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Corrigendum and extension to "Hamiltonicity in directed Toeplitz graphs $T_n\langle 1,2;t_1,t_2\rangle$"

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We use [1] for notation and terminology not defined here. In Theorem 3.3 [1], we proved that $T_n\langle 1,2;3,t\rangle$ is hamiltonian for all n and t. Unfortunately, this theorem does not hold for $n = 8$ when $t = 5$. Here, we correct this error by proving that $T_8\langle 1,2;3,5\rangle$ is non-hamiltonian. i Then we generalize Theorem 3.9 of [1], for all t_1. Finally, we address the conjecture stated in [1], which completes the hamiltonicity investigation in directed Toeplitz graphs $T_n\langle 1,2;t_1,t_2\rangle$.

The corrected version of Theorem 3.3 in [1] can be restated as follows.

Theorem 1 *$T_n\langle 1,2;3,t\rangle$ is hamiltonian if and only if $n \neq 8$ and $t \neq 5$.*

Proof. Let $n \neq 8$ and $t \neq 5$, then by Theorem 3.3 in [1], $T_n\langle 1,2;3,t\rangle$ is hamiltonian.

Conversely, we prove that $T_8\langle 1,2;3,5\rangle$ is non-hamiltonian. Assume, to the contrary, that $T_8\langle 1,2;3,5\rangle$ is hamiltonian. Let $H = H_{1\to 8} \cup H_{8\to 1}$ be a hamiltonian cycle in $T_8\langle 1,2;3,5\rangle$. Then, for every vertex v in H, we have $d^-(v) = 1 = d^+(v)$. Since the path $H_{1\to 8}$ is hamiltonian in the subgraph of $T_8\langle 1,2;3,5\rangle$ induced by $V(H_{8\to 1}\setminus\{1,8\})$, the vertices which are not covered by $H_{1\to 8}$ would be covered by $H_{8\to 1}$. Since increasing edges in $H_{1\to 8}$ are of length one, and two only, $H_{8\to 1}$ contains no pair of successive vertices different from $\{1,2\}$ or $\{7,8\}$. Thus $H_{8\to 1}$ would not be using any increasing edge of length one. The set of all decreasing edges in $T_8\langle 1,2;3,5\rangle$ is $\{(8,3),(7,2),(6,1),(8,5),(7,4),(6,3),(5,2),(4,1)\}$. Now $d^-(1) = d^+(8) = 2$ in $T_8\langle 1,2;3,5\rangle$, so $\{(8,3),\ (4,1)\} \subseteq E(H_{8\to 1})$ or $\{(8,3),\ (6,1)\} \subseteq E(H_{8\to 1})$ or $\{(8,5),\ (4,1)\} \subseteq E(H_{8\to 1})$ or $\{(8,5),(6,1)\} \subseteq E(H_{8\to 1})$. The only possible case is $\{(8,3),\ (6,1)\} \subseteq E(H_{8\to 1})$, because all the others will result in a pair of successive vertices in $H_{8\to 1}$. But then again $\{(8,3),\ (6,1)\}$ cannot be the case, because the edge $(8,3)$ would be stuck at vertex 3 as it can use only the edge of length two which results in successive vertices in $H_{8\to 1}$, say $\{5,6\}$; see Figure 1. This is a contradiction. $\square$

There was a conjecture stated in [1] that, for odd $t_1 \geq 7$ and odd $t_2 < 2t_1 + 1$, $T_n\langle 1,2;t_1,t_2\rangle$ is non-hamiltonian for $n \in \{t_2+3, t_2+5, \ldots, 2t_1+2\}$. Here we prove this conjecture in Theorem 2.

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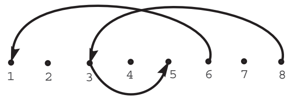

Figure 1.

