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Constructing $(3, b)$-Sudoku pair Latin squares

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Abstract

An (a, b)-Sudoku pair Latin square is a Latin square that is simultaneously an (a, b)-Sudoku Latin square and a (b, a)-Sudoku Latin square. While (a, b)-Sudoku Latin squares are known to exist for any positive integers a and b, the pairs $\{a, b\}$ for which an (a, b)-Sudoku pair Latin square exists are largely unknown. In this article we establish the existence of (a, b)-Sudoku pair Latin squares for an infinite collection of pairs (a, b). Our results show that a $(3, b)$-Sudoku pair Latin square can be constructed for any positive integer b.

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1 Introduction

The popular Sudoku puzzle was designed by Howard Garns and first appeared in a 1975 Dell Magazine as a puzzle called "Number place". A solution to a Sudoku puzzle is a 9×9 Latin square with symbols 1 through 9 so that the 3×3 blocks (tiled in the natural fashion) also contain each symbol exactly once [6]. More generally, an (a, b)-Sudoku Latin square is a Latin square of order ab with the additional property that there is no repetition of symbols within any $a \times b$ tiling region. Quite a bit is known about (a, b)-Sudoku Latin squares: They exist for every pair (a, b) of positive integers and for many pairs (a, b) there are robust methods for constructing mutually orthogonal collections of (a, b)-Sudoku Latin squares (for example, [1, 10, 11]). We let SLS (a, b) denote an (a, b)-Sudoku Latin square.

Certain variations of Sudoku Latin squares have proven useful, such as those related to retransmission permutation arrays [5], orthogonal Sudoku array codes [9], and pooling data from multiple sources [13]. One such variation is the collection of factor pair Latin squares, which was introduced by J. Hammer and D. Hoffman [8]. A factor pair Latin square of order n is a Latin square of order n that is simultaneously an SLS (a, b) for every pair (a, b) of positive integers with $ab = n$. These designs are a specific type of multiple gerechte design [1, 12]. One may surmise from reading [8] that results on factor pair Latin squares are not easily forthcoming. For example, the values of n for which a factor pair Latin square exists are unknown, and there are infinitely many values of n for which no factor pair Latin square of order n exists (e.g., when n is a multiple of 12). However, when n is a prime power, factor pair Latin squares of order n are known to exist [8] and one can construct mutually orthogonal families of such squares [7].

In this paper we examine Sudoku pair Latin squares, which are designs obtained by relaxing the conditions required of a factor pair Latin square. For positive integers a, b we say that a Latin square of order ab is a Sudoku pair Latin square, denoted SPLS (a, b), if it is simultaneously an SLS (a, b) and an SLS (b, a). An example of an SPLS $(3, 5)$ is shown in Figure 1.

Sudoku pair Latin squares are somewhat more tractable than factor pair Latin squares. It is conjectured that an SPLS (a, b) exists for all positive integers a and b, and to prove this conjecture it is sufficient to show that an SPLS (a, b) exists for every pair of relatively prime integers a, b (see [3]). Also, it is known that an SPLS (a, b) exists when $a \in \{1, 2\}$, when $a = b$, and when ab is a prime power (see [3, 8]). However, in general, the construction of an SPLS (a, b) is unknown and the results in [8] illustrate the difficulty of finding such a construction. In this paper we establish the existence of an SPLS $(3, b)$, for $b > 0$, using a case by case analysis on the residue classes of $b \pmod 3$.

11	0	3	6	7	8	10	2	5	12	13	14	9	1	4
4	14	1	9	10	11	3	13	0	6	7	8	5	12	2
2	5	8	12	13	14	1	4	7	9	10	11	0	3	6
6	9	12	4	5	0	8	11	14	3	1	2	7	10	13
7	10	13	2	3	1	6	9	12	4	5	0	8	11	14
8	11	14	0	1	2	7	10	13	5	3	4	6	9	12
10	2	5	8	6	7	9	1	4	14	12	13	11	0	3
3	13	0	11	9	10	5	12	2	8	6	7	4	14	1
1	4	7	14	12	13	0	3	6	11	9	10	2	5	8
12	6	9	3	4	5	14	8	11	2	0	1	13	7	10
13	7	10	5	0	4	12	6	9	1	2	3	14	8	11
14	8	11	1	2	3	13	7	10	0	4	5	12	6	9
9	1	4	7	8	6	11	0	3	13	14	12	10	2	5
5	12	2	10	11	9	4	14	1	7	8	6	3	13	0
0	3	6	13	14	12	2	5	8	10	11	9	1	4	7

Figure 1: SPLS $(3,5)$

2 Constructing SPLS (a, b) where $b \equiv 0 \pmod{a}$

In this section we construct an SPLS (a, b) when b is a multiple of a. While [3] provides a product construction of SPLS $(a, \ell a)$, here we provide a direct construction. Suppose that $b = \ell a$. Let A be an $a \times b$ array whose entries are distinct. Write

$$A = \left[\begin{array}{cccc} A_0 & A_1 & \cdots & A_{\ell-1} \end{array}\right]$$

where each A_i is an $a \times a$ array.

We permute rows and columns of an $a \times a$ array using operators ρ (the down-shift operator) and χ (the right-shift operator), defined as follows: For an $a \times a$ matrix C let ρC denote the $a \times a$ matrix whose i-th row is row $i-1$ of C (modulo a) and let χC denote the $a \times a$ matrix whose i-th column is column $i-1$ of C (modulo a). Here and throughout, rows are counted top to bottom beginning with 0 while columns are counted left to right beginning with 0.

For $0 \leq i, j < a$, define the $b \times b$ array $B^{i,j}$ by

$$B^{i,j} = \left[\begin{array}{ccccc} \rho^i\chi^j A_0 & \rho^i\chi^j A_1 & \cdots & \rho^i\chi^j A_{\ell-2} & \rho^i\chi^j A_{\ell-1} \\ \rho^i\chi^j A_1 & \rho^i\chi^j A_2 & \cdots & \rho^i\chi^j A_{\ell-1} & \rho^i\chi^j A_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^i\chi^j A_{\ell-1} & \rho^i\chi^j A_0 & \cdots & \rho^i\chi^j A_{\ell-3} & \rho^i\chi^j A_{\ell-2} \end{array}\right].$$

That is, $\rho^i\chi^j A_{k+r}$ is the $a \times a$ tiling square of $B^{i,j}$ that lies in the k-th row and r-th column of such squares, where the addition in the subscript is computed modulo ℓ. It is evident that $B^{i,j}$ has no repetition in any row, column, $a \times b$ tiling region, or

$b \times a$ tiling region.[1]

These $B^{i,j}$ arrays can be cobbled together to form an SPLS (a, b). Let M denote the $ab \times ab$ matrix

$$M = \begin{bmatrix} B^{0,0} & B^{1,0} & \cdots & B^{a-1,0} \\ B^{0,1} & B^{1,1} & \cdots & B^{a-1,1} \\ \vdots & \vdots & \vdots & \vdots \\ B^{0,a-1} & B^{1,a-1} & \cdots & B^{a-1,a-1} \end{bmatrix}.$$

Note that M is a Latin square and that M has no repetition in any $a \times b$ or $b \times a$ tiling regions. Therefore we have shown:

Theorem 2.1. *An* SPLS $(a, \ell a)$ *exists for all positive integers* a *and* ℓ.

For example, suppose $a = 2$ and $b = a\ell = 4$ with

$$A = [A_0 \mid A_1] = \left[\begin{array}{cc|cc} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{array}\right].$$

Then

$$M = \begin{bmatrix} B^{0,0} & B^{1,0} \\ B^{0,1} & B^{1,1} \end{bmatrix} = \begin{array}{|cccc|cccc|} \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}$$

is an SPLS $(2, 4)$.

We remark that when a divides b, many members of SPLS (a, b) can be constructed using the regional Kronecker product $\otimes$ described in [3]. Suppose $a, b \in \mathbb{Z}^+$ with $b = \ell a$ for some $\ell \in \mathbb{Z}^+$. If L is a Latin square of order ℓ and A is an SLS (a, a), then $L \otimes A$ is an SPLS (a, b).

3 Constructing SPLS $(3, b)$ where $b \equiv 1 \pmod 3$

In this section we will construct an SPLS $(3, 3k + 1)$ whenever $k \geq 1$. We first establish terminology and notation. For $0 \leq \ell \leq 2$, an array cell location (i, j) lies in **diagonal class** ℓ if $j - i \equiv \ell \pmod 3$.

Let M be an array of order $3(3k + 1)$. The rows of M may be partitioned into groups of 3 consecutive rows; these groups are called **bands** of M. These bands are enumerated from top to bottom, with counting beginning at 0. The **stacks** of M are formed similarly by partitioning the columns into groups of three, and are enumerated from left to right. We say that a $3 \times 3(3k + 1)$ array with symbols $\{0, 1, \ldots, 3(3k + 1) - 1\}$ is an **ideal band** if the following conditions are satisfied:

[1] Note that $B^{0,0}$ can be any Latin square constructed by tiling the arrays $A_0, \ldots, A_{\ell-1}$.

(i) There is no repetition of symbols in any row, nor in any $3 \times (3k+1)$ tiling region.

(ii) Within each 3×3 tiling square A, the symbols within each diagonal class increase incrementally by 3 as one moves from top to bottom. Specifically, if α is a symbol in A lying in row $0 \leq i < 2$ and diagonal class ℓ, then $\alpha + 3 \pmod{3(3k+1)}$ is the symbol in A lying in row $i+1$ and diagonal class ℓ.

(iii) Every residue class modulo 3 is represented within each 3×3 tiling square.

Finally, if B is a $3 \times 3(3k+1)$ array with symbols $\{0, 1, \ldots 3(3k+1)-1\}$ and ℓ is an integer, then the array $B + \ell$ is formed by adding ℓ to each entry of B, modulo $3(3k+1)$.

An example of an ideal band, in the case $k = 2$, is shown in Figure 2. In the figure, the single gridlines indicate a partition into 3×3 tiling squares, whereas the double gridlines indicate the $3 \times (3 \cdot 2 + 1)$ tiling regions.

0	1	2	9	10	11	18	13	17	6	4	5	15	14	16	3	8	7	12	20	19
5	3	4	14	12	13	20	0	16	8	9	7	19	18	17	10	6	11	1	15	2
7	8	6	16	17	15	19	2	3	10	11	12	20	1	0	14	13	9	5	4	18

Figure 2: A 3×21 ideal band.

Lemma 3.1. *If B is an ideal band and A is a 3×3 tiling square in B, then each residue class modulo 3 is represented exactly once in each row and column of A.*

Proof. This follows immediately from Property (ii) and Property (iii) of ideal bands. □

Lemma 3.2. *If B is an ideal band of size $3 \times 3(3k+1)$ and M is the array of order $3(3k+1)$ whose i-th band is $B + 9i$ for $0 \leq i < 3k+1$, then M is an* SPLS$(3, 3k+1)$.

