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Homogeneous edge-disjoint K_{2s} and $T_{st,t}$ unions

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Abstract

Let $r > 2$ and $\sigma \in (0, r-1)$ be integers. We require $t < 2s$, where $t = 2^{\sigma+1}-1$ and $s = 2^{r-\sigma-1}$. Generalizing a known $\{K_4, T_{6,3}\}$-ultrahomogenous graph G_3^1, we find that a finite, connected, undirected, arc-transitive graph G_r^σ exists each of whose edges is shared by just two maximal subgraphs, namely a clique $X_0 = K_{2s}$ and a t-partite regular-Turán graph $X_1 = T_{st,t}$ on s vertices per part. Each copy Y of X_i $(i = 0, 1)$ in G_r^σ shares each edge with just one copy of X_{1-i} and all such copies of X_{1-i} are pairwise distinct. Moreover, G_r^σ is an edge-disjoint union of copies of X_i, for $i = 0, 1$. We prove that G_r^σ is $\{K_{2s}, T_{st,t}\}$-homogeneous if $t < 2s$, and just $\{T_{st,t}\}$-homogeneous otherwise, meaning that there is an automorphism of G_r^σ between any two such copies of X_i relating two preselected arcs.

1 Introduction

In applications of combinatorics to networks and telecommunications, it is desirable to have connected-graph models with the most homogeneity available on collections of vertices forming edge-disjoint cliques of fixed order, (e.g., the 42 K_4 in the 12-regular $\{K_4, T_{6,3}\}$-ultrahomogeneous graph G of [3]), and also forming edge-disjoint complements of forbidden small cliques in larger cliques of fixed order, namely edge-disjoint regular Turán graphs, (e.g., the 21 $K_{2,2,2}$ in G), each edge as the intersection of a clique and a regular Turán graph, a desirable feature given as two-way interpretation of the models.

Our aim is to explore a natural generalization of the construction in [3] to the construction of a family of graphs G_r^σ, $(r, \sigma \in \mathbb{Z},\ r > 2;\ \sigma \in (0, r-1))$, based, as in [3], on binary projective geometry (for example, the Fano plane in [3]). Only the cited graph G in [3] is $\mathcal{C}$-ultrahomogeneous (or $\mathcal{C}$-UH), among the claimed G_r^σ, ($\mathcal{C}$, any graph family). It was found that the strongest homogeneity conditions that the graphs G_r^σ satisfy are those in Definition 1.1, below. First, some known graph homogeneity notions are recalled.

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According to Sheehan [10], a finite, undirected, simple graph Γ is said to be *homogeneous* (respectively, *ultrahomogeneous*) if, whenever two induced subgraphs Y_0 and Y_1 of Γ are isomorphic, then some isomorphism (respectively, every isomorphism) of Y_0 onto Y_1 extends to an automorphism of Γ. Gardiner [4], Gol'fand and Klin [5, 7], and Reichard [8] gave explicit characterizations of ultrahomogeneous graphs. Isaksen et al. [6] defined a graph Γ to be $\mathcal{C}$-*UH*, where $\mathcal{C}$ is a class of graphs, if every isomorphism between any two induced subgraphs of Γ in $\mathcal{C}$ extends to an automorphism of Γ.

Definition 1.1. Let $\mathcal{C}$ be a set of arc-transitive graphs. We say that a finite, undirected, simple graph G is $\mathcal{C}$-*homogeneous* (or $\mathcal{C}$-*H*) if, for any two isomorphic induced subgraphs $Y_0, Y_1 \in \mathcal{C}$ of G and any two arcs (v_i, w_i) of Y_i $(i = 0, 1)$, there exists an automorphism f of G such that $f(Y_0) = Y_1$, $f(v_0) = v_1$ and $f(w_0) = w_1$.

Every $\mathcal{C}$-UH graph is $\mathcal{C}$-H. Since $\mathcal{C}$ in Definition 1.1 is formed by arc-transitive graphs, all arcs in a member of $\mathcal{C}$ behave similarly. If $\mathcal{C}$ consists of two non-isomorphic arc-transitive graphs X_0 and X_1, then a $\mathcal{C}$-H graph is said to be $\{X_0, X_1\}$-*homogeneous* (or $\{X_0, X_1\}$-*H*).

Let $s = 2^{r-\sigma-1}$ and $t = 2^{\sigma+1} - 1$. The graph G of [3] can be expressed as $G = G_r^\sigma$, with $(r, \sigma) = (3, 1)$ and seen as a $\{K_{2s}, T_{st,t}\}$-H graph ([3] showed it is $\{K_4, T_{6,2}\}$-UH), where $T_{st,t}$ is the maximal induced t-partite regular-Turán subgraph on s vertices per part. We conjecture the following (see Subsection 1.1).

Conjecture 1.2. *The claimed generalization G_r^σ of G_3^1 is isomorphic to a graph G that is $\{K_{2s}, T_{st,t}\}$-H if $t < 2s$ and just $\{T_{st,t}\}$-H otherwise.*

Ronse [9] showed that a graph is homogeneous if and only if it is ultrahomogeneous. On the other hand, $\mathcal{C}$-H graphs, which are the case in Theorems 3.3 and 4.10 below, are not necessarily $\mathcal{C}$-UH. The above claimed generalization of the construction of G_3^1 to that of the graphs G_r^σ is given below from Section 3 on, guaranteeing an answer to the following question.

Question 1.3. *For integers $r > 3$, $\sigma \in (0, r-1)$, $s = 2^{r-\sigma-1}$ and $t = 2^{\sigma+1} - 1$, does there exist a connected $\{K_{2s}, T_{st,t}\}$-H graph G_r^σ that is not $\{K_{2s}, T_{st,t}\}$-UH?*

1.1 Cliques and Regular-Turán Subgraphs

In [6], $\mathcal{C}$-UH graphs for the following four classes $\mathcal{C}$ of subgraphs were considered:

(A) the complete graphs;

(B) their complements, that is the empty graphs;

(C) the disjoint unions of complete graphs;

(D) their complements, that is the complete multipartite graphs.

Since $K_{2s} \in$**(A)** and $T_{st,t} \in$**(D)**, then both K_{2s} and $T_{st,t}$ belong to $\mathcal{D} =$**(A)**$\cup$**(D)**. In fact, each G_r^σ will coincide with a connected graph G that Conjecture 1.2 claims is $\mathcal{D}$-H if $t < 2s$ and **(D)**-H otherwise.

Conjecture 1.4. *The graph G of* Conjecture 1.2 *can be set in a unique way both as an edge-disjoint union of a family J_0 of cliques $X_0 = K_{2s}$ and as an edge-disjoint union of a family J_1 of maximal induced regular-Turán subgraphs $X_1 = T_{st,t}$ and with:*

(i) *$\mathcal{D}$ (respectively,* **(D)***) as the smallest class of graphs containing both X_0 and X_1, if $t < 2s$ (respectively, X_1, otherwise);*

(ii) *all copies of X_1 in G present in J_1 and every copy of X_0 in G either contained in a copy of X_1 or present in J_0;*

(iii) *no two copies of X_i in G sharing more than one vertex, for $i \in \{0, 1\}$;*

(iv) *each edge ξ in G shared by exactly one copy H_0 of X_0 and one copy H_1 of X_1, so that ξ is the only edge in $H_0 \cap H_1$.*

A graph G as in these items **(i)**–**(iv)** is arc-transitive with the same number, say $m_i(G) = m_i(G, v)$, of copies of X_i incident to each vertex v of G, independently of v, for $i \in \{0, 1\}$.

If $t < 2s$, then G is claimed (Conjecture 1.2) to be $\mathcal{D}$-H, where $\mathcal{D} = \{X_0, X_1\}$, since no copy of X_0 is contained in a copy of X_1 in G, and we say G is $\mathcal{D}_{K_2}$-*homogeneous* (or $\mathcal{D}_{K_2}$-*H*). If $\mathcal{D} = \{X_0, X_1\}$, then a $\mathcal{D}_{K_2}$-H graph is said to be $\mathcal{D}^{m_0,m_1}_{\ell_0,\ell_1}$-*H*, or $\{X_0\}^{m_0}_{\ell_0}\{X_1\}^{m_1}_{\ell_1}$-*H*, where ℓ_i is the number of copies of X_i in G and $m_i = m_i(G)$, for $i \in \{0, 1\}$.

