathbb{Z}_4$, and Boolean functions, *Des. Codes Crypto.* **87** (2019), 1447–1455.

[27] C. Tang, N. Li, Y. Qi, Z. Zhou and T. Helleseth, Linear Codes With Two or Three Weights From Weakly Regular Bent Functions, *IEEE Trans. Inform. Theory* **62**(3) (2016), 1166–1176.

[28] V. D. Tonchev, Codes, in: *Handbook of Combinatorial Designs*, 2nd edn., (Eds.: C.J. Colbourn and J.H. Dinitz), Chapman & Hall/CRC Press, Boca Raton, 2007, 667–702.

[29] M. Yamada, Self-dual $\mathbb{Z}_4$-codes of Type IV generated by skew-Hadamard matrices and conference matrices, *Australas. J. Combin.* **43** (2008), 177–188.

(Received 3 Nov 2021; revised 4 May 2022, 9 Sep 2022)