Proof. Property (i) of B together with the construction of M guarantee that M has no repetition in rows, nor in $3 \times (3k+1)$ tiling regions. It remains to show that there is no repetition in the columns of M, nor in the $(3k+1) \times 3$ tiling regions.

Let $[i, j]$ denote the symbol lying in location (i, j) of M. Suppose $[i, \ell]$ and $[j, \ell]$ are symbols in the ℓ-th column of M with $[i, \ell] = [j, \ell]$. Lemma 3.1 says that there is a single symbol $[r, \ell] \in B$ with $[i, \ell] \equiv [j, \ell] \equiv [r, \ell] \pmod 3$. Therefore, the construction of M tells us that $i = r + 3s$, $j = r + 3t$, $[i, \ell] = [r, \ell] + 9s$, and $[j, \ell] = [r, \ell] + 9t$ for some $0 \leq s, t < 3k+1$. The latter two equations imply $9(t-s) \equiv 0 \pmod{3(3k+1)}$, which in turn says $9(t-s) \equiv 0 \pmod{3k+1}$. Hence $t - s \equiv 0 \pmod{3k+1}$ because 3 and $3k+1$ are relatively prime. However, $0 \leq s, t < 3k+1$, so $s = t$ which implies $i = j$. Thus we can conclude that there is no repetition in the columns of M.

Let R be any $(3k+1)\times 3$ region contained within a stack of M, and let $\alpha_0, \alpha_1, \alpha_2$ be the entries in the top row of R. It follows from property (ii) of B and the construction of M that $\alpha_i + 3j \in R$ for $0 \le i < 3$ and $0 \le j < 3k+1$. Now, suppose that $\alpha_i + 3j = \alpha_r + 3s$ for some $0 \le i, r < 3$ and $0 \le j, s < 3k+1$. Then $\alpha_i - \alpha_r = 3(r-j)$, so α_i and α_r lie in the same residue class modulo 3. But Lemma 3.1 together with the construction of M implies that $\alpha_0, \alpha_1, \alpha_2$ lie in distinct residue classes modulo 3. Therefore $i = r$, and hence $3(r-j) \equiv 0 \pmod{3(3k+1)}$. Arguing as in the previous paragraph, we conclude that $r = j$. Therefore there are $3(3k+1)$ distinct symbols lying in R, and this exhausts the set of locations in R. Therefore there is no repetition of symbols in R. $\square$

Note that any SPLS $(3, 3k+1)$ formed via Lemma 3.2 has the stronger property that there is non-repetition in *any* $(3k+1)\times 3$ region lying within a stack, not just in $(3k+1)\times 3$ tiling regions.

Using Lemma 3.2 requires us to construct ideal bands. To aid in this task, we employ auxiliary arrays called *pre-bands*. We describe these presently. Given a $3 \times (3k+1)$ array P and $\ell \in \{0,1,2\}$, a **partition of type** ℓ splits P into three disjoint regions, each with $3k+1$ locations, as shown in Figure 3, with types 0, 1, and 2 being shown top to bottom, respectively.

Let $k \ge 1$ and let P be a $3\times(3k+1)$ Latin rectangle with symbols $\{0,1,\dots,3k\}$. For $\ell \in \{0,1,2\}$ we say that P is a **pre-band of type** ℓ if it satisfies the following properties:

(i) Each column of P consists of consecutive symbols, ordered from top to bottom, with 0 succeeding $3k$.

(ii) In each of the three regions of P given by a partition of type ℓ, every symbol appears exactly once.

Lemma 3.3. *Let k be a positive integer and $\ell \in \{0,1,2\}$. A $3\times(3k+1)$ pre-band of type ℓ exists.*

Proof. Examples of pre-bands of types 0, 1, and 2, from top to bottom, respectively, are shown in Figure 4. In the figure, an ellipsis represents incremental increase by 3 from cell to cell as one moves left to right. Figure 4 presents some uncertainty about what the pre-bands look like when $k = 1$. These are shown in Figure 5. $\square$

Lemma 3.4. *Let $k \ge 1$. Given three $3\times(3k+1)$ pre-bands, one of each type, it is possible to construct a $3\times 3(3k+1)$ ideal band.*

Proof. Let P_0, P_1, P_2 be three $3\times(3k+1)$ pre-bands of types 0, 1, and 2, respectively. Consider the arrays $Q_\ell = 3P_\ell + \ell$ $(0 \le \ell \le 2)$, the entries of which are obtained by multiplying each entry of P_ℓ by 3 and adding ℓ. Each Q_ℓ inherits a partition of type ℓ from P_ℓ. Examples of Q_0, Q_1, Q_2 are shown in Figure 6 in the case $k = 2$ using the pre-bands from the proof of Lemma 3.3.

$0,0$	$0,1$	$\cdots$	$0,k-1$	$0,k$	$0,k+1$	$\cdots$	$0,2k-1$	$0,2k$	$0,2k+1$	$\cdots$	$0,3k$
$1,0$	$1,1$	$\cdots$	$1,k-1$	$1,k$	$1,k+1$	$\cdots$	$1,2k-1$	$1,2k$	$1,2k+1$	$\cdots$	$1,3k$
$2,0$	$2,1$	$\cdots$	$2,k-1$	$2,k$	$2,k+1$	$\cdots$	$2,2k-1$	$2,2k$	$2,2k+1$	$\cdots$	$2,3k$

$0,0$	$0,1$	$\cdots$	$0,k-1$	$0,k$	$0,k+1$	$\cdots$	$0,2k-1$	$0,2k$	$0,2k+1$	$\cdots$	$0,3k$
$1,0$	$1,1$	$\cdots$	$1,k-1$	$1,k$	$1,k+1$	$\cdots$	$1,2k-1$	$1,2k$	$1,2k+1$	$\cdots$	$1,3k$
$2,0$	$2,1$	$\cdots$	$2,k-1$	$2,k$	$2,k+1$	$\cdots$	$2,2k-1$	$2,2k$	$2,2k+1$	$\cdots$	$2,3k$

$0,0$	$0,1$	$\cdots$	$0,k-1$	$0,k$	$0,k+1$	$\cdots$	$0,2k-1$	$0,2k$	$0,2k+1$	$\cdots$	$0,3k$
$1,0$	$1,1$	$\cdots$	$1,k-1$	$1,k$	$1,k+1$	$\cdots$	$1,2k-1$	$1,2k$	$1,2k+1$	$\cdots$	$1,3k$
$2,0$	$2,1$	$\cdots$	$2,k-1$	$2,k$	$2,k+1$	$\cdots$	$2,2k-1$	$2,2k$	$2,2k+1$	$\cdots$	$2,3k$

Figure 3: Partitions of types 0, 1, and 2, top to bottom, respectively. Entries indicate cell locations.

Let $A_0, A_1, \ldots, A_{3k}$ denote the 3×3 tiling squares of our proposed $3 \times 3(3k+1)$ ideal band B, listed in order from left to right. We fill the arrays A_j $(0 \leq j \leq 3k)$ using Q_0, Q_1, Q_2 as follows:

(i) For $0 \leq j \leq 2k-1$ the entries of the j-th column of Q_ℓ, from top to bottom, are placed in the ℓ-th diagonal class of A_j, from top to bottom.

(ii) For $2k \leq j \leq 3k$, the entries of the j-th column of Q_ℓ, from top to bottom, are placed in the $(3-\ell)$-th diagonal class of A_j (modulo 3), from top to bottom.

Using Q_0, Q_1, Q_2 as in Figure 6, this method produces the ideal band B shown in Figure 2.

It is evident that this method of filling B accounts for all of the diagonal classes of the A_j, and hence all the available locations in B. Also, from Property (i) of pre-bands it follows that entries in any column of Q_ℓ increase incrementally by 3 from top to bottom, and hence B satisfies Property (ii) of ideal bands. Because each A_j contains entries from every Q_ℓ, we know that B satisfies Property (iii) of ideal

0	3	$\cdots$	$3k-3$	$3k$	2	$\cdots$	$3k-4$	$3k-1$	1	$\cdots$	$3k-2$
1	4	$\cdots$	$3k-2$	0	3	$\cdots$	$3k-3$	$3k$	2	$\cdots$	$3k-1$
2	5	$\cdots$	$3k-1$	1	4	$\cdots$	$3k-2$	0	3	$\cdots$	$3k$

0	3	$\cdots$	$3k-3$	$3k-2$	1	$\cdots$	$3k-5$	$3k-1$	2	$\cdots$	$3k-4$	$3k$
1	4	$\cdots$	$3k-2$	$3k-1$	2	$\cdots$	$3k-4$	$3k$	3	$\cdots$	$3k-3$	0
2	5	$\cdots$	$3k-1$	$3k$	3	$\cdots$	$3k-3$	0	4	$\cdots$	$3k-2$	1

0	3	$\cdots$	$3k-3$	$3k-1$	1	$\cdots$	$3k-5$	$3k-2$	2	$\cdots$	$3k-4$	$3k$
1	4	$\cdots$	$3k-2$	$3k$	2	$\cdots$	$3k-4$	$3k-1$	3	$\cdots$	$3k-3$	0
2	5	$\cdots$	$3k-1$	0	3	$\cdots$	$3k-3$	$3k$	4	$\cdots$	$3k-2$	1

Figure 4: Pre-bands of types 0, 1, and 2, from top to bottom.

0	3	2	1
1	0	3	2
2	1	0	3

0	1	2	3
1	2	3	0
2	3	0	1

0	2	1	3
1	3	2	0
2	0	3	1

Figure 5: The case $k = 1$. Pre-bands of type 0, 1, and 2, from left to right, respectively.

bands. Further, because each of $0, 1, \ldots, 3k$ appears exactly once in each row of a pre-band (due to its Latin rectangle property), the entries in the i-th row of all the Q_ℓ, taken collectively, account for each member of $\{0, 1, \ldots, 3(3k+1)-1\}$ exactly once. Because these entries comprise the i-th row of B, there is no repetition within rows of B.