From [3], we know that the line graph of the n-cube is a $\mathcal{D}^{m_0,m_1}_{\ell_0,\ell_1}$-UH graph with $\mathcal{D} = \{K_n, K_{2,2}\}$, $\ell_0 = 2^n$, $\ell_1 = 2^{n-3}n(n-1)$, $m_0 = 2$ and $m_1 = n - 1$, for $3 \leq n \in \mathbb{Z}$. As in [3], we say that G is *line-graphical* if $\min(m_0, m_1) = 2 = m_i$ and X_i is complete for just one of $i = 0, 1$. In [3], it was shown that G^1_3 is non-line-graphical $\{K_4, T_{6,2}\}_{K_2}$-UH. Our construction yields each of G^1_3 and G^1_4 as $\{K_4, T_{6,3}\}^{4,3}_{42,21}$-H [3] and $\{K_8, T_{12,3}\}^{8,7}_{2520,1470}$-H, of vertex order 42 and 2520 and regular degree 12 and 56, respectively. If $t \geq 2s$, then $X_1 = T_{st,t}$ contains K_{2s}, so G^σ_r cannot be $\mathcal{D}$-H. In this case, we still say that G^σ_r is a $\{X_0 \not\subset X_1, X_1\}$-*H graph*, meaning that G^σ_r is X_1-H with an automorphism f of G^σ_r between any two copies of X_0 not contained in any copy of X_1 and so that f relates two preselected arcs. Here, m_0 and ℓ_0 are taken only for copies of $K_{2s} \not\subset T_{st,t}$. For example, G^2_4 is a $\{K_4 \not\subset T_{14,7}, T_{14,7}\}^{12,3}_{630,45}$-H graph of vertex order 210 and degree 36.

Question 1.5. *Does there exist a connected non-line-graphical graph G^σ_r answering* Question 1.3 *that is either $\mathcal{D}^{m_0,m_1}_{\ell_0,\ell_1}$-H or $\{X_0 \not\subset X_1, X_1\}^{m_0,m_1}_{\ell_0,\ell_1}$-H, with* $\min\{m_0, m_1\} > 2$*?*

Graphs G^σ_r answering Question 1.5 affirmatively are constructed in Section 3, with their claimed properties established in Theorem 3.3 and more specifically in Theorem 4.10.

Theorem 3.3 yields the existence of such graphs G^σ_r. Theorem 4.10 establishes order, diameter and other parameters of such graphs for $r \leq 8$ (Proposition 3.7) and $\rho = r - \sigma \leq 5$ (Proposition 4.8), a total of 18 initial cases. Furthering the concept of $\mathcal{C}$-UH graph in [6], a graph G is said to be $\{K_{2s} \not\subset T_{st,t}\}$-*UH* if every isomorphism between cliques K_{2s} not contained in any induced copy of $T_{st,t}$ extends to an automorphism of G. Surpassing Theorem 4.10, if $t \geq 2s$ and $r - \sigma = 2$, then it can be shown that G_r^σ is $\{K_4 \not\subset T_{4t,t}\}$-UH.

2 Binary Projective Geometry

In order to prove the results claimed above, we need some notions of binary projective geometry. An *incidence structure* is a triple (P, L, I) consisting of a set P of *points*, a set L of *lines* and an *incidence relation* I indicating which points lie on which lines. Let $c, d, m, n \in \mathbb{Z}$ such that $0 < c < n$, $d < m$ and $cm = dn$. A *configuration* $R = (m_c, n_d)$ is an incidence structure of m points and n lines with c lines through each point and d points on each line [2]. Its *Levi graph* $L = L(R) = L(m_c, n_d)$ is the bipartite graph with:

(a) m "black" vertices representing the points of R;

(b) n "white" vertices representing the lines of R; and

(c) an edge joining each pair composed by a "black" vertex and a "white" vertex representing respectively a point and a line (in which that point lies) incident in R.

To any $R = (m_c, n_d)$ we associate its *Menger graph*, whose vertices are the points of R; and each two, say u and v, are joined by an edge $e = uv$ whenever u and v are in a common line.

If $m = n$ and $c = d$, in which case R is said to be *symmetric*, the dual configuration $\overline{R}$ is defined by reversing the roles of points and lines in R. In this case, both R and $\overline{R}$ share the same Levi graph, but the black-white coloring of their vertices is reversed. If R is isomorphic to $\overline{R}$, in which case R is said to be *self-dual*, an isomorphism between R and $\overline{R}$ is called a *duality* and we simply denote this by $R = (n_d)$.

Remark 2.1. Theorem 4.10 in Section 4.9 below asserts that each G_r^σ is the Menger graph of a configuration $(|V(G_r^\sigma)|_{m_0}, (\ell_0)_{2s})$ whose points and lines are the vertices and copies of K_{2s} in G_r^σ, respectively. For example, G_4^2 is the Menger graph of a configuration $(210_{12}, 630_4)$. On the other hand, if $(r, \sigma) = (3, 1)$ then the said configuration is self-dual and its Menger graph coincides with the corresponding dual Menger graph [3].

A *projective space* $\mathbf{P}$ is an incidence structure (P, L, I) satisfying three conditions [1]:

(1) each two distinct points a and b are in exactly one line, called the *line through* ab;

(2) (Veblen-Young) if a, b, c, d are distinct points and the lines through ab and cd meet, then so do the lines through ac and bd;

(3) any line has at least three points on it.

A *subspace* of $\mathbf{P}$ is a subset X such that any line containing two points of X is a subset of X, where the full and empty spaces are considered as subspaces of $\mathbf{P}$.

The *dimension* of $\mathbf{P}$ is the largest number r for which there is a strictly ascending chain of subspaces in $\mathbf{P}$ of the form $\{\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^r = \mathbf{P}\}$; in this case, $\mathbf{P}$ is said to be a *projective r-space* $\mathbf{P}^r$.

An *affine r-space* $A(r)$ is obtained by removing a copy of $\mathbf{P}^{r-1}$ from $\mathbf{P}^r$. Conversely, an affine r-space $A(r)$ leads to a projective r-space $\mathbf{P}^r$, the *closure* of $A(r)$, by adding the corresponding $(r-1)$-subspace $\mathbf{P}^{r-1}$ (said to be a *subspace at* ∞) whose points correspond to the classes of parallel lines, taken as the *directions of parallelism* of $A(r)$.

The projective space over the field $\mathbf{F}_2^r$ of 2^r elements ($r > 2$) is said to be the *binary projective* $(r-1)$*-space* $\mathbf{P}_2^{r-1}$ (Fano plane, if $r = 3$). Since each point of $\mathbf{P}_2^{r-1}$ represents a line ℓ of $\mathbf{F}_2^r$, that consists of two elements f, g of $\mathbf{F}_2^r$ one of which, say f, is the null element $\mathbf{0} \in \mathbf{F}_2^r$, then we represent each point of $\mathbf{P}_2^{r-1}$ by the r-tuple that stands for g, say $g = a_0a_1 \ldots a_{r-1}(\neq \mathbf{0})$, written without parentheses and commas. Since this r-tuple is composed by 0s and 1s, then it may be read from left to right as a binary number by removing the zeros preceding its leftmost 1, and we represent g by the resulting integer.

If $a, b \in \mathbb{Z}$ with $a < b$, we denote the integer interval $\{a, a+1, \ldots, b\}$ by $[a, b] = (a-1, b] = [a, b+1) = (a-1, b+1) \subset \mathbb{Z}$. We say that the empty set of $\mathbf{P}_2^{r-1}$ is a $(-j)$-*subspace* of $\mathbf{P}_2^{r-1}$ ($0 \leq j \in \mathbb{Z}$). Otherwise, $\mathbf{P}_2^{r-1}$ can be taken as the nonzero part of $\mathbf{F}_2^r$. So, if $j \in [0, r-2]$, then each j-subspace of $\mathbf{P}_2^{r-1}$ is taken as the intersection of $\mathbf{F}_2^r \backslash \{\mathbf{0}\}$ with an $\mathbf{F}_2$-linear j-subspace of $\mathbf{F}_2^r$.

Each of the $n = 2^r - 1$ points $a_0a_1 \ldots a_{r-1}$ in $\mathbf{P}_2^{r-1}$ is re-denoted by the integer it represents as a binary r-tuple (with hexadecimal read-out, if $r \leq 4$) in which case the reading must be started at the leftmost $a_i \neq 0$ ($i \in [0, r)$). In this wayi, $(0, 2^r)$ is taken to represent $\mathbf{P}_2^{r-1}$.

We identify $\mathbf{P}_2^{r-2}$ with the $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$ represented by the integer interval $(0, 2^{r-1})$ and call it the *initial copy* $\mathcal{P}_2^{r-2}$ of $\mathbf{P}_2^{r-2}$ in $\mathbf{P}_2^{r-1}$.

