AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **82(2)** (2022), Pages 197–200

Note on sets of hyperplane-type $(m, n)_{r-1}$ in $\mathrm{PG}(r, q)$

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Abstract

It is shown that if $\mathcal{K}$ is a set of k points of $\mathrm{PG}(r, q)$ of hyperplane-type $(m, n)_{r-1}$, $q = p^h$, p an odd prime and $h \geq 1$, then either $n - m \leq q^{(r-1)/2}$ or p divides m, n and k or p divides $m - 1, n - 1$ and $k - 1$.

1 Introduction

Let k, m and n be three integers satisfying $k > 0$, $0 \leq m < n \leq q^{r-1} + q^{r-2} + \cdots + q + 1$. Then a k-set $\mathcal{K}$ of points of $\mathrm{PG}(r, q)$ of *hyperplane-type* $(m, n)_{r-1}$ is a set of points of $\mathrm{PG}(r, q)$ of size k intersected by any hyperplane of $\mathrm{PG}(r, q)$ either in m or n points, and both the sizes m and n of these intersections occur. The integers m and n are the *intersection numbers* of $\mathcal{K}$ with respect to the hyperplanes. The interest in the study of such sets is motivated not only by the fact that many classical and nice objects of $\mathrm{PG}(3, q)$ have exactly two intersection numbers with respect to the hyperplanes, but also because of their connection with coding theory: the points of a k-set of hyperplane-type $(m, n)_{r-1}$ seen as columns of a matrix give rise to a linear (projective) code of dimension r and with two weights, and vice versa (see e.g. [1]).

Such sets have been intensively investigated, especially for $r \leq 3$, and for any r one has that $(n - m) \mid q^{r-1}$ and k fulfils the equation

$$\theta_{r-2}k^2 - k[\theta_{r-2} + (m + n - 1)\theta_{r-1}] + mn\theta_r = 0 \tag{1.1}$$

where $\theta_r = q^r + q^{r-1} + \cdots + q + 1$.

* This research was partially supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA-INdAM). The author was also supported by the project "VALERE: Vanvitelli pEr la RicErca" of the University of Campania "Luigi Vanvitelli".

ISSN: 2202-3518

Since, if $m = 0$ then the set $\mathcal{K}$ is either a point or the complement of a hyperplane, and if $n = \theta_{r-1}$, then the set $\mathcal{K}$ is either the complement of a point or a hyperplane, we may assume that $1 \leq m < \theta_{r-1}$.

As already remarked, the most studied cases are $r = 2$ and $r = 3$, and in the 2-dimensional case not only desarguesian planes have been considered. When $r = 3$, most attention has been devoted to the case $n - m \leq q$, which certainly occurs if $1 \leq m \leq q$ (see e.g. [5]). Some examples of such sets with $n - m = q$ and $r = 3$ are hyperbolic quadrics, Baer subgeometries and Hermitian surfaces. If $r = 3$ and $n - m = q$ then the possible sizes of $\mathcal{K}$ are $k = m(q + 1)$ and $k = \dfrac{(q^2 + 1)(q + m)}{q + 1}$. In [4] or [5] two subsets of $\mathrm{PG}(3, 3)$ of plane-type $(3, 6)_2$ and size $k = 15 = \frac{(3^2+1)(3+3)}{3+1}$ are given. For a more general overview of k-sets of $\mathrm{PG}(3, q)$ of plane-type $(m, n)_2$ with $n - m = q$ we refer the interested reader to [2]. When $n - m \neq q$, a triple (k, m, n) with k fulfilling Equation (1.1) is called *non-standard*, and recently in [3] we can find results concerning the existence of admissible non-standard triples for k-sets of plane-type $(m, n)_2$ in $\mathrm{PG}(3, q)$.