To show that B is an ideal band, it remains to show that each symbol in B appears exactly once in each $3 \times (3k+1)$ tiling region. We begin by examining the left $3 \times (3k+1)$ tiling region of B. This tiling region consists of $A_0, \ldots, A_{k-1}$ together with the left column of A_k. Property (ii) of pre-bands together with the method of assigning entries of the Q_ℓ $(0 \leq \ell \leq 2)$ to B implies that it suffices to show that each entry in this left tiling region originates from the left region of some Q_ℓ (determined by the type-ℓ partition of Q_ℓ). For $0 \leq j \leq k-1$, an entry of A_j originates from the the j-th column of some Q_ℓ, which lies within the left region of Q_ℓ. Now, let α, β, γ denote the entries in the left column of A_k, from top to bottom. Note that the top

0	3	6	2	5	1	4
1	4	0	3	6	2	5
2	5	1	4	0	3	6

0	3	4	1	5	2	6
1	4	5	2	6	3	0
2	5	6	3	0	4	1

0	3	5	1	4	2	6
1	4	6	2	5	3	0
2	5	0	3	6	4	1

0	9	18	6	15	3	12
3	12	0	9	18	6	15
6	15	3	12	0	9	18

1	10	13	4	16	7	19
4	13	16	7	19	10	1
7	16	19	10	1	13	4

2	11	17	5	14	8	20
5	14	20	8	17	11	2
8	17	2	11	20	14	5

Figure 6: Pre-bands P_0, P_1, and P_2 top left to right, and arrays Q_0, Q_1, Q_2 bottom left to right.

entry α lies in diagonal class 0 of A_k, and hence it originates from the top entry of the k-th column of Q_0, which lies in the left region of Q_0. Meanwhile, middle entry β lies in diagonal class 2 of A_k, and hence it originates from the middle entry of the k-th column of Q_2, which lies in the left region of Q_2. Finally, the bottom entry γ lies in diagonal class 1 of A_k, and hence originates from the bottom entry of the k-th column of Q_1, which lies in the left region of Q_1. We conclude that all entries in the left $3 \times (3k+1)$ tiling region of B originate from the left region of some Q_ℓ. Therefore the left $3 \times (3k+1)$ tiling region of B contains each symbol of B exactly once. A similar argument involving $A_{2k+1}, \ldots, A_{3k}$ together with the right column of A_{2k} shows that the right $3 \times (3k+1)$ tiling region of B contains each symbol of B exactly once. Finally, because B has no repetition in rows, we know each symbol of B appears exactly three times in B. This, together with our previous work, shows that the central $3 \times (3k+1)$ tiling region of B must contain each symbol exactly once. □

From Lemmas 3.2, 3.3, and 3.4, we have:

Theorem 3.5. *An* SPLS $(3, 3k+1)$ *exists for all non-negative integers* k.

Theorem 3.5 is illustrated by the SPLS $(3, 4)$ which is given in Figure 7, using the pre-bands shown in Figure 5. Also, note that Theorem 3.5 holds trivially when $k = 0$ because an SPLS $(3, 1)$ is simply a Latin square of order 3.

4 Constructing SPLS $(3, b)$ where $b \equiv 2 \pmod 3$

In this section we show that an SPLS $(3, 3k+2)$ can be constructed for any positive integer k. We reprise the following notation from Section 2, where C is an $n \times n$ array: Arrays ρC and χC are obtained from C by cycling the rows and columns of C downward and rightward, respectively. Specifically, the i-th row of ρC is row $i-1$ of C (modulo n) and likewise for columns of χC and C. We will let $C^{i,j}$ denote $\rho^i \chi^j C$. Also, we let C^t denote the transpose of C.

0	1	2	9	4	8	6	5	7	3	11	10
5	3	4	11	0	7	10	9	8	1	6	2
7	8	6	10	2	3	11	1	0	5	4	9
9	10	11	6	1	5	3	2	4	0	8	7
2	0	1	8	9	4	7	6	5	10	3	11
4	5	3	7	11	0	8	10	9	2	1	6
6	7	8	3	10	2	0	11	1	9	5	4
11	9	10	5	6	1	4	3	2	7	0	8
1	2	0	4	8	9	5	7	6	11	10	3
3	4	5	0	7	11	9	8	10	6	2	1
8	6	7	2	3	10	1	0	11	4	9	5
10	11	9	1	5	6	2	4	3	8	7	0

Figure 7: An SPLS $(3, 4)$

Proposition 4.1. [2] *There is an idempotent quasigroup (i.e., Latin square) of order n for each $n > 2$.*

Lemma 4.2. *Suppose that A is an $n \times n$ array whose n^2 entries are distinct.*

(i) If M is the $n^2 \times n^2$ array whose (i, j)-th $n \times n$ tiling region is $A^{j,i}$ for $0 \le i, j < n$, then M is an SLS (n, n).

(ii) If N is the $n^2 \times n^2$ array whose (i, j)-th $n \times n$ tiling region is $A^{i+j,i+j}$ for $0 \le i, j < n$, then N is an SLS (n, n).

The main goal of this section is to prove the following theorem:

Theorem 4.3. *There is an* SPLS $(3, 3k + 2)$ *for all non-negative integers k.*

Example 4.4. Figure 1 shows an SPLS $(3, 3k + 2)$ when $k = 1$, but here we present and justify its construction, which will be useful when we address the general construction of an SPLS $(3, 3k + 2)$ in the proof of Theorem 4.3. The following arrays will be employed in our construction: Begin with the SPLS $(3, 2)$

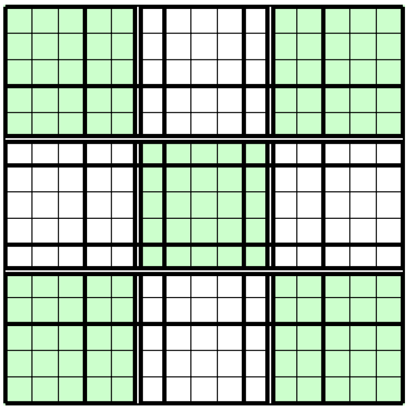
$$A = \left[\begin{array}{cc|c||c|cc} 4 & 5 & 0 & 3 & 1 & 2 \\ 2 & 3 & 1 & 4 & 5 & 0 \\ \hline 0 & 1 & 2 & 5 & 3 & 4 \\ \hline\hline 3 & 4 & 5 & 2 & 0 & 1 \\ \hline 5 & 0 & 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 & 4 & 5 \end{array}\right] = \left[\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array}\right] \tag{1}$$

where A_1, A_2, A_3, A_4 are the 3×3 quadrants of A. Also let

$$B = \begin{bmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} & 0 & 3 \\ 4 & & 1 \\ 2 & 5 & \end{bmatrix}.$$

To avoid a profusion of superscripts, we let β denote B^t.

Note that an SPLS $(3, 5)$ grid has the form

$G =$  (2)

where the single and double grid lines partition the rows/columns into groups of size 3 and 5, respectively. The horizontal single grid lines and vertical double gridlines indicate a partition into 3×5 regions, while horizontal double grid lines and vertical single grid lines indicate a partition into 5×3 regions. The single grid lines partition the grid G into 3×3 tiling regions G_{ij} $(0 \leq i, j < 5)$, where each G_{ij} is classified into three types:

- Type I regions lie wholly within both a 5×3 tiling region and a 3×5 tiling region.
- Type II regions lie wholly within a 5×3 tiling region but not within a 3×5 tiling region, or vice versa.
- Type III regions lie neither within a 5×3 tiling region nor within a 3×5 tiling region.

The 5×5 subgrids in (2) have been colored to help emphasize the various types of regions $G_{i,j}$.

Observe that G_{ij} is of Type I when i, j are even, G_{ij} is of Type II when exactly one of i, j is odd, and G_{ij} is of Type III when i, j are odd. The following array pairs each 3×3 region $G_{i,j}$ with its type:

$$\begin{array}{lllll} (G_{00}, \mathrm{I}) & (G_{01}, \mathrm{II}) & (G_{02}, \mathrm{I}) & (G_{03}, \mathrm{II}) & (G_{04}, \mathrm{I}) \\ (G_{10}, \mathrm{II}) & (G_{11}, \mathrm{III}) & (G_{12}, \mathrm{II}) & (G_{13}, \mathrm{III}) & (G_{14}, \mathrm{II}) \\ (G_{20}, \mathrm{I}) & (G_{21}, \mathrm{II}) & (G_{22}, \mathrm{I}) & (G_{23}, \mathrm{II}) & (G_{24}, \mathrm{I}) \\ (G_{30}, \mathrm{II}) & (G_{31}, \mathrm{III}) & (G_{32}, \mathrm{II}) & (G_{33}, \mathrm{III}) & (G_{34}, \mathrm{II}) \\ (G_{40}, \mathrm{I}) & (G_{41}, \mathrm{II}) & (G_{42}, \mathrm{I}) & (G_{43}, \mathrm{II}) & (G_{44}, \mathrm{I}). \end{array}$$

Using symbols in the set $\{0, 1, \ldots, 14\}$, we fill the cells of G in a multi-step process, first filling regions of Type III, then those of Type II, and finally those of Type I. As we complete each step we check that the SPLS $(3, 5)$ conditions have not been violated (i.e., no repetitions in rows, columns, 5×3 tiling regions, and 3×5 tiling regions).

First, we insert A_1 in region G_{11}, A_2 in region G_{13}, A_3 in region G_{31}, and A_4 in region G_{33}, as shown below:

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & A_1 & & A_2 & \\ \hline & & & & \\ \hline & A_3 & & A_4 & \\ \hline & & & & \\ \hline \end{array} . \tag{3}$$

Observe that the bottom row of A_1 and the top row of A_3 lie in the same 5×3 tiling region, and similarly for A_2 and A_4. Meanwhile, the right column of A_1 and the left column of A_2 lie in the same 3×5 tiling region, and similarly for A_3 and A_4.

Because A is a Latin square, the partially filled array in (3) has no repetition in rows and columns and further, because A is an SPLS $(3, 2)$, we see from (2) that the partially filled array has no repetition in 3×5 or 5×3 tiling regions. This accounts for all of the Type III regions.

Moving to regions of Type II, we insert $B^{j,i}$ into region $G_{2i,2j+1}$ for $0 \le i \le 2$ and $0 \le j \le 1$, as shown below:

$$\begin{array}{|c|c|c|c|c|}\hline \phantom{B^{0,0}} & B^{0,0} & & B^{1,0} & \\ \hline & A_1 & & A_2 & \\ \hline & B^{0,1} & & B^{1,1} & \\ \hline & A_3 & & A_4 & \\ \hline & B^{0,2} & & B^{1,2} & \\ \hline \end{array} . \tag{4}$$

Part (i) of Lemma 4.2, together with the fact that the set of symbols in B is disjoint from those in A, imply that the array (4) has no repetitions in rows and columns. Observe that distinct regions $G_{2i,2j+1}$ and $G_{2k,2\ell+1}$ never meet the same 5×3 tiling region, so this, together with the fact that the symbols of A and the symbols of B are disjoint, indicates that the array (4) has no repetitions within any 5×3 tiling region. Regarding 3×5 tiling regions, upon examining the grid G in (2) we see that the right column of $B^{0,i}$ and the left column of $B^{1,i}$ will make contributions to the same 3×5 tiling region $(0 \le i \le 2)$, but because the symbols in the right column of $B^{0,i}$ are distinct from those in the left column of $B^{1,i}$, this does not induce any repetition in the affected 3×5 tiling regions. We conclude that the array (4) has no repetitions in any 3×5 tiling region. Similarly, we may insert $\beta^{j,i}$ in region $G_{2i+1,2j}$ for $0 \le i \le 1$ and $0 \le j \le 2$ without inducing repetitions in any row, column, 3×5 tiling region or 5×3 tiling region. This accounts for all of the Type II regions, giving the following array:

$$\begin{array}{|c|c|c|c|c|}\hline & B^{0,0} & & B^{1,0} & \\ \hline \beta^{0,0} & A_1 & \beta^{1,0} & A_2 & \beta^{2,0} \\ \hline & B^{0,1} & & B^{1,1} & \\ \hline \beta^{0,1} & A_3 & \beta^{1,1} & A_4 & \beta^{1,2} \\ \hline & B^{0,2} & & B^{1,2} & \\ \hline \end{array}\,.$$