The points of $\mathbf{P}_2^{r-2}$ are taken as the *directions of parallelism* of the affine space $A(r-1)$ obtained from $\mathbf{P}_2^{r-1} \setminus \mathbf{P}_2^{r-2}$ by *puncturing* the first entry $a_0 = 1$ of its points $a_0a_1 \ldots a_{r-1} = 1a_1 \ldots a_{r-1}$. Each of the $2^{r-2} - 1$ $(r-3)$-subspaces S of the initial copy $\mathcal{P}_2^{r-2}$ in $\mathbf{P}_2^{r-1}$ yields exactly two non-initial $(r-2)$-subspaces of $\mathbf{P}_2^{r-1}$, namely:

(i) an $(r-2)$-subspace formed by the points of S and the *complements in* $n = 2^r - 1$ of the points $i \in \mathbf{P}_2^{r-2} \setminus S$, namely the points $n - i$;

(ii) an $(r-2)$-subspace formed by the point $n = 2^r - 1$, the points i of S and their complements $n - i$ in n.

This representation of the $(r-3)$-subspaces of $\mathbf{P}_2^{r-1}$ determines a representation of the corresponding subspaces of $A(r)$ and that of their complementary subspaces in $\mathbf{P}_2^{r-1}$, these considered as subspaces at ∞.

Any subspace of $\mathbf{P}_2^{r-1}$ of positive dimension is presentable via an initial copy of a lower-dimensional subspace, by an immediate generalization of items **(i)**–**(ii)**.

Example 2.2. $\mathbf{P}_2^2$ is formed by the nonzero binary 3-tuples 001, 010, 011, 100, 101, 110, 111, re-denoted respectively by their hexadecimal integer forms: $1, 2, 3, 4, 5, 6, 7$. So $\mathbf{P}_2^2 \subset \mathbf{P}_2^3$ is represented as $\{1, \ldots, 7\}$ immersed into $\{1, \ldots, f = 15\}$ by sending $1 := 001$ onto $1 := 0001$; $2 := 010$ onto $2 := 0010$, etc., that is, by prefixing a zero to each 3-tuple.

Now, puncturing the first entry of the 4-tuples of $\mathbf{P}_2^3$ (but writing the punctured entry between parentheses) yields: (0)001 as the direction of parallelism of the affine lines of $A(3)$ with point sets (rewritten in hexadecimal notation in $\mathbf{P}_2^3$ without delimiting braces or separating commas): $\{(1)000, (1)001\} = 89$, $\{(1)010, (1)011\} = ab$, $\{(1)100, (1)101\} = cd$, $\{(1)110, (1)111\} = ef$.

We mention now the binary projective 1-space $\mathbf{P}_2^1$, formed by the points 1, 2, 3 and the line 123. Items **(i)** and **(ii)** apply to $\mathbf{P}_2^1$ and determine the respective planes $123ba98 = 123(f-4)(f-5)(f-6)(f-7)$ and $123fedc = 123f(f-1)(f-2)(f-3)$ in $\mathbf{P}_2^3$.

By writing the members of the initial copy $\mathcal{P}_2^{r-2}$ in $\mathbf{P}_2^{r-1}$ in their numerical order, then writing the complement of each of them immediately underneath, and finally the symbol n under the two resulting rows, we distinguish the codimension-1 subspaces (also called projective hyperplanes) of $\mathbf{P}_2^{r-1}$, other than $\mathcal{P}_2^{r-2}$, with their elements in bold font, in contrast with the remaining elements, in normal font, as shown here for $r = 3$, and partially for $r = 4$:

$$
\left|\begin{array}{cccccc|cccccc}
\mathbf{1}23 & \mathbf{1}\mathbf{45} & 1\mathbf{2}3 & 1\mathbf{2}3 & 12\mathbf{3} & 12\mathbf{3} & \mathbf{123}4567 & \mathbf{123}4567 & \mathbf{1}23\mathbf{45}67 & \mathbf{1}23\mathbf{45}67 & \cdots & 12\mathbf{34}56\mathbf{7} & 12\mathbf{34}56\mathbf{7} \\
6\mathbf{54} & \mathbf{6}54 & \mathbf{6}5\mathbf{4} & 6\mathbf{5}4 & \mathbf{65}4 & 65\mathbf{4} & edc\mathbf{ba98} & \mathbf{edc}ba98 & e\mathbf{dc}ba\mathbf{98} & \mathbf{e}dc\mathbf{ba}98 & \cdots & \mathbf{ed}cb\mathbf{a9}8 & ed\mathbf{cb}a9\mathbf{8} \\
7 & \mathbf{7} & 7 & \mathbf{7} & 7 & \mathbf{7} & f & \mathbf{f} & f & \mathbf{f} & \cdots & f & \mathbf{f}
\end{array}\right|
$$

Let $(r, s) \in \mathbb{Z}^2$, $r > 2$, $\sigma \in (0, r-1)$ and A_0 be a σ-subspace of $\mathbf{P}_2^{r-1}$. The set of $(\sigma+1)$-subspaces of $\mathbf{P}_2^{r-1}$ that contain A_0 is said to be the (r, σ)-*pencil* of $\mathbf{P}_2^{r-1}$ *through* A_0. A linearly ordered presentation of such a set is an (r, σ)-*ordered pencil* of $\mathbf{P}_2^{r-1}$ *through* A_0. There are $(2^{r-\sigma} - 1)!$ (r, σ)-ordered pencils of $\mathbf{P}_2^{r-1}$ through A_0, since there are $2^{r-\sigma} - 1$ $(\sigma+1)$-subspaces containing A_0 in $\mathbf{P}_2^{r-1}$.

An (r, σ)-ordered pencil v of $\mathbf{P}_2^{r-1}$ through A_0 has the form $v = (A_0 \cup A_1, \ldots, A_0 \cup A_{m_0})$, where $A_1, \ldots, A_{m_0}$ are the nontrivial cosets of $\mathbf{F}_2^r$ mod its subspace $A_0 \cup \{\mathbf{0}\}$, with $m_0 = 2^{r-\sigma} - 1$. As a shorthand for this, we just write $v = (A_0, A_1, \ldots, A_{m_0})$ and consider $A_1, \ldots, A_{m_0}$ as the *non-initial* entries of v.

3 Graphs of Ordered Pencils

Definition 3.1. *Let $\mathcal{G}_r^\sigma$ be the graph whose vertices are the (r, σ)-ordered pencils $v = (A_0, A_1, \ldots, A_{m_0}) = (A_0(v), A_1(v), \ldots, A_{m_0}(v))$ of $\mathbf{P}_2^{r-1}$, with an edge precisely between each two vertices $v = (A_0, A_1 \ldots, A_{m_0})$ and $v' = (A'_0, A'_1 \ldots, A'_{m_0})$ that satisfy the following three conditions:*

1. *$A_0 \cap A_0'$ is a $(\sigma-1)$-subspace of $\mathbf{P}_2^{r-1}$;*

2. *$A_i \cap A_i'$ is a nontrivial coset of $\mathbf{F}_2^r$ mod $(A_0 \cap A_0') \cup \{\mathbf{0}\}$, for each $1 \leq i \leq m_0$;*

3. *$\cup_{i=1}^{m_0}(A_i \cap A_i')$ is an $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$.*

Let v_r^σ be the lexicographically smallest (r,σ)-ordered pencil which is a vertex of $\mathcal{G}_r^\sigma$ and let G_r^σ be the component of $\mathcal{G}_r^\sigma$ containing v_r^σ.

Item **3** in Definition 3.1 is needed only if $(r,\sigma) \neq (3,1)$; in this case, let us denote the $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$ required in such item **3** by $U(v,v') = \cup_{i=1}^{m_0}(A_i \cap A_i')$.

Example 3.2. Let u_r^σ be the lexicographically smallest neighbor of v_r^σ in G_r^σ. Then:

$$\begin{array}{lll} v_3^1 = (1,23,45,67), & u_3^1 = (2,13,46,57), & U(v_3^1,u_3^1) = 347; \\ v_4^1 = (1,23,45,67,89,ab,cd,ef), & u_4^1 = (2,13,46,57,8a,9b,ce,df), & U(v_4^1,u_4^1) = 3478bcf; \\ v_4^2 = (123,4567,89ab,cdef), & u_4^2 = (145,2367,89cd,abef), & U(v_4^2,u_4^2) = 16789ef. \end{array}$$

Theorem 3.3. *Both $\mathcal{G}_r^\sigma$ and G_r^σ are $\{K_{2s}, T_{st,t}\}_{K_2}$-H (respectively, $\{K_{2s} \not\subset T_{st,t}, T_{st,t}\}_{K_2}$-H), if $t < 2s$ (respectively, $t \geq 2s$). Moreover, $\mathcal{G}_r^\sigma$ is:* **(a)** *of order $\binom{r}{\sigma}_2 m_0!$, where $\binom{r}{\sigma}_2 = \Pi_{i=1}^{r-\sigma}\frac{2^{i+\sigma}-1}{2^i-1}$ is Gaussian binomial coefficient (number of σ-subspaces A_0 in $\mathbf{P}_2^{r-1}$);* **(b)** *$s(t-1)m_0$-regular;* **(c)** *uniquely representable as an edge-disjoint union of $m_0|V(\mathcal{G}_r^\sigma)|s^{-1}t^{-1}$ (respectively, $(2^\sigma-1)|V(\mathcal{G}_r^\sigma)|$) copies of K_{2s} (respectively, $T_{st,t}$), with exactly m_0 (respectively, m_1) such copies incident to each vertex, no two sharing more than one vertex, and each edge of $\mathcal{G}_r^\sigma$ present in exactly one such copy. In case $r-\sigma=2$, then G_r^σ is K_4-UH.*

A proof of Theorem 3.3 is given in Subsection 3.1 prior to Proposition 3.7. The following conjecture is confirmed for $r \leq 8$ (via Proposition 3.7) and for $\rho = r-\sigma \leq 5$ (via Propositiom 4.8) in Theorem 4.10.