If $r = 3$ and $n-m = q^2$, then $m \geq q+1$ and so $n = q^2+q+1$ and $m = q+1$; that is, $\mathcal{K}$ is a plane. Note that, in this case, since $q = p^h$, p prime, we have that p divides $m - 1, n - 1$ and $k - 1$. If $r = 3$ and $n - m = 1$, the discriminant of Equation (1.1) is $\Delta = (q + 1)^2 - 4mq^2(q(q + 1) - m)$, and from $\Delta \geq 0$ it follows that $m = 0$. Thus, a set of points of $\mathrm{PG}(3, q)$ of plane type $(m, m + 1)_2$ is a point.

In the planar case, independently of the plane being desarguesian, in [7] it has been proved that *if* $(m, n) = 1 = (m-1, n-1)$, $m \geq 2$, *then either* $n-m < \sqrt{q}$ *or* q *is a square,* $n-m = \sqrt{q}$ *and* $k = m(q+\sqrt{q}+1)$ *or* $k = q\sqrt{q}+\sqrt{q}(\sqrt{q}-1)(m-1)+m$.

Sets of points of $\mathrm{PG}(r, q)$ with two intersection numbers m and n with respect to the family of all d-dimensional subspaces of $\mathrm{PG}(r, q)$ are defined similarly, and we recall that such sets with $d \leq r - 2$ also have exactly two intersection numbers with respect to hyperplanes.

In this note, it is shown that if $\mathcal{K}$ is a k-set of hyperplane-type $(m, n)_{r-1}$, $r \geq 2$, then either $n - m \leq q^{(r-1)/2}$ or p divides m and n or p divides $m - 1$ and $n - 1$, where p is the prime number such that $q = p^h$, with p odd and $h \geq 1$. Indeed, this will follow from the proof of the following result.

Theorem 1.1 *Let* $\mathcal{K}$ *be a* k*-set of points of* $\mathrm{PG}(r, q)$ *of hyperplane-type* $(m, n)_{r-1}$, $r \geq 2$, $q = p^h$ *and* $h \geq 1$. *Assume* $n - m > q^{\frac{r-1}{2}}$. *Then either* $m \equiv n \equiv k \equiv 0 \mod p$ *or* $m \equiv n \equiv k \equiv 1 \mod p$.

2 Proof of Theorem 1.1

Let us start by recalling some useful properties for k-sets of points of $\mathrm{PG}(r, q)$ of hyperplane-type $(m, n)_{r-1}$.

An *m-hyperplane* (or *n-hyperplane*) is a hyperplane intersecting $\mathcal{K}$ in exactly m (or n) points.

If P is a point of $\mathcal{K}$, denote by $v_m(P)$ and $v_n(P)$, respectively, the number of m-hyperplanes and n-hyperplanes through P. Then $v_m(P) + v_n(P) = \theta_{r-1}$ and (see e.g. [4, 6])

$$v_m(P) = \frac{n\theta_{r-1} - k\theta_{r-2}}{n-m} - \frac{q^{r-1}}{n-m}, \tag{2.1}$$

$$v_n(P) = \frac{k\theta_{r-2} - m\theta_{r-1}}{n-m} + \frac{q^{r-1}}{n-m}. \tag{2.2}$$

Let P be a point of $\mathrm{PG}(r, q)$ not in $\mathcal{K}$, and denote by $u_m(P)$ and $u_n(P)$, respectively, the number of m-hyperplanes and n-hyperplanes through P. Then $u_m(P)+u_n(P) = \theta_{r-1}$ and

$$u_m(P) = \frac{n\theta_{r-1} - k\theta_{r-2}}{n-m}, \quad u_n(P) = \frac{k\theta_{r-2} - m\theta_{r-1}}{n-m}. \tag{2.3}$$

Thus the integers $v_m(P)$, $v_n(P)$, $u_m(P)$, $u_n(P)$ are independent from the point P, and comparing Equations (2.1), (2.2) and (2.3) gives

$$(n-m) \mid q^{r-1}$$

and, from $u_n = \dfrac{(k-m)\theta_{r-2} - mq^{r-1}}{n-m}$, it follows that $(n-m) \mid (k-m)$. So $(n-m) \mid (k-n)$ since $k-n = k-m-(n-m)$.