Regarding regions of Type I, we insert $C^{i+j,i+j}$ in region $G_{2i,2j}$ for $0 \le i, j \le 2$, as shown below:

$$\begin{array}{|c|c|c|c|c|}\hline C^{0,0} & B^{0,0} & C^{1,1} & B^{1,0} & C^{2,2} \\ \hline \beta^{0,0} & A_1 & \beta^{1,0} & A_2 & \beta^{2,0} \\ \hline C^{1,1} & B^{0,1} & C^{2,2} & B^{1,1} & C^{0,0} \\ \hline \beta^{0,1} & A_3 & \beta^{1,1} & A_4 & \beta^{1,2} \\ \hline C^{2,2} & B^{0,2} & C^{0,0} & B^{1,2} & C^{1,1} \\ \hline \end{array}\,. \tag{5}$$

Due to part (ii) of Lemma 4.2 and the fact that the symbols of C are disjoint from those of B, we see that the array (5) has no repetition in any row or column. Further, upon examining the grid G in (2), we see that no $C^{i,i}$ intersects any 3×5 or 5×3 tiling region that also intersects one of A_1, A_2, A_3, A_4. This, together with the fact that the symbols of C are disjoint from those of B, tells us that there is no repetition of symbols in any 3×5 or 5×3 tiling region in (5).

It remains to fill the diagonals of each $C^{i,i}$ appearing in (5). An examination of (2) and (5) tells us that to avoid repetition in 3×5 and 5×3 regions, the diagonal entries of $C^{i,i}$ $(0 \le i \le 2)$ must be filled by the symbols in the $(2-i)$-th column of B. Further, there is exactly one way to assign these entries so as to avoid repetition in rows and columns, as shown below:

$$C^{0,0} \longrightarrow \begin{bmatrix} 11 & 0 & 3 \\ 4 & 14 & 1 \\ 2 & 5 & 8 \end{bmatrix} \quad C^{1,1} \longrightarrow \begin{bmatrix} 10 & 2 & 5 \\ 3 & 13 & 0 \\ 1 & 4 & 7 \end{bmatrix} \quad C^{2,2} \longrightarrow \begin{bmatrix} 9 & 1 & 4 \\ 5 & 12 & 2 \\ 0 & 3 & 6 \end{bmatrix}.$$

Upon assigning these diagonal entries in $C^{i,i}$ $(0 \le i \le 2)$ we obtain the SPLS $(3, 5)$ shown in Figure 1.

This example will find application in the construction of an SPLS $(3, 3k+2)$ when $k > 2$. Further, we have provided a step-by-step illustration of the construction of an SPLS $(3, 11)$, displayed in the figures of Appendix A, to coincide with the construction given in the proof that follows.

Proof of Theorem 4.3. Equation (1), Figure 1, and Figure 14 (Appendix B) establish the result for $k \le 2$, so we assume $k \ge 3$ for our present construction.[2] There are several ingredients that must be assembled prior to the construction. First, let $G = (G_{ij})$ denote the empty SPLS $(3, 3k+2)$ grid, partitioned into 3×3 regions (G_{ij}) $(0 \le i, j < 3k+2)$. Each G_{ij} is one of three types:

- Type I regions lie wholly within both a $(3k+2) \times 3$ tiling region and a $3 \times (3k+2)$ tiling region.
- Type II regions lie wholly within a $(3k+2) \times 3$ tiling region but not within a $3 \times (3k+2)$ tiling region, or vice versa.
- Type III regions lie neither within a $(3k+2) \times 3$ tiling region nor within a $3 \times (3k+2)$ tiling region.

Observe that G_{ij} is of Type I when neither i nor j lies in $\{k, 2k+1\}$, that G_{ij} is of Type II when exactly one of i, j lies in $\{k, 2k+1\}$, and that G_{ij} is of Type III when i, j both lie in $\{k, 2k+1\}$.

We endeavor to fill the G_{ij} with various 3×3 arrays, which are described as follows: Let A and C be as in Example 4.4. For $0 \le \ell < k$ let B_ℓ be the 3×3 array whose (i, j)-th entry is $6 + 9\ell + 3i + j$ for $0 \le i, j \le 2$. Observe that the entries in any B_ℓ are distinct, that the entries in B_i are disjoint from those in B_j whenever $i \ne j$, and that $\{0, 1, \ldots, 3(3k+2) - 1\}$ is the disjoint union of $\{0, 1, 2, 3, 4, 5\}$ (the symbols in A_i $(1 \le i \le 4)$ and C) with the set of symbols that lie in some B_ℓ $(0 \le \ell < k)$. We let β_ℓ denote B_ℓ^t. Let C_ℓ^i (with $0 \le \ell < k$ and $0 \le i \le 2$) denote $C^{i,i}$ with diagonal entries filled with the symbols in the $(2-i)$-th column of B_ℓ, as follows:

$$
\begin{aligned}
C_\ell^0 &= \begin{bmatrix} 6+9\ell+5 & 0 & 3 \\ 4 & 6+9\ell+8 & 1 \\ 2 & 5 & 6+9\ell+2 \end{bmatrix} \\
C_\ell^1 &= \begin{bmatrix} 6+9\ell+4 & 2 & 5 \\ 3 & 6+9\ell+7 & 0 \\ 1 & 4 & 6+9\ell+1 \end{bmatrix} \\
C_\ell^2 &= \begin{bmatrix} 6+9\ell+3 & 1 & 4 \\ 5 & 6+9\ell+6 & 2 \\ 0 & 3 & 6+9\ell \end{bmatrix}.
\end{aligned} \tag{6}
$$

[2] Our strategy for constructing an SPLS $(3, 3k+2)$ does not apply in the case $k = 2$, because no idempotent quasigroup of order 2 exists. The SPLS $(3, 8)$ given in Figure 14 was obtained using a stochastic backtracking computer program.

Finally, by Proposition 4.1 there is an idempotent Latin square $D = (D_{i,j})$ $(0 \le i, j < k)$ of order k with symbol set $\{B_0, B_1, \ldots, B_{k-1}\}$ satisfying $D_{i,i} = B_i$ for $0 \le i < k$.

We begin filling the grid G in the following manner: Place A_1 in $G_{k,k}$, A_2 in $G_{k,2k+1}$, A_3 in $G_{2k+1,k}$, and A_4 in $G_{2k+1,2k+1}$. Place $B_\ell^{i,j}$ in $G_{j(k+1)+\ell,(i+1)(k+1)-1}$ for $0 \le i \le 1$, $0 \le j \le 2$, and $0 \le \ell \le k-1$. Likewise, place $\beta_\ell^{i,j}$ in $G_{(j+1)(k+1)-1,i(k+1)+\ell}$ for $0 \le i \le 2$, $0 \le j \le 1$, and $0 \le \ell \le k-1$. Then place C_ℓ^i in $G_{(i+j)(k+1)+\ell,[-(i+j)](k+1)+\ell}$ for $0 \le i, j \le 2$ and $0 \le \ell \le k-1$, where $[-(i+j)]$ in the subscript is to be reduced modulo 3. These placements account for all of the regions of Type II and Type III, and some of the Type I regions. This is illustrated in Figure 8.

C_0^0				$B_0^{0,0}$	C_0^1				$B_0^{1,0}$	C_0^2			
	C_1^0			$B_1^{0,0}$		C_1^1			$B_1^{1,0}$		C_1^2		
		$\ddots$		$\vdots$			$\ddots$		$\vdots$			$\ddots$	
			C_{k-1}^0	$B_{k-1}^{0,0}$				C_{k-1}^1	$B_{k-1}^{1,0}$				C_{k-1}^2
$\beta_0^{0,0}$	$\beta_1^{0,0}$	$\cdots$	$\beta_{k-1}^{0,0}$	A_1	$\beta_0^{1,0}$	$\beta_1^{1,0}$	$\cdots$	$\beta_{k-1}^{1,0}$	A_2	$\beta_0^{2,0}$	$\beta_1^{2,0}$	$\cdots$	$\beta_{k-1}^{2,0}$
C_0^1				$B_0^{0,1}$	C_0^2				$B_0^{1,1}$	C_0^0			
	C_1^1			$B_1^{0,1}$		C_1^2			$B_1^{1,1}$		C_1^0		
		$\ddots$		$\vdots$			$\ddots$		$\vdots$			$\ddots$	
			C_{k-1}^1	$B_{k-1}^{0,1}$				C_{k-1}^2	$B_{k-1}^{1,1}$				C_{k-1}^0
$\beta_0^{0,1}$	$\beta_1^{0,1}$	$\cdots$	$\beta_{k-1}^{0,1}$	A_3	$\beta_0^{1,1}$	$\beta_1^{1,1}$	$\cdots$	$\beta_{k-1}^{1,1}$	A_4	$\beta_0^{2,1}$	$\beta_1^{2,1}$	$\cdots$	$\beta_{k-1}^{2,1}$
C_0^2				$B_0^{0,2}$	C_0^0				$B_0^{1,2}$	C_0^1			
	C_1^2			$B_1^{0,2}$		C_1^0			$B_1^{1,2}$		C_1^1		
		$\ddots$		$\vdots$			$\ddots$		$\vdots$			$\ddots$	
			C_{k-1}^2	$B_{k-1}^{0,2}$				C_{k-1}^0	$B_{k-1}^{1,2}$				C_{k-1}^1

Figure 8: Partially filled SPLS $(3, 3k+2)$

We claim that the resulting partially filled array shown in Figure 8 does not violate any of the SPLS $(3, 3k+2)$ conditions (i.e., no repetitions in rows, columns, $3 \times (3k+2)$ tiling regions, and $(3k+2) \times 3$ tiling regions). The subarray of our current completion of G (shown in Figure 8) in cells $G_{i,j}$ with $i, j \in \{\ell, k, k+\ell+$

$1, 2k+1, 2k+2+\ell\}$ yields the following array[3]:

$$\begin{array}{|c|c|c|c|c|}\hline C_\ell^0 & B_\ell^{0,0} & C_\ell^1 & B_\ell^{1,0} & C_\ell^2 \\ \hline \beta_\ell^{0,0} & A_1 & \beta_\ell^{1,0} & A_2 & \beta_\ell^{2,0} \\ \hline C_\ell^1 & B_\ell^{0,1} & C_\ell^2 & B_\ell^{1,1} & C_\ell^0 \\ \hline \beta_\ell^{0,1} & A_3 & \beta_\ell^{1,1} & A_4 & \beta_\ell^{2,1} \\ \hline C_\ell^2 & B_\ell^{0,2} & C_\ell^0 & B_\ell^{1,2} & C_\ell^1 \\ \hline \end{array}. \tag{7}$$

Due to the construction in Example 4.4, the array (7) is an SPLS $(3, 5)$ on symbols $\{0, 1, \ldots, 5, 6+9\ell, \ldots, 6+9\ell+8\}$. Putting this together with the grid G, the arrangement of the array in Figure 8, and the fact that the set of symbols in the B_i is disjoint from the set of symbols in B_j whenever $i \neq j$, it follows that the array in Figure 8 does not violate any of the SPLS $(3, 3k+2)$ conditions.