Conjecture 3.4. *A graph G_r^σ answering affirmatively* Question 1.5 *(or modified for $\{X_0 \not\subset X_1, X_1\}$-H graphs, if $t \geq 2s$) has $m_0 = 2^\rho - 1$, $m_1 = 2s(2^\sigma-1)$, $\ell_0 = \frac{m_0}{st}|V(G_r^\sigma)|$, $\ell_1 = (2^\sigma-1)|V(G_r^\sigma)|$, with $|V(G_r^\sigma)| = \Pi_{i=1}^\rho(2^{i-1}(2^{i+\sigma}-1)) = \binom{r}{\sigma}_2\Pi_{i=1}^\rho(2^{i-1}(2^i-1))$.*

Remark 3.5. For each $(r-1,\sigma-1)$-ordered pencil $U = (U_0,U_1,\dots,U_{m_0})$ of an $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$ (where m_0 is as in Definition 3.1, and $U_0 = \emptyset$ if $\sigma = 1$) there is a copy $[U]_r^\sigma = [U_0,U_1,\dots,U_{m_0}]_r^\sigma$ of K_{2s} in $\mathcal{G}_r^\sigma$ induced by those $(A_0,A_1,\dots,A_{m_0}) \in V(\mathcal{G}_r^\sigma)$ with $A_i \supset U_i$, for $1 \leq i \leq m_0$. For example, the induced copies of K_8 in G_4^1 incident to v_4^1 are:

$$\begin{array}{llll} [\emptyset,2,4,6,8,a,c,e]_4^1, & [\emptyset,3,4,7,8,b,c,f]_4^1, & [\emptyset,2,5,7,8,a,d,f]_4^1, & [\emptyset,3,5,6,8,b,d,e]_4^1, \\ [\emptyset,2,4,6,9,b,d,f]_4^1, & [\emptyset,3,4,7,9,a,d,e]_4^1, & [\emptyset,2,5,7,9,b,c,\ e]_4^1 & [\emptyset,3,5,6,9,a,c,f]_4^1. \end{array}$$

As an additional example, the induced copies of K_4 in G_4^2 incident to v_4^2 are:

$$\begin{array}{l} [1,45,89,cd]_4^2,\ [1,67,89,ef]_4^2,\ [2,57,8a,df]_4^2,\ [2,46,8a,ce]_4^2,\ [3,47,8b,cf]_4^2,\ [3,56,8b,de]_4^2, \\ [1,45,ab,ef]_4^2,\ [1,67,ab,cd]_4^2,\ [2,57,9b,ce]_4^2,\ [2,46,9b,df]_4^2,\ [3,47,9a,de]_4^2,\ [3,56,9a,cf]_4^2. \end{array}$$

Remark 3.6. For each $(\sigma+1)$-subspace W of $\mathbf{P}_2^{r-1}$ and each $i \in [1, m_0]$, there is a copy $[(W)_i]_r^\sigma$ of $T_{st,t}$ induced in $\mathcal{G}_r^\sigma$ by the vertices $(A_0, A_1, \ldots, A_{m_0})$ of $\mathcal{G}_r^\sigma$ having A_0 as a σ-subspace of W and $A_i \subset \mathbf{P}_2^{r-1} \setminus A_0$, for $i = 1, \ldots, m_0$. For example, the three 4-vertex parts of the lexicographically first and last (of the seven) copies of $T_{12,3}$ in G_4^1 incident to v_4^1, namely $[(\mathbf{P}_2^1)_1]_4^1 = [(123)_1]_4^1$ and $[(\mathbf{P}_2^1)_7]_4^1 = [(1ef)_7]_4^1$, are denoted (columnwise):

$[(123)_1]_4^1$	$(1,23,45,67,89,ab,cd,ef)$	$(2,13,46,57,8a,9b,ce,df)$	$(3,12,47,56,8b,9a,cf,de)$
	$(1,23,45,67,ab,89,ef,cd)$	$(2,13,46,57,9b,8a,df,ce)$	$(3,12,47,56,9a,8b,de,cf)$
	$(1,23,67,45,89,ab,ef,cd)$	$(2,13,57,46,8a,9b,df,ce)$	$(3,12,56,47,8b,9a,de,cf)$
	$(1,23,67,45,ab,89,cd,ef)$	$(2,13,57,46,9b,8a,ce,df)$	$(3,12,56,47,9a,8b,cf,de)$
......	......	......	......
......	......	......	......
$[(1ef)_7]_4^1$	$(1,23,45,67,89,ab,cd,ef)$	$(e,2c,4a,68,79,5b,3d,1f)$	$(f,2d,4b,69,78,5a,3c,1e)$
	$(1,23,ab,89,67,45,cd,ef)$	$(e,2c,5b,79,68,4a,3d,1f)$	$(f,2d,5a,78,69,4b,3c,1e)$
	$(1,cd,45,89,67,ab,23,ef)$	$(e,3d,4a,79,68,5b,2c,1f)$	$(f,3c,4b,78,69,5a,2d,1e)$
	$(1,cd,ab,67,89,45,23,ef)$	$(e,3d,5b,68,79,4a,2c,1f)$	$(f,3c,5a,69,78,4b,2d,1e)$

The first and last (of the 14) 2-vertex parts of the three (columnwise) copies of $T_{14,7}$ in G_4^2 incident to v_4^2, namely $[(\mathbf{P}_2^2)_1]_4^2 = [(1234567)_1]_4^2$, $[(12389ab)_2]_4^2$ and $[(123cdef)_3]_4^2$, are:

$[(1234567)_1]_4^2$	$[(12389ab)_2]_4^2$	$[(123cdef)_3]_4^2$
$(123,4567,89ab,cdef)$	$(123,4567,89ab,cdef)$	$(123,4567,89ab,cdef)$
$(123,4567,cdef,89ab)$	$(123,cdef,89ab,4567)$	$(123,89ab,4567,cdef)$
......	......	......
......	......	......
$(356,1247,8bde,9acf)$	$(39a,47de,128b,56cf)$	$(3de,479a,568b,12cf)$
$(356,1247,9acf,8bde)$	$(39a,56cf,128b,47de)$	$(3de,568b,479a,12cf)$

3.1 Automorphisms

Recall from Definition 3.1 that G_r^σ is the connected component containing the vertex $v_r^\sigma = (A_0(v_r^\sigma), \ldots, A_{m_0}(v_r^\sigma))$ in $\mathcal{G}_r^\sigma$. Let $W_r^\sigma = \{w \in V(G_r^\sigma) : A_0(w) = A_0(v_r^\sigma)\}$. For each $w \in W_r^\sigma$, let ϵ_w be an automorphism of G_r^σ such that $\epsilon_w(v_r^\sigma) = w$, a permutation of the non-initial entries of v_r^σ. Let the composition of permutations be taken from left to right, that is with the leftmost factor acting first. The *open neighborhood* $N_{G_r^\sigma}(w)$ of w in G_r^σ is the subgraph induced by the neighbors of w. To characterize the automorphisms of G_r^σ, it suffices to determine the subgroup $\mathcal{N}_r^\sigma = \mathcal{A}(N_{G_r^\sigma}(v_r^\sigma))$ of $\mathcal{A}(G_r^\sigma)$ as well as the automorphisms ϵ_w. Proposition 3.7 below establishes the cardinality of $\mathcal{N}_r^\sigma$ for $r \le 8$, and thus the corresponding cardinality of $\mathcal{A}(G_r^\sigma)$. The automorphisms ϵ_w yield important information about the order and diameter of the graphs G_r^σ. In items **(A)**-**(C)** below, a set of generators for $\mathcal{N}_r^\sigma$ is given by means of products of transpositions of the form $(\alpha\ \beta)$, where α and β are two affine σ-subspaces of $\mathbf{P}_2^{r-1}$ that have a common $(\sigma-1)$-subspace $\theta_{\alpha,\beta}$ at ∞. For any such $(\alpha\ \beta)$, we define the *affine difference* $\chi_{\alpha,\beta}$ to be the affine σ-subspace of $\mathbf{P}_2^{r-1}$ formed by the third points c in the lines determined by each two points $a \in \alpha$, $b \in \beta$. Here, it suffices to take all such c for a fixed $a \in \alpha$ and a variable $b \in \beta$. Needed in display (1) below, we denote $(\alpha\ \beta)$ by $[\theta_{\alpha,\beta} \cdot \chi_{\alpha,\beta}(\alpha\ \beta)]$. If $0 < h \in \mathbb{Z}$, then a permutation ϕ of the affine σ-subspaces of $\mathbf{P}_2^{r-1}$ that is a product of transpositions $(\alpha_i\ \beta_i)$, for