Now for the proof of Theorem 1.1, we are going to rewrite Equation (1.1). Let

$$\theta_{r-2}k^2 - k[\theta_{r-2} + (m+n-1)\theta_{r-1}] + mn\theta_r = 0.$$

Thus,

$$\begin{aligned}
\theta_{r-2}k^2 - k((m+n)q^{r-1} + (m+n)\theta_{r-2} - q^{r-1}) + mn\theta_r &= 0,\\
\theta_{r-2}k^2 - kmq^{r-1} - knq^{r-1} - km\theta_{r-2} - kn\theta_{r-2} + kq^{r-1} &\\
+ mnq^r + mnq^{r-1} + mn\theta_{r-2} &= 0,\\
\theta_{r-2}k(k-n) - mq^{r-1}(k-n) - m\theta_{r-2}(k-n) - knq^{r-1} &\\
+kq^{r-1} + mnq^r - mnq^{r-1} + mnq^{r-1} &= 0,\\
\theta_{r-2}(k-n)(k-m) - mq^{r-1}(k-n) - nq^{r-1}(k-m) &\\
+kq^{r-1} + mnq^r - mnq^{r-1} - mq^{r-1} + mq^{r-1} &= 0,
\end{aligned}$$

$$\begin{aligned}
&\theta_{r-2}(k-n)(k-m) - mq^{r-1}(k-n) - nq^{r-1}(k-m) + (k-m)q^{r-1}\\
&\qquad = \; -mnq^r + mnq^{r-1} - mq^{r-1},\\
&\theta_{r-2}(k-n)(k-m) - mq^{r-1}(k-n) - nq^{r-1}(k-m) + (k-m)q^{r-1}\\
&\qquad = \; -mnq^r + mq^{r-1}(n-1),\\
&\theta_{r-2}(k-n)(k-m) - mq^{r-1}(k-n) - nq^{r-1}(k-m) + (k-m)q^{r-1}\\
&\qquad = \; mq^{r-1}(n-1-nq).
\end{aligned}$$

The left-hand side is divisible by $(n-m)^2$ and so $(n-m)^2$ divides $mq^{r-1}(n-1-nq)$. Hence, since $n-m > q^{\frac{r-1}{2}}$, a factor of $n-m$ has to divide $m(n-1)$; in particular $p \mid m(n-1)$. Thus, either p divides m, n and k or p divides $m-1$, $n-1$ and $k-1$, and so the theorem is proved.

References

[1] R. Calderbank and W. M. Kantor, The geometry of two-weight codes, *Bull. London Math. Soc.* **18** (2) (1986), 97–122.

[2] M. de Finis, On k-sets of type (m, n) in PG$(3, q)$ with respect to planes, *Ars Combin.* **21** (1986), 119–136.

[3] S. Innamorati and F. Zuanni, On the parameters of two-intersection sets in PG$(3, q)$, *AAPP Atti della Accademia Peloritana dei Pericolanti Classe di Scienze Fisiche, Matematiche e Naturali* **96** (S2) (2018), A7(1)–A7(12). doi:10.1478/AAPP.96S2A7.

[4] V. Napolitano and D. Olanda, Sets of type $(3, h)$ in PG$(3, q)$, *Rend. Lincei Mat. Appl.* **23** (2012), 395–403. doi:10.4171/RLM/635.

[5] V. Napolitano, On sets of type $(m, h)_2$ in PG$(3, q)$ with $m \leq q$, *Note Mat.* **35** (1) (2015), 109–123. doi:10.1285/i15900932v35n1p109.

[6] V. Napolitano, Ruled sets of plane-type $(m, h)_2$ in PG$(3, q)$ with $m \leq q$, *J. Geom.* **108** (2017), 953–960.

[7] G. Tallini, Some new results on sets of type (m, n) in projective planes, *J. Geom* **29** (1987), 411–422.

(Received 7 Jan 2021; revised 17 Jan 2022)