We now fill the remaining Type I regions. We begin by filling region $G_{i,j}$ with $D_{i,j}$ whenever $i \neq j$ and $0 \le i, j \le k-1$. The upper-left portion of array in Figure 8 becomes:

$$\begin{array}{ccccc|c} C_0^0 & D_{0,1} & \cdots & D_{0,k-2} & D_{0,k-1} & B_0^{0,0} \\ D_{1,0} & C_1^0 & \cdots & D_{1,k-2} & D_{1,k-1} & B_1^{0,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D_{k-2,0} & D_{k-2,1} & \cdots & C_{k-2}^0 & D_{k-2,k-1} & B_{k-2}^{0,0} \\ D_{k-1,0} & D_{k-1,1} & \cdots & D_{k-1,k-2} & C_{k-1}^0 & B_{k-1}^{0,0} \\ \hline \beta_0^{0,0} & \beta_1^{0,0} & \cdots & \beta_{k-2}^{0,0} & \beta_{k-1}^{0,0} & A_1 \end{array}. \tag{8}$$

Because the Latin square $(D_{i,j})$ is idempotent on $\{B_0, \ldots, B_{k-1}\}$, no symbols of B_ℓ appear in any member of $\{D_{\ell,0}, D_{\ell,1}, \ldots, D_{\ell,\ell-1}, D_{\ell,\ell+1}, \ldots, D_{\ell,k-1}\}$, and likewise no symbols of β_ℓ appear in any member of $\{D_{0,\ell}, D_{1,\ell}, \ldots, D_{\ell-1,\ell}, D_{\ell+1,\ell}, \ldots, D_{,k-1,\ell}\}$, for $0 \le \ell \le k-1$. As a direct consequence, the insertion of these $D_{i,j}$ arrays in Figure 8 does not induce repetition in any row, column, $(3k+2) \times 3$ tiling region, or $3 \times (3k+2)$ tiling region.

For $0 \le i, j \le k-1$ with $i \neq j$, we place $D_{i,j}^{r,s}$ in region $G_{i+s(k+1), j+r(k+1)}$ for $0 \le r, s \le 2$, giving the array in Figure 9. This accounts for all of the remaining Type I regions. Appealing to the argument in the previous paragraph, there is no repetition within any $(3k+2) \times 3$ tiling region, or $3 \times (3k+2)$ tiling region. Further, by Part (i) of Lemma 4.2 there is no repetition in any row or column of Figure 9. We conclude that the array in Figure 9 is an SPLS $(3, 3k+2)$. □

[3]Informally, the array in (7) is the set of all filled cells $G_{i,j}$ that are ℓ units from the edge of the array in Figure 8.

C_0^0	$D_{0,1}^{0,0}$	$\cdots$	$D_{0,k-1}^{0,0}$	$B_0^{0,0}$	C_0^1	$D_{0,1}^{1,0}$	$\cdots$	$D_{0,k-1}^{1,0}$	$B_0^{1,0}$	C_0^2	$D_{0,1}^{2,0}$	$\cdots$	$D_{0,k-1}^{2,0}$
$D_{1,0}^{0,0}$	C_1^0	$\cdots$	$D_{1,k-1}^{0,0}$	$B_1^{0,0}$	$D_{1,0}^{1,0}$	C_1^1	$\cdots$	$D_{1,k-1}^{1,0}$	$B_1^{1,0}$	$D_{1,0}^{2,0}$	C_1^2	$\cdots$	$D_{1,k-1}^{2,0}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$D_{k-1,0}^{0,0}$	$D_{k-1,1}^{0,0}$	$\cdots$	C_{k-1}^0	$B_{k-1}^{0,0}$	$D_{k-1,0}^{1,0}$	$D_{k-1,1}^{1,0}$	$\cdots$	C_{k-1}^1	$B_{k-1}^{1,0}$	$D_{k-1,0}^{2,0}$	$D_{k-1,1}^{2,0}$	$\cdots$	C_{k-1}^2
$\beta_0^{0,0}$	$\beta_1^{0,0}$	$\cdots$	$\beta_{k-1}^{0,0}$	A_1	$\beta_0^{1,0}$	$\beta_1^{1,0}$	$\cdots$	$\beta_{k-1}^{1,0}$	A_2	$\beta_0^{2,0}$	$\beta_1^{2,0}$	$\cdots$	$\beta_{k-1}^{2,0}$
C_0^1	$D_{0,1}^{0,1}$	$\cdots$	$D_{0,k-1}^{0,1}$	$B_0^{0,1}$	C_0^2	$D_{0,1}^{1,1}$	$\cdots$	$D_{0,k-1}^{1,1}$	$B_0^{1,1}$	C_0^0	$D_{0,1}^{2,1}$	$\cdots$	$D_{0,k-1}^{2,1}$
$D_{1,0}^{0,1}$	C_1^1	$\cdots$	$D_{1,k-1}^{0,1}$	$B_1^{0,1}$	$D_{1,0}^{1,1}$	C_1^2	$\cdots$	$D_{1,k-1}^{1,1}$	$B_1^{1,1}$	$D_{1,0}^{2,1}$	C_1^0	$\cdots$	$D_{1,k-1}^{2,1}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$D_{k-1,0}^{0,1}$	$D_{k-1,1}^{0,1}$	$\cdots$	C_{k-1}^1	$B_{k-1}^{0,1}$	$D_{k-1,0}^{1,1}$	$D_{k-1,1}^{1,1}$	$\cdots$	C_{k-1}^2	$B_{k-1}^{1,1}$	$D_{k-1,0}^{2,1}$	$D_{k-1,1}^{2,1}$	$\cdots$	C_{k-1}^0
$\beta_0^{0,1}$	$\beta_1^{0,1}$	$\cdots$	$\beta_{k-1}^{0,1}$	A_3	$\beta_0^{1,1}$	$\beta_1^{1,1}$	$\cdots$	$\beta_{k-1}^{1,1}$	A_4	$\beta_0^{2,1}$	$\beta_1^{2,1}$	$\cdots$	$\beta_{k-1}^{2,1}$
C_0^2	$D_{0,1}^{0,2}$	$\cdots$	$D_{0,k-1}^{0,2}$	$B_0^{0,2}$	C_0^0	$D_{0,1}^{1,2}$	$\cdots$	$D_{0,k-1}^{1,2}$	$B_0^{1,2}$	C_0^1	$D_{0,1}^{2,2}$	$\cdots$	$D_{0,k-1}^{2,2}$
$D_{1,0}^{0,2}$	C_1^2	$\cdots$	$D_{1,k-1}^{0,2}$	$B_1^{0,2}$	$D_{1,0}^{1,2}$	C_1^0	$\cdots$	$D_{1,k-1}^{1,2}$	$B_1^{1,2}$	$D_{1,0}^{2,2}$	C_1^1	$\cdots$	$D_{1,k-1}^{2,2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$D_{k-1,0}^{0,2}$	$D_{k-1,1}^{0,2}$	$\cdots$	C_{k-1}^2	$B_{k-1}^{0,2}$	$D_{k-1,0}^{1,2}$	$D_{k-1,1}^{1,2}$	$\cdots$	C_{k-1}^0	$B_{k-1}^{1,2}$	$D_{k-1,0}^{2,2}$	$D_{k-1,1}^{2,2}$	$\cdots$	C_{k-1}^1

Figure 9: Completed SPLS $(3, 3k+2)$

5 An extension to SPLS $(a, ma-1)$

It is not known whether an SPLS $(a, a-1)$ exists for every $a \geq 3$. However, if an SPLS $(a, a-1)$ exists for some $a \geq 3$, then the strategy used to prove Theorem 4.3 can also be used to construct an SPLS $(a, ma-1)$ whenever $m \neq 3$. In this section we outline the construction, closely paralleling the Section 4 construction[4].

As in Section 4, we begin our construction with an empty SPLS $(a, ma-1)$ grid G partitioned into $a \times a$ regions G_{ij} $(0 \leq i, j < ma-1)$. Each G_{ij} is one of three types:

- Type I regions lie wholly within both an $(ma-1) \times a$ tiling region and an $a \times (ma-1)$ tiling region.
- Type II regions lie wholly within an $(ma-1) \times a$ tiling region but not within

[4] Regarding $a = 2$, recall from the introduction that an SPLS $(2, t)$ exists for every $t \geq 1$.

an $a \times (ma-1)$ tiling region, or vice versa.

- Type III regions lie neither within an $(ma-1) \times a$ tiling region nor within an $a \times (ma-1)$ tiling region.

If $T = \{m-1, 2m-1, \dots, (a-1)m-1\}$, then observe that G_{ij} is of Type I when neither i nor j lies in T, that G_{ij} is of Type II when exactly one of i, j lies in T, and that G_{ij} is of Type III when i, j both lie in T.

We now set about filling the grids G_{ij} with symbols. First, suppose $a \geq 3$ and an SPLS $(a, a-1)$ A exists with symbol set $\{0, 1, \dots, a(a-1)-1\}$. We write $A = (A_{ij})$ where each A_{ij} is an $a \times a$ array $(0 \leq i, j < a-1)$. We place A_{ij} in region $G_{m(i+1)-1, m(j+1)-1}$ for $0 \leq i, j \leq a-1$. This accounts for all of the Type III regions in G.

Next, for $0 \leq \ell < m-1$, let B_ℓ be the $a \times a$ array whose (i, j) entry is

$$a(a-1) + a^2\ell + ai + j$$

for $0 \leq i, j \leq a-1$. Note that the symbols in B_s are disjoint from the symbols in B_t (and from those in A) whenever $s \neq t$. The union of these symbols, those from the B_ℓ and those from A, is the complete set of symbols $\{0, 1, \dots, a(ma-1)-1\}$ needed for an SPLS $(a, ma-1)$. We let β_ℓ denote B_ℓ^t. Place $B_\ell^{i,j}$ in $G_{mj+\ell, m(i+1)-1}$ for $0 \leq i \leq a-2$, $0 \leq j \leq a-1$, and $0 \leq \ell \leq m-2$. Likewise, place $\beta_\ell^{i,j}$ in $G_{m(j+1)-1, mi+\ell}$ for $0 \leq i \leq a-1$, $0 \leq j \leq a-2$, and $0 \leq \ell \leq m-2$, where subscripts are reduced modulo $ma-1$. These placements account for all Type II regions in G.