$1 \leq i \leq h$, with a common $\theta = \theta_{\alpha_i,\beta_i}$ and a common $\chi = \chi_{\alpha_i,\beta_i}$ will be indicated

$$\phi = \prod_{i=1}^{h} (\alpha_i\ \beta_i) = [\theta.\chi \prod_{i=1}^{h} (\alpha_i\ \beta_i)], \tag{1}$$

where the points of $\mathbf{P}_2^{r-1}$ that are not in the pairs of parentheses $(\alpha_1\ \beta_1), \cdots, (\alpha_h\ \beta_h)$ are fixed points of ϕ. To each such ϕ we associate a permutation ψ of $\mathbf{P}_2^{r-1}$ that permutes the non-initial entries of the ordered pencils that are vertices of G_r^σ. Concretely, ψ is obtained by replacing each number a in the entries of the transpositions of ϕ by the integer $\lfloor a/2^\sigma \rfloor$ and setting only one representative of each set of repeated resulting transpositions in expressing ψ. We write $\omega = (\Pi\phi).\psi$ to express a product of permutations ϕ of G_r^σ having a common associated permutation ψ. It is convenient to write $\Pi\phi = \phi^\omega$ and $\psi = \psi^\omega$. In this context, () stands for the identity permutation. Now, a set of generators of $\mathcal{A}(G_r^\sigma)$ is formed by those $\omega = \phi^\omega.\psi^\omega$ expressible as follows:

(A) Given a point $\pi \in \mathbf{P}_2^\rho = \mathbf{P}_2^{r-\sigma}$ and an $(r-2)$-subspace α of $\mathbf{P}_2^{r-1}$ containing $\{\pi\} \cup \mathbf{P}_2^{\sigma-1}$, let $\phi^\omega = \phi^\omega(\pi, \alpha)$ be the product of all transpositions of affine σ-spaces of $\mathbf{P}_2^{r-1}$ with a common $(\sigma-1)$-subspace at ∞ in $\mathbf{P}_2^{\sigma-1}$ and a common affine difference containing π and contained in $(\alpha \setminus \mathbf{P}_2^{\sigma-1})$. Let $\psi^\omega(\pi, \alpha)$ be the ψ^ω associated to ϕ^ω. Some examples of triples (ω, π, α) here are (in hexadecimal notation or its continuation in the English alphabet, from $10 = a$ passing through $15 = f$ and up to $31 = v$) as follows:

$$\begin{array}{llll}
G_3^1: & (\omega{=}[\emptyset.2(4\ 6)(5\ 7)].1(2\ 3), & \pi{=}2, & \alpha{=}123); \\
 & (\omega{=}[\emptyset.3(4\ 7)(5\ 6)].1(2\ 3), & \pi{=}3, & \alpha{=}123); \\
 & (\omega{=}[\emptyset.6(2\ 4)(3\ 5)].3(1\ 2), & \pi{=}6, & \alpha{=}167); \\
 & (\omega{=}[\emptyset.1(2\ 3)(6\ 7)].(), & \pi{=}1, & \alpha{=}145).
\end{array}$$

$$\begin{array}{llll}
G_4^1: & (\omega{=}[\emptyset.2(8\ a)(9\ b)(c\ e)(d\ f)].1(4\ 5)(6\ 7), & \pi{=}2, & \alpha{=}1234567); \\
 & (\omega{=}[\emptyset.4(8\ c)(9\ d)(a\ e)(b\ f)].2(4\ 6)(5\ 7), & \pi{=}4, & \alpha{=}1234567); \\
 & (\omega{=}[\emptyset.2(4\ 6)(5\ 7)(8\ a)(9\ b)].1(2\ 3)(4\ 5), & \pi{=}2, & \alpha{=}123cdef); \\
 & (\omega{=}[\emptyset.c(4\ 8)(5\ 9)(6\ a)(7\ b)].6(2\ 4)(3\ 5), & \pi{=}c, & \alpha{=}123cdef); \\
 & (\omega{=}[\emptyset.5(8\ d)(9\ c)(a\ f)(b\ e)].2(4\ 6)(5\ 7); & \pi{=}5, & \alpha{=}1234567); \\
 & (\omega{=}[\emptyset.1(2\ 3)(6\ 7)(a\ b)(e\ f)].(); & \pi{=}1, & \alpha{=}14589cd); \\
 & (\omega{=}[\emptyset.6(2\ 4)(3\ 5)(a\ c)(b\ d)].3(1\ 2)(5\ 6); & \pi{=}6, & \alpha{=}16789ef).
\end{array}$$

$$\begin{array}{llll}
G_4^2: & (\omega{=}[1.45(89\,cd)(ab\ ef)][2.46(8a\ ce)(9b\ df)][3.47(8b\ cf)(9a\ de)].1(2\,3), & \pi{=}4,; & \alpha{=}1234567) \\
 & (\omega{=}[1.cd(45\,89)(67\ ab)][2.ce(46\ 8a)(57\,9b)][3.cf(47\,8b)(56\,9a)].3(1\,2), & \pi{=}c, & \alpha{=}123cdef). \\
G_5^2: & (\omega{=}[1.op(89\ gh)(ab\ ij)(cd\ kl)(ef\ mn)][2.oq(8a\ gi)(9b\ hj)(ce\ km)(df\ ln)] & & \\
 & \quad [3.or(8b\ gj)(9a\ hi)(cf\ kn)(de\ lm)].6(2\ 4)(3\ 5), & \pi{=}o, & \alpha{=}1234567opqrstuv). \\
G_5^3 & (\omega{=}[123.89ab(ghij\ opqr)(klmn\ stuv)][145.89cd(ghkl\ opst)(ijmn\ qruv)] & & \\
 & \quad [167.89ef(ghmn\ opuv)(ijkl\ qrst)][246.8ace(gikm\ oqsu)(hjln\ prtv)] & & \\
 & \quad [257.8adf(giln\ oqtv)(hjkm\ prsu)][347.8bcf(gjkn\ orsv)(hilm\ pqtu)] & & \\
 & \quad [356.8bde(gjkn\ ortu)(hilm\ pqsv)].\,1(2\ 3), & \pi{=}8, & \alpha{=}123456789abcdef).
\end{array}$$

(B) Given a point $\pi \in \mathbf{P}_2^{\sigma-1}$ and an $(r-2)$-subspace α of $\mathbf{P}_2^{r-1}$ containing $\mathbf{P}_2^{\rho-1}$, let $\phi^\omega = \phi(\pi, \alpha)$ be the product of the transpositions of pairs of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ not contained in $(\alpha \setminus \mathbf{P}_2^{\sigma-1})$ with a common $(\sigma-1)$-subspace π at ∞ and a common affine difference $(\mathbf{P}_2^{\sigma-1} \setminus \pi)$. In each case, $\psi^\omega = ()$. Some triples (ω, π, α)

are:

G_4^2: $(\omega=[1.23(89\ ab)(cd\ ef)].()$, $\pi=1$, $\alpha=1234567)$;
$(\omega=[1.23(45\ 67)(cd\ ef)].()$, $\pi=1$, $\alpha=12389ab)$;
$(\omega=[1.23(45\ 67)(89\ ab)].()$, $\pi=1$, $\alpha=123cdef)$;
$(\omega=[2.13(8a\ 9b)(ce\ df)].()$, $\pi=2$, $\alpha=1234567)$;
$(\omega=[2.13(46\ 57)(ce\ df)].()$, $\pi=2$, $\alpha=12389ab)$;
$(\omega=[2.13(46\ 57)(8a\ 9b)].()$, $\pi=2$, $\alpha=123cdef)$;
$(\omega=[3.12(8b\ 9a)(cf\ de)].()$, $\pi=3$, $\alpha=1234567)$;
$(\omega=[3.12(47\ 56)(cf\ de)].()$, $\pi=3$, $\alpha=12389ab)$;
$(\omega=[3.12(47\ 56)(8b\ 9a)].()$, $\pi=3$, $\alpha=123cdef)$.