Theorem 2 *For odd $t_1 \geq 7$ and odd $t_2 < 2t_1 + 1$, if $n \in \{t_2 + 3, t_2 + 5, \ldots, 2t_1 + 2\}$, then $T_n\langle 1, 2; t_1, t_2\rangle$ is non-hamiltonian*

Proof. Theorem 2.10 in [1], asserts that, for odd $t_2 \geq 7$, $T_n\langle 1, 2; t_2\rangle$ is non-hamiltonian if $n \in \{t_2 + 3, t_2 + 5, \ldots, 2t_2 + 2\}$. For odd t_1 and t_2 such that $t_1 < t_2 < 2t_1 + 1$, we have $t_2 + 3 \leq 2t_1 + 2$ (because $t_2 < 2t_1 + 1$ implies that $t_2 + 3 < 2t_1 + 4 \leq 2t_1 + 2$) and $2t_1 + 2 < 2t_2 + 2$ (because $t_1 < t_2$). So by Theorem 2.10 in [1], for odd t_1 and t_2 such that $t_1 < t_2 < 2t_1 + 1$ and $t_2 \geq 7$, $T_n\langle 1, 2; t_2\rangle$ is non-hamiltonian if $n \in \{t_2+3, t_2+5, \ldots, 2t_1+2\}$. Now we show that for odd t_1 and t_2 such that $t_1 < t_2 < 2t_1+1$ and $t_2 \geq 7$, $T_n\langle 1, 2; t_1, t_2\rangle$ is non-hamiltonian if $n \in \{t_2+3, t_2+5, \ldots, 2t_1+2\}$.

Assume, to the contrary, that for odd $t_1 \geq 7$ and odd $t_2 < 2t_1+1$, $T_n\langle 1, 2; t_1, t_2\rangle$ is hamiltonian if $n \in \{t_2+3, t_2+5, \ldots, 2t_1+2\}$. Let $H = H_{1\to n} \cup H_{n\to 1}$ be a hamiltonian cycle in $T_n\langle 1, 2; t_1, t_2\rangle$. This hamiltonian cycle H in $T_n\langle 1, 2; t_1, t_2\rangle$ cannot have all the decreasing edges of the same length, say t_1 or t_2, because by Theorem 2.10 in [1], for odd $t_{i\in\{1,2\}} \geq 7$, $T_n\langle 1, 2; t_i\rangle$ is non-hamiltonian if $n \in \{t_i + 3, t_i + 5, \ldots, 2t_i + 2\}$. Thus H needs to use the decreasing edges of both length t_1 and t_2. Since $n \leq 2t_1 + 2$ and $n \leq t_1 + t_2$ (because $n \leq 2t_1 + 2 = t_1 + t_1 + 2 \leq t_1 + t_2$, as $t_1 + 2 \leq t_2$), $H_{n\to 1}$ cannot use more than two decreasing edges, because otherwise $H_{n\to 1}$ contains pairs of successive vertices but, as explained in the proof of Theorem 3.3, $H_{n\to 1}$ contains no pair of successive vertices. Thus $H_{n\to 1}$ can have exactly two decreasing edges of different lengths. Since $d^-(v) = 1 = d^+(v)$, for every vertex v in H, we have either $(n, n - t_2), (t + 1, 1) \in E(H_{n\to 1})$ or $(n, n - t_1), (t + 2, 1) \in E(H_{n\to 1})$. Since the increasing edges in $H_{n\to 1}$ are of length two only, but $n - t_2$ is odd while $t_1 + 1$ is even, it follows that there is no path $P_{n-t_2\to t_1+1}$ in $H_{n\to 1}$ between $n - t_2$ and $t_1 + 1$. Similarly there is no path $P_{n-t_1\to t_2+1}$ in $H_{n\to 1}$ between $n - t_1$ and $t_2 + 1$. This is a contradiction. □

Now by using Theorems 1 and 2, along with Theorem 3.8 in [1], and using some results in [2, 3], we can generalize Theorem 3.9 in [1] for all t_1, as follows.

Theorem 3 *Let $G = T_n\langle 1, 2; t_1, t_2\rangle$.*

1. *If t_1 and t_2 both are even, then G is hamiltonian if and only if n is odd.*
2. *If t_1 and t_2 are of opposite parity, then G is hamiltonian for all n.*
3. *If t_1 and t_2 both are odd, and*
   (a) if $t_2 \geq 2t_1 + 1$, then G is hamiltonian for all n.
   (b) if $t_2 < 2t_1 + 1$, then G is hamiltonian if and only if $n \notin \{t_2 + 3, t_2 + 5, \ldots, 2t_1 + 2\}$.

Proof. This is left to the reader. □

References

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