Now we begin filling some of the Type I regions. Let C denote the $a \times a$ array with no entries along the diagonal and entry $(k-1)a+i$ in location $(i, (i+k) \bmod a)$, where $1 \leq k \leq a-1$ and $0 \leq i \leq a-1$. (Essentially we are placing the symbols $0, \dots, a(a-1)-1$ in order along the broken diagonals of C.) Then, for $0 \leq r \leq a-1$ and $0 \leq \ell \leq m-2$ we let $C_\ell^r = C^{r,r}$ with entry

$$a^2(\ell+1) + [i+1]a - r - 1$$

in diagonal location (i, i) for $0 \leq i \leq a-1$, where $[i+1]$ is reduced modulo a. Note that the diagonal symbols of C_ℓ^r are precisely the symbols in the $(a-1-r)$-th column of B_ℓ. Place C_ℓ^r in region $G_{(r+j)m+\ell, [-(r+j)]m+\ell}$ for $0 \leq r, j \leq a-1$ and $0 \leq \ell \leq m-2$, where again the $[-(r+j)]$ in the subscript is reduced modulo a.

Finally we fill the remaining Type I regions. Let $D = (D_{i,j})$ $(0 \leq i, j < m-1)$ be an idempotent Latin square of order $m-1$ with symbol set $\{B_0, \dots, B_{m-2}\}$ satisfying $D_{i,i} = B_i$ for $0 \leq i < m-1$. Such a Latin square exists by Proposition 4.1, except when $m = 3$. For $0 \leq i, j < m-1$ with $i \neq j$ we place $D_{i,j}^{r,s}$ in region $G_{i+sm, j+rm}$ for $0 \leq r, s \leq a-1$.

This construction specializes to the construction in Section 4. Further, the reader may wish to check that when $a = 4$, $m = 2$, and A is the SPLS $(4, 3)$ given in Figure 7, this construction gives rise to the SPLS $(4, 7)$ shown in Figure 15.

Given an SPLS $(a, a-1)$, we assert that the construction described above yields an SPLS $(a, ma-1)$. The argument in favor of this construction closely parallels the argument culminating in Theorem 4.3, so the majority of the details will not be repeated here. However, there is an aspect of the present construction that needs a closer look. Recall that the existence of an SPLS $(3, 2\cdot 3-1)$ (Example 4.4) was crucial in the proof of Theorem 4.3. In Example 4.4 the diagonal entries in C_0^r $(0 \le r \le 2)$ are drawn from the $(2-r)$-th column of B_0 to prevent repetition in 3×5 and 5×3 tiling regions. However, these entries need to be placed carefully on the diagonal of C_0^r to avoid repetition in rows and columns. This placement (6) was checked by inspection. In our present circumstances, it is crucial that when $m = 2$ our construction yields an SPLS $(a, 2a-1)$. As in Example 4.4, the diagonal entries of C_0^r are drawn from the $(a-1-r)$-th column of B_0 to prevent repetition in $a\times(2a-1)$ and $(2a-1)\times a$ tiling regions, but we can no longer use inspection to check that our placement of these symbols on the diagonal of C_0^r does not induce repetition in rows and columns. We address this problem here.

Let M be the proposed SPLS $(a, 2a-1)$ produced through the construction and let $0 \le r \le a-1$. Any row of M containing a diagonal element of C_0^r must lie in an $a \times a(2a-1)$ subarray of M of the form

$$M_j = \begin{array}{|c|c|c|c|c|c|c|c|}\hline C_0^j & B_0^{0,j} & C_0^{j+1} & B_0^{1,j} & \cdots & C_0^{j+a-2} & B_0^{a-2,j} & C_0^{j+a-1} \\ \hline\end{array}.$$

where $0 \le j \le a-1$ and superscripts on C_0 are reduced modulo a. Now, let us consider the entry $(C_0^r)_{i,i} = a^2 + [i+1]a - r - 1$ lying in the (i,i) location of C_0^r. (Again, $[i+1]$ denotes the residue of $i+1$ modulo a.) We want to show that this entry makes no other appearance in the i-th row of M_j. For reasons that will be made clear momentarily, it is convenient to express this entry as

$$(C_0^r)_{i,i} = a^2 + a([i+1]-1) + (a-r-1). \tag{9}$$

First note that no other entry in the i-th row of C_0^r can be equal to $(C_0^r)_{i,i}$ because $(C_0^r)_{i,i} \ge a(a-1)$ while non-diagonal entries of C_0^r are no larger than $a(a-1)-1$. Next, consider entries in the i-th row of C_0^s, where $s \ne r$. Once again, the non-diagonal entries of C_0^s are strictly smaller than $(C_0^r)_{i,i}$. Further, because $r \ne s$ we have

$$(C_0^r)_{i,i} = a^2 + a([i+1]-1) + (a-r-1) \ne a^2 + a([i+1]-1) + (a-s-1) = (C_0^s)_{i,i}.$$

Finally, note that the symbols in the i-th row of $B_0^{t,j}$ are

$$\{a^2 + a([i-t]-1) + k \mid 0 \le k \le a-1\}$$

and that $0 \le t \le a-2$. By (9) and the Division Algorithm, $(C_0^r)_{i,i}$ is equal to one of these symbols if and only if $[i+1]-1 = [i-t]-1$, where the brackets indicate reduction modulo a. However, this requires $t \equiv -1 \pmod a$, which is impossible because $0 \le t \le a-2$. Therefore, $(C_0^r)_{i,i}$ appears exactly once within the i-th row of M_j. A similar argument (using β instead of B) applies to columns of M.

Therefore, we have:

Theorem 5.1. *Let* $a, m \in \mathbb{Z}^+$. *If* $a \geq 3$ *and an* SPLS $(a, a-1)$ *exists, then so does an* SPLS $(a, ma-1)$ *whenever* $m \neq 3$.

An SPLS $(4, 3)$ is given in Figure 7 and an SPLS $(6, 5)$ is given in Figure 16, so we have[5]:

Corollary 5.2. *If* $m \in \mathbb{Z}^+$, *then an* SPLS $(4, 4m-1)$ *and an* SPLS $(6, 6m-1)$ *exists whenever* $m \neq 3$.

Also, it is known [4] that an SPLS $(p, p-1)$ exists when p is prime. This implies the following corollary.

Corollary 5.3. *If* $p, m \in \mathbb{Z}^+$ *and* p *is prime, then an* SPLS $(p, mp-1)$ *exists whenever* $m \neq 3$.

The Appendices A through D follow, on the subsequent pages.

[5]The SPLS $(6, 5)$ in Figure 16 was found using an adaptation of an SAT sudoku solver.

A Constructing an SPLS $(3, 11)$

Step 1: Place arrays A_1, A_2, A_3, A_4 in the Type III regions of G.

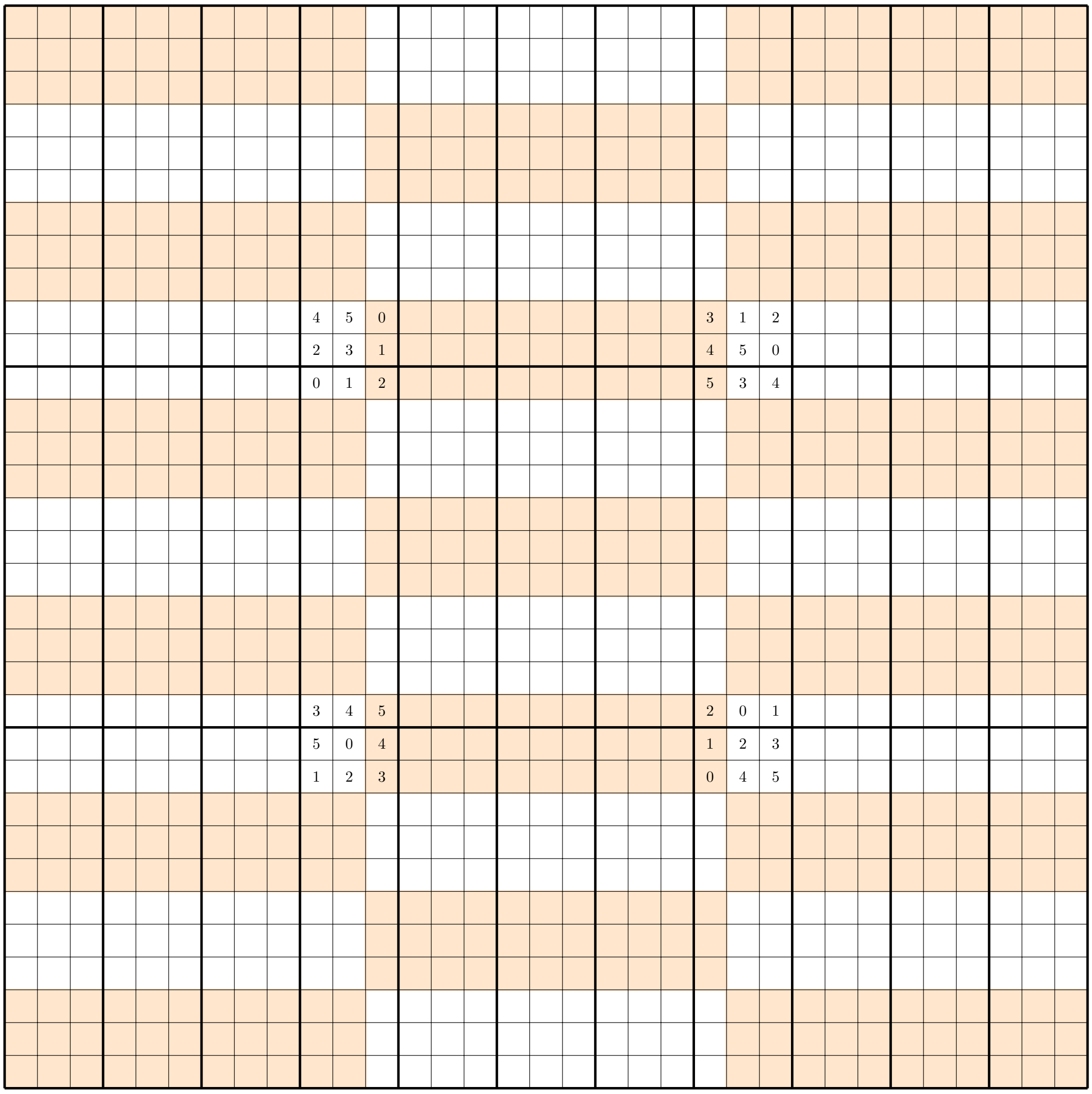


Figure 10: Step 1

Step 2: Use the B_ℓ and β_ℓ arrays to fill the Type II regions of G.