G_5^2: $(\omega=[2.13(gi\ hj)(km\ ln)(oq\ pr)(su\ tv)].()$, $\pi=2$, $\alpha=123456789abcdef)$;
$(\omega=[2.13(8a\ 9b)(ce\ df)(pr\ oq)(su\ tv)].()$, $\pi=2$, $\alpha=1234567ghijklmn)$;
$(\omega=[1.23(89\ ab)(cd\ ef)(op\ qr)(st\ uv)].()$, $\pi=1$, $\alpha=1234567ghijklmn)$;
$(\omega=[1.23(gh\ ij)(kl\ mn)(op\ qr)(st\ uv)].()$, $\pi=1$, $\alpha=123456789abcdef)$;
$(\omega=[3.12(8b\ 9a)(cf\ de)(or\ pq)(sv\ tu)].()$, $\pi=3$, $\alpha=1234567ghijklmn)$;
$(\omega=[3.12(gj.hi)(kn\ ln)(or\ pq)(st\ tu)].()$, $\pi=3$, $\alpha=123456789abcdef)$.

G_5^3: $\omega=([347.1256(gjkn\ hilm)(pqsv\ ortu)].()$, $\pi=347$, $\alpha=123456789abcdef)$.

(C) Given a point $\pi \in \mathbf{P}_2^{\sigma-1}$ and an $(r-2)$-subspace α of $\mathbf{P}_2^{r-1}$ with $\pi \in \alpha$, let ϕ^ω be the product of the transpositions of pairs of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ not contained in α with common $(\sigma-1)$-subspace at ∞ contained in α and common affine difference contained in α and containing π. Again, $\psi^\omega = ()$. Some triples (ω, π, α) here are:

G_4^2: $(\omega=[4.15(26\ 37)][5.14(27\ 36)][8.19(2a\ 3b)][9.18(2b\ 3a)][c.1d(2e\ 3f)][d.1c(2f\ 3e)].()$, $\pi=1$, $\alpha=3478bcf)$;
$(\omega=[4.37(15\ 26)][7.34(16\ 25)][8.3b(19\ 2a)][b.38(1a\ 29)][c.3f(1d\ 2e)][f.3c(1e\ 2d)].()$, $\pi=3$, $\alpha=1459cd)$.

G_5^2: $(\omega=[4.37(15\ 26)][7.34(16\ 25)][8.3b(19\ 2a)][b.38(1a\ 29)][c.3f(1d\ 2e)]$
$[f.3c(1e\ 2d)][g.3j(1h\ 2i)][j.3g(1i\ 2h)][k.3n(1l\ 2m)][n.3k(1m\ 2l)]$
$[o.3r(1p\ 2q)][r.3o(1a\ 2p)][v.3s(1u\ 2t)][s.3v(1t\ 2u)].()$, $\pi=3$, $\alpha=3478bcfgjknorsv)$.

G_5^3: $(\omega=[189.67ef(23ab\ 45cd)][1ef.6789(23cd\ 45ab)][1gh.67mn(23ij\ 45kl)]$
$[1mn.67gh(23kl\ 45ij)][1op.67uv(23qr\ 45st)][1uv.67op(23st\ 45qr)].()$, $\pi=167,\alpha=16789efghmnopuv)$;

$(\omega=[189.23ab(45cd\ 67ef)][1ab.2389(45ef\ 67cd)]1kl.23mn(45gh\ 67ij)]$
$[1mn.23kl(45ij\ 67gh)][1st.23uv(45op\ 67qr)][1uv.23st(45qr\ 67op)].(),i$ $\pi=123$, $\alpha=12389abklmnstuv)$.

Proof. (of Theorem 3.3). We treat the case $t < 2s$ and leave remaining details to the reader. Because of the properties of $\mathbf{P}_2^{r-1}$ provided in Sections 2-3, $\mathcal{G}_r^\sigma$ and its connected components satisfy Definition 1.1 with $\mathcal{C} = \mathcal{C}'$, where $\mathcal{C}'$ is formed by members of the class (A), namely the copies of $X_0 = K_{2s}$ in Remark 3.5, and members of the class (D), namely the copies of $X_1 = T_{st,t}$ in Remark 3.6. Moreover, $\mathcal{G}_r^\sigma$ is a $\mathcal{C}'$-H graph uniquely expressible as an edge-disjoint union U_0 of copies of X_0, namely those in Remark 3.5, and as an edge-disjoint union U_1 of copies of X_1, namely those in Remark 3.6. Let us see it satisfies items **(i)**-**(iv)** of Conjecture 1.4:

Item **(i)** holds because $\mathcal{C}'$ contains just one copy of K_{2s} and one copy of $T_{st,t}$. Item **(iv)** holds because each edge of $\mathcal{G}_r^\sigma$ is a projective line of $\mathbf{P}_2^{r-1}$ and thus the intersection of the projective subspaces represented by a corresponding copy of K_{2s} as in Remark 3.5 and a corresponding copy of $T_{st,t}$ as in Remark 3.6.

For example for $(r, \sigma) = (4, 1)$, the copies of $K_{2s} = K_8$ and $T_{st,t} = T_{12,3}$ denoted respectively by $[\emptyset, 2, 4, 6, 8, a, c, e\,]_4^1$ and $[(123)_1]_4^1$ intersect just at the adjacent vertices $v_4^1 = (1, 23, 45, 67, 89, ab, cd, ef)$ and $w_4^1 = (3, 12, 47, 56, 8b, 9a, cf, de)$, and at the edge (v_4^1, w_4^1). For $(r, \sigma) = (4, 2)$, the copies of $K_{2s} = K_4$ and $T_{st,t} = T_{14,2}$ denoted respectively by $[1,45,89,cd\,]^2_4$ and $[(1234567)_1]^2_4$ intersect just at the adjacent vertices $v^2_4=(123,4567,89ab,cdef)$ and $w^2_4=(145,2367,89cd,abef)$, and at the edge v^2_4,w^2_4.

This way, the lexicographically smallest edge in $\mathcal{G}^\sigma_r$ belongs both to the lexicographically smallest copies of X_0 and X_1 and constitutes their intersection. This situation is carried out to the remaining edges of $\mathcal{G}^\sigma_r$ via the automorphisms presented above. Item **(ii)** holds since no copy of $T_{st,t}$ is obtained other than those in Remark 3.6. In fact, no copy of K_{2s} in Remark 3.5 contains a copy of $T_{st,t}$, and no copy of K_{2s} not contained in a copy of $T_{st,t}$ exists in $\mathcal{G}^\sigma_r$ other than those in Remark 3.5, this fact concluded from item **(iv)**. Here we took care of cases such as $(r,\sigma)=(4,1)$ for which copies or K_{2s} may be found inside any copy of $T_{st,t}$. As for item **(iii)**, recall that a vertex A of $\mathcal{G}^\sigma_r$ is an (r,σ)-ordered pencil of $\mathbf{P}^{r-1}_2$ through some σ-subspace A_0 of $\mathbf{P}^{r-1}_1$. This contains the $(r-1,\sigma-1)$-ordered pencils $U=(U_0,U_1,\dots,U_{m_0})$ of the $(r-2)$-subspaces of $\mathbf{P}^{r-1}_2$ induced by the vertices $(A_0,A_1,\dots,A_{m_0})$ of $\mathcal{G}^\sigma_r$ with $A_i\supset U_i$, for $1\le i\le m_0$, The corresponding copies $[U]^\sigma_r=[U_0,U_1,\dots,U_{m_0}]^\sigma_r$ of K_{2s} are just all those containing A. Clearly, A is the only vertex of $\mathcal{G}^\sigma_r$ that these copies have in common, which takes care of half of item **(iii)**. As for the other half, the vertex A is contained in the $(\sigma+1)$-subspaces W of $\mathbf{P}^{r-1}_2$ with an index $i=i_W\in[1,m_0]$ determining a copy $[(W)_i]^\sigma_r$ of $T_{st,t}$ induced in $\mathcal{G}^\sigma_r$ by the vertices $(A_0,A_1,\dots,\,A_{m_0})$ of $\mathcal{G}^\sigma_r$ with A_0 as a σ-subspace of W and $A_i\subset W\setminus A_0$, for $i=1,\dots,m_0$. Clearly, A is the only vertex of $\mathcal{G}^\sigma_r$ that these copies have in common, proving item **(iii)**. Now, the first sentence in the statement holds, as G^σ_r is a component of $\mathcal{G}^\sigma_r$. However, if $r-\sigma=2$, then in Definition 3.1, item **3** is implied by items **1-2**, which insures that both $\mathcal{G}^\sigma_r$ and G^σ_r are K_4-UH. On the other hand, the number of σ-subspaces F' in $\mathbf{P}^{r-1}_2$ is $\#F'=\binom{r}{\sigma}_2$. For each such F' taken as initial entry A_0 of some vertex v of $\mathcal{G}^\sigma_r$, there are m_0 classes mod $F'\cup\mathbf{0}$ permuted and distributed from left to right into the remaining positions A_i of v. Thus,

$$|\mathcal{G}^\sigma_r|=(\#F')m_0!\,.$$

Each vertex v of $\mathcal{G}^\sigma_r$ is the intersecting vertex of exactly m_0 copies of $T_{st,t}$. Since the regular degree of $T_{st,t}$ is $s(t-1)$, then the regular degree of $\mathcal{G}^\sigma_r$ is $s(t-1)m_0$. The edge numbers of $T_{st,t}$ and $\mathcal{G}^\sigma_r$ are respectively $s^2t(t-1)/2$ and $s(t-1)m_0|V(\mathcal{G}^\sigma_r)|/2$, so $\mathcal{G}^\sigma_r$ is the edge-disjoint union of $m_0|V(\mathcal{G}^\sigma_r)|s^{-1}t^{-1}$ copies of $T_{st,t}$. □