									6	7	8										12	13	14									
									9	10	11										6	7	8									
									12	13	14										9	10	11									
									15	16	17										21	22	23									
									18	19	20										15	16	17									
									21	22	23										18	19	20									
									24	25	26										30	31	32									
									27	28	29										24	25	26									
									30	31	32										27	28	29									
6	9	12	15	18	21	24	27	30	4	5	0	8	11	14	17	20	23	26	29	32	3	1	2	7	10	13	16	19	22	25	28	31
7	10	13	16	19	22	25	28	31	2	3	1	6	9	12	15	18	21	24	27	30	4	5	0	8	11	14	17	20	23	26	29	32
8	11	14	17	20	23	26	29	32	0	1	2	7	10	13	16	19	22	25	28	31	5	3	4	6	9	12	15	18	21	24	27	30
									8	6	7										14	12	13									
									11	9	10										8	6	7									
									14	12	13										11	9	10									
									17	15	16										23	21	22									
									20	18	19										17	15	16									
									23	21	22										20	18	19									
									26	24	25										32	30	31									
									29	27	28										26	24	25									
									32	30	31										29	27	28									
12	6	9	21	15	18	30	24	27	3	4	5	14	8	11	23	17	20	32	26	29	2	0	1	13	7	10	22	16	19	31	25	28
13	7	10	22	16	19	31	25	28	5	0	4	12	6	9	21	15	18	30	24	27	1	2	3	14	8	11	23	17	20	32	26	29
14	8	11	23	17	20	32	26	29	1	2	3	13	7	10	22	16	19	31	25	28	0	4	5	12	6	9	21	15	18	30	24	27
									7	8	6										13	14	12									
									10	11	9										7	8	6									
									13	14	12										10	11	9									
									16	17	15										22	23	21									
									19	20	18										16	17	15									
									22	23	21										19	20	18									
									25	26	24										31	32	30									
									28	29	27										25	26	24									
									31	32	30										28	29	27									

Figure 11: Step 2

Step 3: Fill the "diagonal" Type I regions of G with the C^i_ℓ arrays.

11	0	3							6	7	8	10	2	5							12	13	14	9	1	4						
4	14	1							9	10	11	3	13	0							6	7	8	5	12	2						
2	5	8							12	13	14	1	4	7							9	10	11	0	3	6						
			20	0	3				15	16	17				19	2	5				21	22	23				18	1	4			
			4	23	1				18	19	20				3	22	0				15	16	17				5	21	2			
			2	5	17				21	22	23				1	4	16				18	19	20				0	3	15			
						29	0	3	24	25	26							28	2	5	30	31	32							27	1	4
						4	32	1	27	28	29							3	31	0	24	25	26							5	30	2
						2	5	26	30	31	32							1	4	25	27	28	29							0	3	24
6	9	12	15	18	21	24	27	30	4	5	0	8	11	14	17	20	23	26	29	32	3	1	2	7	10	13	16	19	22	25	28	31
7	10	13	16	19	22	25	28	31	2	3	1	6	9	12	15	18	21	24	27	30	4	5	0	8	11	14	17	20	23	26	29	32
8	11	14	17	20	23	26	29	32	0	1	2	7	10	13	16	19	22	25	28	31	5	3	4	6	9	12	15	18	21	24	27	30
10	2	5							8	6	7	9	1	4							14	12	13	11	0	3						
3	13	0							11	9	10	5	21	2							8	6	7	4	14	1						
1	4	7							14	12	13	0	3	6							11	9	10	2	5	8						
			19	2	5				17	15	16				18	1	4				23	21	22				20	0	3			
			3	22	0				20	18	19				5	21	2				17	15	16				4	23	1			
			1	4	16				23	21	22				0	3	15				20	18	19				2	5	17			
						28	2	5	26	24	25							27	1	4	32	30	31							29	0	3
						3	31	0	29	27	28							5	30	2	26	24	25							4	32	1
						1	4	25	32	30	31							0	3	24	29	27	28							2	5	26
12	6	9	21	15	18	30	24	27	3	4	5	14	8	11	23	17	20	32	26	29	2	0	1	13	7	10	22	16	19	31	25	28
13	7	10	22	16	19	31	25	28	5	0	4	12	6	9	21	15	18	30	24	27	1	2	3	14	8	11	23	17	20	32	26	29
14	8	11	23	17	20	32	26	29	1	2	3	13	7	10	22	16	19	31	25	28	0	4	5	12	6	9	21	15	18	30	24	27
9	1	4							7	8	6	11	0	3							13	14	12	10	2	5						
5	12	2							10	11	9	4	14	1							7	8	6	3	13	0						
0	3	6							13	14	12	2	5	8							10	11	9	1	4	7						
			18	1	4				16	17	15				20	0	3				22	23	21				19	2	5			
			5	21	2				19	20	18				4	23	1				16	17	15				3	22	0			
			0	3	15				22	23	21				2	5	17				19	20	18				1	4	16			
						27	1	4	25	26	24							29	0	3	31	32	30							28	2	5
						5	30	2	28	29	27							4	32	1	25	26	24							3	31	0
						0	3	24	31	32	30							2	5	26	28	29	27							1	4	25

Figure 12: Step 3

Step 4: Fill the remaining Type I regions with the $D_{i,j}$ arrays.

11	0	3	30	24	27	21	15	18	6	7	8	10	2	5	32	26	29	23	17	20	12	13	14	9	1	4	31	25	28	22	16	19
4	14	1	28	31	25	19	22	16	9	10	11	3	13	0	27	30	24	18	21	15	6	7	8	5	12	2	29	32	26	20	23	17
2	5	8	26	29	32	17	20	23	12	13	14	1	4	7	25	28	31	16	19	22	9	10	11	0	3	6	24	27	30	15	18	21
30	24	27	20	0	3	12	6	9	15	16	17	32	26	29	19	2	5	14	8	11	21	22	23	31	25	28	18	1	4	13	7	10
28	31	25	4	23	1	10	13	7	18	19	20	27	30	24	3	22	0	9	12	6	15	16	17	29	32	26	5	21	2	11	14	8
26	29	32	2	5	17	8	11	14	21	22	23	25	28	31	1	4	16	7	10	13	18	19	20	24	27	30	0	3	15	6	9	12
21	15	18	12	6	9	29	0	3	24	25	26	23	17	20	14	8	11	28	2	5	30	31	32	22	16	19	13	7	10	27	1	4
19	22	16	10	13	7	4	32	1	27	28	29	18	21	15	9	12	6	3	31	0	24	25	26	20	23	17	11	14	8	5	30	2
17	20	23	8	11	14	2	5	26	30	31	32	16	19	22	7	10	13	1	4	25	27	28	29	15	18	21	6	9	12	0	3	24
6	9	12	15	18	21	24	27	30	4	5	0	8	11	14	17	20	23	26	29	32	3	1	2	7	10	13	16	19	22	25	28	31
7	10	13	16	19	22	25	28	31	2	3	1	6	9	12	15	18	21	24	27	30	4	5	0	8	11	14	17	20	23	26	29	32
8	11	14	17	20	23	26	29	32	0	1	2	7	10	13	16	19	22	25	28	31	5	3	4	6	9	12	15	18	21	24	27	30
10	2	5	32	26	29	23	17	20	8	6	7	9	1	4	31	25	28	22	16	19	14	12	13	11	0	3	30	24	27	21	15	18
3	13	0	27	30	24	18	21	15	11	9	10	5	21	2	29	32	26	20	23	17	8	6	7	4	14	1	28	31	25	19	22	16
1	4	7	25	28	31	16	19	22	14	12	13	0	3	6	24	27	30	15	18	21	11	9	10	2	5	8	26	29	32	17	20	23
32	26	29	19	2	5	14	8	11	17	15	16	31	25	28	18	1	4	13	7	10	23	21	22	30	24	27	20	0	3	12	6	9
27	30	24	3	22	0	9	12	6	20	18	19	29	32	26	5	21	2	11	14	8	17	15	16	28	31	25	4	23	1	10	13	7
25	28	31	1	4	16	7	10	13	23	21	22	24	27	30	0	3	15	6	9	12	20	18	19	26	29	32	2	5	17	8	11	14
23	17	20	14	8	11	28	2	5	26	24	25	22	16	19	13	7	10	27	1	4	32	30	31	21	15	18	12	6	9	29	0	3
18	21	15	9	12	6	3	31	0	29	27	28	20	23	17	11	14	8	5	30	2	26	24	25	19	22	16	10	13	7	4	32	1
16	19	22	7	10	13	1	4	25	32	30	31	15	18	21	6	9	12	0	3	24	29	27	28	17	20	23	8	11	14	2	5	26
12	6	9	21	15	18	30	24	27	3	4	5	14	8	11	23	17	20	32	26	29	2	0	1	13	7	10	22	16	19	31	25	28
13	7	10	22	16	19	31	25	28	5	0	4	12	6	9	21	15	18	30	24	27	1	2	3	14	8	11	23	17	20	32	26	29
14	8	11	23	17	20	32	26	29	1	2	3	13	7	10	22	16	19	31	25	28	0	4	5	12	6	9	21	15	18	30	24	27
9	1	4	31	25	28	22	16	19	7	8	6	11	0	3	30	24	27	21	15	18	13	14	12	10	2	5	32	26	29	23	17	20
5	12	2	29	32	26	20	23	17	10	11	9	4	14	1	28	31	25	19	22	16	7	8	6	3	13	0	27	30	24	18	21	15
0	3	6	24	27	30	15	18	21	13	14	12	2	5	8	26	29	32	17	20	23	10	11	9	1	4	7	25	28	31	16	19	22
31	25	28	18	1	4	13	7	10	16	17	15	30	24	27	20	0	3	12	6	9	22	23	21	32	26	29	19	2	5	14	8	11
29	32	26	5	21	2	11	14	8	19	20	18	28	31	25	4	23	1	10	13	7	16	17	15	27	30	24	3	22	0	9	12	6
24	27	30	0	3	15	6	9	12	22	23	21	26	29	32	2	5	17	8	11	14	19	20	18	25	28	31	1	4	16	7	10	13
22	16	19	13	7	10	27	1	4	25	26	24	21	15	18	12	6	9	29	0	3	31	32	30	23	17	20	14	8	11	28	2	5
20	23	17	11	14	8	5	30	2	28	29	27	19	22	16	10	13	7	4	32	1	25	26	24	18	21	15	9	12	6	3	31	0
15	18	21	6	9	12	0	3	24	31	32	30	17	20	23	8	11	14	2	5	26	28	29	27	16	19	22	7	10	13	1	4	25

Figure 13: Step 4 - SPLS $(3, 11)$

B A Computer-Generated SPLS $(3, 8)$

12	21	14	16	11	23	0	1	2	13	15	18	3	20	22	19	17	4	8	6	9	5	10	7
17	22	2	13	8	10	3	4	5	6	7	14	21	23	1	9	11	16	20	15	18	12	0	19
20	19	9	15	18	5	6	7	8	10	16	17	0	11	4	12	21	23	2	22	3	1	13	14
18	6	23	19	1	14	9	10	11	4	3	21	12	5	13	7	20	8	16	0	17	2	22	15
3	4	8	17	0	20	12	13	14	9	23	19	2	15	16	22	18	1	7	10	5	6	21	11
7	11	5	2	21	22	15	16	17	20	1	0	8	10	18	6	3	14	12	19	13	4	23	9
10	0	15	9	4	6	18	19	20	11	12	22	14	17	7	13	5	2	23	21	1	16	3	8
1	16	13	3	12	7	21	22	23	5	2	8	6	19	9	0	10	15	14	11	4	17	20	18
8	20	11	23	5	2	17	14	16	21	4	10	15	18	3	1	9	22	13	7	6	19	12	0
2	15	22	8	6	21	1	18	9	16	20	7	11	14	12	3	13	0	10	4	19	23	5	17
14	9	17	11	7	16	23	0	10	19	13	5	1	4	21	18	15	6	3	12	22	20	8	2
5	13	19	4	10	12	20	3	22	23	8	15	17	6	0	2	14	7	18	9	21	11	1	16
23	10	4	14	17	15	2	11	19	0	18	9	22	13	8	21	16	12	5	1	20	7	6	3
21	1	0	18	22	19	7	6	4	12	14	3	20	16	2	5	8	10	11	17	23	9	15	13
16	12	3	20	13	9	8	5	15	1	17	6	23	7	10	11	4	19	0	14	2	22	28	21
6	18	7	0	3	1	13	12	21	2	22	11	5	9	19	20	23	17	15	8	16	14	4	10
19	2	10	21	20	11	5	17	18	15	6	16	4	8	23	14	1	3	9	13	12	0	7	22
4	23	16	22	9	8	14	15	7	3	0	1	13	12	17	10	2	20	21	5	11	18	19	6
13	7	12	1	19	18	16	9	0	14	11	20	10	21	15	17	6	5	22	23	8	3	2	4
15	17	20	5	23	4	22	21	3	18	9	12	7	2	6	8	0	11	19	16	10	13	14	1
0	14	6	10	2	3	11	8	13	22	19	4	16	1	5	23	7	18	17	20	15	21	9	12
22	8	21	12	14	0	10	23	6	17	5	13	18	3	11	4	19	9	1	2	7	15	16	20
11	3	1	6	15	13	19	2	12	7	10	23	9	0	20	16	22	21	4	18	14	8	17	5
9	5	18	7	16	17	4	20	1	8	21	2	19	22	14	15	12	13	6	3	0	10	11	23

Figure 14: An SPLS $(3, 8)$ produced by a stochastic backtracking algorithm.

C An SPLS (4, 7) Constructed by Algorithm

19	0	4	8	12	13	14	15	18	3	7	11	24	25	26	27	17	2	6	10	20	21	22	23	16	1	5	9
9	23	1	5	16	17	18	19	8	22	0	4	12	13	14	15	11	21	3	7	24	25	26	27	10	20	2	6
6	10	27	2	20	21	22	23	5	9	26	1	16	17	18	19	4	8	25	0	12	13	14	15	7	11	24	3
3	7	11	15	24	25	26	27	2	6	10	14	20	21	22	23	1	5	9	13	16	17	18	19	0	4	8	12
12	16	20	24	0	1	2	9	15	19	23	27	4	8	6	5	14	18	22	26	7	3	11	10	13	17	21	25
13	17	21	25	5	3	4	11	12	16	20	24	0	7	10	9	15	19	23	27	8	1	6	2	14	18	22	26
14	18	22	26	7	8	6	10	13	17	21	25	2	3	11	1	12	16	20	24	0	5	4	9	15	19	23	27
15	19	23	27	9	10	11	6	14	18	22	26	1	5	3	2	13	17	21	25	4	0	8	7	12	16	20	24
18	3	7	11	15	12	13	14	17	2	6	10	27	24	25	26	16	1	5	9	23	20	21	22	19	0	4	8
8	22	0	4	19	16	17	18	11	21	3	7	15	12	13	14	10	20	2	6	27	24	25	26	9	23	1	5
5	9	26	1	23	20	21	22	4	8	25	0	19	16	17	18	7	11	24	3	15	12	13	14	6	10	27	2
2	6	10	14	27	24	25	26	1	5	9	13	23	20	21	22	0	4	8	12	19	16	17	18	3	7	11	15
24	12	16	20	2	0	1	8	27	15	19	23	9	4	7	6	26	14	18	22	5	10	3	11	25	13	17	21
25	13	17	21	4	5	3	7	24	12	16	20	11	0	8	10	27	15	19	23	9	2	1	6	26	14	18	22
26	14	18	22	6	7	8	3	25	13	17	21	10	2	0	11	24	12	16	20	1	9	5	4	27	15	19	23
27	15	19	23	11	9	10	5	26	14	18	22	6	1	4	3	25	13	17	21	2	7	0	8	24	12	16	20
17	2	6	10	14	15	12	13	16	1	5	9	26	27	24	25	19	0	4	8	22	23	20	21	18	3	7	11
11	21	3	7	18	19	16	17	10	20	2	6	14	15	12	13	9	23	1	5	26	27	24	25	8	22	0	4
4	8	25	0	22	23	20	21	7	11	24	3	18	19	16	17	6	10	27	2	14	15	12	13	5	9	26	1
1	5	9	13	26	27	24	25	0	4	8	12	22	23	20	21	3	7	11	15	18	19	16	17	2	6	10	14
20	24	12	16	1	2	0	4	23	27	15	19	8	9	5	7	22	26	14	18	6	11	10	3	21	25	13	17
21	25	13	17	3	4	5	0	20	24	12	16	7	11	9	8	23	27	15	19	10	6	2	1	22	26	14	18
22	26	14	18	8	6	7	2	21	25	13	17	3	10	1	0	20	24	12	16	11	4	9	5	23	27	15	19
23	27	15	19	10	11	9	1	22	26	14	18	5	6	2	4	21	25	13	17	3	8	7	0	20	24	12	16
16	1	5	9	13	14	15	12	19	0	4	8	25	26	27	24	18	3	7	11	21	22	23	20	17	2	6	10
10	20	2	6	17	18	19	16	9	23	1	5	13	14	15	12	8	22	0	4	25	26	27	24	11	21	3	7
7	11	24	3	21	22	23	20	6	10	27	2	17	18	19	16	5	9	26	1	13	14	15	12	4	8	25	0
0	4	8	12	25	26	27	24	3	7	11	15	21	22	23	20	2	6	10	14	17	18	19	16	1	5	9	13

Figure 15: SPLS (4, 7)

D A Computer-Generated SPLS $(5, 6)$

27	14	4	10	25	30	11	16	3	15	17	12	24	20	1	26	13	5	2	22	18	9	23	21	29	8	19	28	7	6
21	22	11	9	2	23	18	6	10	7	14	5	28	16	29	27	12	30	25	20	8	17	3	19	24	4	1	13	26	15
12	8	29	7	26	17	24	2	28	4	19	27	22	21	25	23	9	14	11	15	13	30	6	1	16	3	20	18	10	5
28	6	3	20	18	5	1	29	13	26	30	23	4	2	15	8	17	19	24	16	10	12	7	27	25	9	11	21	22	14
24	16	15	13	1	19	20	21	22	25	9	8	10	11	3	7	18	6	28	29	26	4	5	14	2	30	27	23	17	12
30	23	19	17	5	27	14	12	9	8	18	26	7	13	6	21	10	4	1	3	22	20	15	11	28	16	29	25	24	2
29	20	28	15	24	26	30	23	27	22	25	1	19	18	17	2	8	16	21	14	12	13	10	9	11	6	7	5	4	3
25	21	22	18	12	7	6	28	29	24	20	16	30	27	26	15	11	3	23	19	4	2	17	5	14	1	13	10	8	9
16	13	10	11	4	2	21	17	19	3	15	7	14	5	9	20	1	24	26	28	25	6	29	8	18	12	22	30	23	27
14	9	8	6	3	1	13	11	5	10	4	2	29	28	23	25	22	12	30	18	7	27	24	16	26	21	17	20	15	19
26	27	30	23	19	25	16	20	18	14	13	10	12	6	8	17	7	9	5	4	15	21	1	3	22	24	2	11	28	29
17	5	2	1	7	15	12	8	4	9	24	22	21	3	11	29	27	13	10	6	30	28	20	23	19	26	25	16	14	18
22	29	24	28	21	20	27	30	26	23	11	25	18	19	16	14	15	10	17	13	9	8	12	2	3	7	5	4	6	1
18	12	16	14	13	11	29	19	21	28	6	3	2	23	5	4	20	1	7	27	24	25	26	22	17	10	15	9	30	8
10	4	9	3	6	8	17	7	2	5	1	15	26	30	28	24	25	22	29	11	19	16	14	18	27	23	21	12	20	13
23	2	7	27	11	24	22	18	15	16	29	21	20	14	12	30	6	28	9	8	5	10	4	13	1	19	3	17	25	26
19	17	5	26	15	12	4	10	25	1	8	24	13	9	22	18	23	21	3	2	20	29	30	7	6	11	14	27	16	28
8	1	25	30	20	6	9	3	14	13	7	17	27	10	4	19	5	26	16	12	28	23	11	15	21	29	24	2	18	22
3	28	21	16	29	18	23	27	30	19	12	20	15	1	7	11	2	17	14	25	6	24	22	26	13	5	10	8	9	4
13	10	14	4	9	22	28	5	11	2	26	6	25	29	24	16	3	8	27	21	1	19	18	17	30	20	23	15	12	7
20	11	26	19	22	29	25	24	17	21	28	18	23	8	27	5	30	7	4	10	3	15	2	12	9	14	6	1	13	16
6	30	23	2	27	16	15	4	1	12	5	14	9	22	21	28	19	20	13	26	11	7	8	25	10	17	18	29	3	24
5	15	12	24	8	10	3	9	7	6	16	30	11	17	13	1	29	18	22	23	21	14	28	20	4	27	26	19	2	25
1	7	18	25	17	14	26	13	8	20	2	19	3	4	10	12	24	15	6	9	27	5	16	29	23	28	30	22	11	21
9	3	13	21	28	4	10	22	23	29	27	11	16	25	14	6	26	2	18	30	17	1	19	24	12	15	8	7	5	20
15	26	27	29	30	28	19	25	24	18	23	13	17	12	20	22	21	11	8	7	16	3	9	6	5	2	4	14	1	10
11	25	20	22	23	21	8	26	16	30	10	9	5	24	2	3	28	29	15	1	14	18	27	4	7	13	12	6	19	17
7	24	17	12	16	13	2	15	6	27	22	4	1	26	19	10	14	25	20	5	23	11	21	28	8	18	9	3	29	30
4	18	6	5	14	9	7	1	12	11	3	29	8	15	30	13	16	27	19	17	2	26	25	10	20	22	28	24	21	23
2	19	1	8	10	3	5	14	20	17	21	28	6	7	18	9	4	23	12	24	29	22	13	30	15	25	16	26	27	11

Figure 16: An SPLS $(5, 6)$ constructed using an adaptation of an SAT sudoku solver.

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