Proposition 3.7. *For $\sigma>0$ and $\rho=r-\sigma>1$ (so $r>2$), let*

$$\begin{aligned}&A=2^{\sigma+1}-1+(\rho-2)(2^\sigma+1)+\max(\rho-3,0),\\&B=\Pi^\rho_{i=1}(2^i-1)\ \textit{and}\\&C=(2^\sigma-1)!\,.\end{aligned}$$

Then, at least for $r\le 8$, the cardinality of $\mathcal{N}^\sigma_r$ is 2^ABC, where the last term in the sum expressing A differs from $\rho-3$ only if $\rho=2$.

Proof. The statement was established computationally from the generators of $\mathcal{N}^\sigma_r$ presented in items **(A)**–**(C)**. □

Question 3.8. *Is the statement of* Proposition 3.7 *valid for all $r>8$?*

4 Order and Diameter

In the terminology of Subsection 3.1, the set of automorphisms ϵ_w, that we will denote $\mathcal{H}_\rho = \{\epsilon_w : w \in W_r^\sigma\}$, admits the structure of a group under composition.

Estimates of the order and diameter of G_r^σ are obtained by considering a graph H_ρ whose vertex set is $\mathcal{H}_\rho = \mathcal{H}_{r-\sigma}$, $(r > 3;\ \sigma \in (0, r-1))$, actually defined in Subsection 4.1 below. The elements of $\mathcal{H}_\rho$ are classified into auxiliary *types* in Subsections 4.2-4.6, presenting a direct relation with the distance to the identity permutation in Theorem 4.1, and less finely (for suitable brevity) into *super-types* in Subsection 4.8. Moreover, a set of $V(H_{\rho-1})$-coset representatives in $V(H_\rho)$ (Subsection 4.7) will allow us to achieve (via Proposition 4.4) the conjectured numerical properties of G_r^σ (Conjecture 3.4) in Theorem 4.10.

4.1 An Auxiliary Graph

The diameter of G_r^σ is realized by the distance from $v_r^\sigma = (A_0(v_r^\sigma), A_1(v_r^\sigma), \ldots, A_{m_0}(v_r^\sigma))$ to some vertex $w \in V(G_r^\sigma) \setminus \{v_r^\sigma\}$. To determine one such w, we use the distance-2 graph $(G_r^\sigma)_2$ with vertex set $V(G_r^\sigma)$ and edge set formed by the pairs of vertices at distance 2 from each other. Consider the subgraph H of $(G_r^\sigma)_2$ induced by W_r^σ. Clearly, $v_r^\sigma \in V(H)$. Moreover, H depends only on $\rho = r - \sigma$. So, we denote $H = H_\rho$. In particular, note that

$$\text{Diameter}(G_r^\sigma) \leq 2 \times \text{Diameter}(H_\rho).$$

Consider the case $(r, \sigma) = (3, 1)$. Denoting $B_1 = 23, B_2 = 45$ and $B_3 = 67$, let us assign to each vertex v of $H_2 = K_{3,3}$ the permutation that maps the sub-indices i of the entries A_i of v $(i = 1, 2, 3)$ into the sub-indices j of the pairs B_j correspondingly filling those entries A_i. This yields the following bijection from $V(H_2) = W_3^1$ onto the group $K = S_3$ of permutations of the point set of the projective line $\mathbf{P}_2^1$:

(1,23,45,67)	$\to$ 123()	(1,23,67,45)	$\to$ 1(23)
(1,45,23,67)	$\to$ 3(12)	(1,45,67,23)	$\to$ (123)
(1,67,23,45)	$\to$ (132)	(1,67,45,23)	$\to$ 2(13)

where each permutation on the right side of the arrow '$\to$' is expressed in cycle notation, presented with its nontrivial cycles written as usual between parentheses (but without separating spaces) and with fixed points, if any, written to the left of the leftmost pair of parentheses, for later convenience.

There exists a bijection from $V(H_\rho)$ onto the group $\mathcal{H}_\rho$ defined at the start of Subsection 3.1. The elements of $\mathcal{H}_\rho$ will be called $\mathcal{A}$-*permutations*. They yield an auxiliary notation for the vertices of H_ρ so we take $V(H_\rho) = \mathcal{H}_\rho$. For example, $v_r^\sigma \in V(H_\rho)$ is taken as the identity permutation $I_\rho = 123\ldots2^\rho$, with fixed-point set $\mathbf{P}_2^{\rho-1} = 123\ldots2^\rho$. In fact, $\mathcal{H}_\rho$ is formed by permutations of the non-initial entries in the ordered pencils that are vertices of G_r^σ, as were the permutations ψ^ω in Subsection 3.1, but now the permutation ϕ^ω composing with each ψ^ω an automorphism ω of G_r^σ is the identity $() = 123\ldots 2^\rho()$, since this automorphism ω takes v_r^σ onto some vertex $w \in W_r^\sigma$.

An ascending sequence $\{V(H_2) \subset V(H_3) \subset \cdots \subset V(H_\rho) \subset \cdots\}$ of $\mathcal{A}$-permutation groups is generated via the embeddings $\Psi_\rho : V(H_{\rho-1}) \to V(H_\rho)$, $(\rho > 2)$, defined by setting $\Psi_\rho(\psi)$ to equal the product of the $\mathcal{A}$-permutation ψ of $\mathbf{P}_2^{\rho-2} \subset \mathbf{P}_2^{\rho-1}$ by the permutation obtained from ψ by replacing each of its symbols i by $m_0 - i$, with m_0 becoming a fixed point of $\Psi_\rho(\psi)$. Let us call this construction of $\Psi_\rho(\psi)$ out of ψ the *doubling* of ψ. For example, $\Psi_3 : V(H_2) \to V(H_3)$ maps the elements of $V(H_2)$ as follows:

123	$\to$ 7123654() =1234567()	1(23)	$\to$ 71(23)6(54)=167(23)(45)
(123)	$\to$ 7(123)(654)=7(123)(465)	3(12)	$\to$ 73(12)4(65)=347(12)(56)
(132)	$\to$ 7(132)(645)=7(132)(456)	2(13)	$\to$ 72(13)5(64)=257(13)(46)

where each resulting $\mathcal{A}$-permutation in $V(H_3)$ is rewritten to the right of the equal sign by expressing, from left to right and lexicographically, first the fixed points and then the cycles. The three $\mathcal{A}$-permutations of $V(H_3)$ displayed to the right of the middle dividing vertical line above are of the form $abc(de)(fg)$, where ade and afg are lines of $\mathbf{P}_2^3$, namely: 123 and 145, for 167(23)(45); 312 and 356, for 347(12)(56); 213 and 246, for 257(13)(46).

A point of $\mathbf{P}_2^{\rho-1}$ playing the role of a in a product Π of $2^{\rho-2}$ disjoint transpositions, as in the three just cited $\mathcal{A}$-permutations, is said to be the *pivot* of Π. For example, for each point $p \in \{a, b, c\}$ of $\mathbf{P}_2^2$ there are three $\mathcal{A}$-permutations in $V(H_3)$ having p as its pivot. The $\mathcal{A}$-permutations in $V(H_3)$ having pivot 1 are: 123(45)(67), 145(23)(67) and 167(23)(45) (respectively, having pivot 7 are: 167(23)(45), 347(12)(56) and 257(13)(46)).

For each $(\rho - 2)$-subspace Q of $\mathbf{P}_2^{\rho-1}$ and each point $a \in Q$, we define a (Q, a)-*transposition* as a permutation $(b\,c)$ such that there is a line $abc \subseteq \mathbf{P}_2^{\rho-1}$ with $bc \cap Q = \emptyset$.

For each pair (Q, a) formed by a $(\rho-2)$-subspace Q and a point a as above, there are exactly $2^{\rho-2}$ (Q, a)-transpositions. The product of these $2^{\rho-2}$ transpositions is an $\mathcal{A}$-permutation in $V(H_\rho)$ called the (Q, a)-*permutation* $p(Q, a)$, with Q as fixed-point set and a as pivot.

These (Q, a)-permutations $p(Q, a)$ in $V(H_\rho)$ act as a set of generators for the group $V(H_\rho)$. In fact, all elements of $V(H_\rho)$ can be obtained from the (Q, a)-permutations by means of reiterated multiplications.

4.2 A Remotest Vertex from the Identity Permutation

For $\rho > 1$, a particular element $J_\rho \in V(H_\rho) \setminus V(H_{\rho-1})$ at maximum distance from I_ρ is obtained as a product $J_\rho = p_\rho q_\rho$ with:

(A) $p_\rho = p(Q, 2^{\rho-1})$, where Q is the $(\rho - 2)$-subspace of $\mathbf{P}_2^{\rho-1}$ containing both $2^{\rho-1}$ and $\mathbf{P}_2^{\rho-3}$. For example:

$$p_2 = 2(13), \quad p_3 = 415(26)(37), \quad p_4 = 81239ab(4c)(5d)(6e)(7f),$$
$$p_5 = g1234567hijklmn(8o)(9p)(aq)(br)(cs)(dt)(eu)(fv), \quad p_6 = \ldots.$$

(B) q_ρ defined inductively by $q_2 = 3(12)$ and $q_{\rho+1} = \Psi_\rho(p_\rho q_\rho)$, for $\rho > 1$, with Ψ_ρ as in Subsection 4.1.

Initial cases of J_ρ with products indicated by means of dots '$\cdot$' are:

$$
\begin{aligned}
J_2 &= 2(13)\cdot 3(12) = (132);\\
J_3 &= 415(26)(37)\cdot 7(132)(645) = (1372456);\\
J_4 &= 81239ab(4c)(5d)(6e)(7f)\cdot f(1372456)(ec8dba9) = (137f248d6c5ba9e);\\
J_5 &= g1234567hijklmn(8o)(9p)(aq)(br)(cs)(dt)(eu)(fv)\cdot(137f248d6c5ba9e)(usogtrnipjqklmh)\\
&= (137fv248gt6codraklmhu)(5bnipes)(9jq).
\end{aligned}
$$

4.3 Vertex Types

A way to express $v = J_2, J_3, J_4, J_5$, etc. (Subsection 4.2) is by accompanying v with an underlying expression u similar in form to v:

J_2:	J_3:	J_4:	J_5:
v=(132)	(1372456)	$(137f248d6c5ba9e)$	$(137fv248gt6codraklmhu)(5bnipes)(9jq)$
u=(213)	(2456137)	$(248d6c5ba9e137f)$	$(248gt6codraklmhu137fv)(es5bnip)(q9j)$

where each b_i in a cycle $(b_0b_1\dots b_{x-1})$ of u located exactly under a symbol a_i of a corresponding cycle $(a_0a_1\dots a_{x-1})$ of v is a point in a line $a_ib_ia_{i+1}$ of $\mathbf{P}_2^{\rho-1}$ (with $i+1$ taken mod x). Each $\mathcal{A}$-permutation v, like those J_2, J_3, J_4, J_5, will be written likewise: accompanied by similar expression u underlying v. This yields a *two-level expression* $\substack{v\\u}$. We say that:

(i) b_i is the *difference symbol* (ds) of a_i and a_{i+1} in v, for $0 \le i < x$, with x equal to the length of the cycle $(b_0b_1\dots b_{x-1})$ of v containing b_i;

(ii) $(b_0b_1\dots b_{x-1})$ is the *ds-cycle* of the cycle $(a_0a_1\dots a_{x-1})$; and

(iii) u is the *difference-symbol level*, or *ds-level*, of v.

Notice that $(a_0a_1\dots a_{x-1})$ and $(b_0b_1\dots b_{x-1})$ differ by a shift of $(b_0b_1\dots b_{x-1})$ to the right with respect to $(a_0a_1\dots a_{x-1})$ in an amount of, say, y positions. For the four displayed examples above, the values of y are: $y = 1$ for J_2; $y = 4$ for J_3; $y = 11$ for J_4; and $y = 16, 2, 1$, one for each of the three cycles of J_5.

For $\rho > 1$, we define the *type* $\tau_\rho(J_\rho)$ of J_ρ as the expression of the parenthesized lengths of the cycles composing J_ρ with each length sub-indexed by its y. Thus, the types of the four examples above are:

$$\tau_2(J_2) = (3_1),\ \ \tau_3(J_3) = (7_4),\ \ \tau_4(J_4) = (15_{11}),\ \ \tau_5(J_5) = (21_{16})(7_2)(3_1).$$

The ds notation is extended to the elements $p(Q,a)$ of $V(H_\rho)$ defined at the end of Subsection 4.1 by expressing the two-level expressions $\substack{v\\u}$ of the $(\mathbf{P}_2^{\rho-2},1)$-permutations $v = p(\mathbf{P}_2^{\rho-2},1)$ as in the following two examples:

$$\left| \rho = 3 : \begin{array}{c} v \\ u \end{array} = \begin{array}{c} 123(45)(67) \\ 123(11)(11) \end{array} \right| \rho = 4 : \begin{array}{c} v \\ u \end{array} = \begin{array}{c} 1234567(89)(ab)(cd)(ef) \\ 1234567(11)(11)(11)(11) \end{array} \right|$$

with the pivot 1 meaning that 1 is the only common point the subspaces 145 and 167 share for $\rho = 3$; respectively 189, $1ab$, $1cd$ and $1ef$ share for $\rho = 4$. In general for $\rho > 1$:

(a) each fixed point in v appears in u under its appearance in v;

(b) ds-cycles of length x are well-defined cycles only if $x > 2$; and

(c) each transposition, say (a_0a_1), in v is said to have *degenerate ds-cycle* $(b\,b)$ (not a well-defined cycle), where ba_0a_1 is a line of $\mathbf{P}_2^{\rho-1}$; the pivot b is said to *dominate* each such (a_0a_1) in v.

The *types* of the $(\mathbf{P}_2^{\rho-2}, 1)$-permutations $p(\mathbf{P}_2^{\rho-2}, 1)$ above are defined as:

$$\begin{array}{rcllll} \tau_3(p(\mathbf{P}_2^1, 1)) & = & \tau_3(123(45)(67)) & = (1(2)(2)) & = (1((2)^2)), \\ \tau_4(p(\mathbf{P}_2^2, 1)) & = & \tau_4(1234567(89)(ab)(cd)(ef)) & = (1(2)(2)(2)(2)) & = (1((2)^4)), \end{array}$$

with an auxiliary concept of *pivot domination of transpositions*, employed to the right of the second equal sign in each of the two cases and expressed in general for each $p(\mathbf{P}_2^{\rho-2}, 1)$ by means of parentheses containing first the length, 1, of the pivot followed by the parenthesized lengths of the transpositions this pivot *dominates*. More generally, if v is of the form $p(Q, a)$ in $V(H_\rho)$, then we take its type to be $\tau_\rho(v) = (1((2)^{2^{\rho-2}}))$.

By employing this concept of domination, the doubling provided by the embeddings $\Psi_\rho : V(H_{\rho-1}) \to V(H_\rho)$ (Subsection 4.1) will allow the expression of other *types* of $\mathcal{A}$-permutations, from Subsection 4.5 on.

For now, define the *type* of $I_\rho = 12\ldots(2^\rho - 1) = 12\ldots m_0$ to be $\tau_\rho(I_\rho) = (1)$.

The following result is fundamental in counting $\mathcal{A}$-permutations and finding the diameter of H_ρ as a function of their fixed-point sets, and thus as a function of their types, that we will keep defining in the sequel. In particular, this result ensures that J_ρ realizes the diameter of H_ρ.

Theorem 4.1. *The distance $d(v, I_\rho)$ from an $\mathcal{A}$-permutation v to the identity I_ρ in the graph H_ρ is related to the cardinality of the fixed-point set F_v of v in $\mathbf{P}_2^{\rho-1}$ as follows:*

$$\log_2(1 + |F_v|) + d(v, I_\rho) = \rho. \tag{2}$$

Proof. $|F_{I_\rho}| = 2^\rho - 1 = m_0$, so (2) holds for I_ρ, as $\log_2(1+(2^\rho - 1)) = \rho$. Adjacent to I_ρ are the elements of the form $p(Q, a)$, each having $2^{\rho-1} - 1$ fixed points, so (1) holds for the vertices at distance 1 from I_ρ. The vertices at distance 2 from I_ρ have $2^{\rho-2} - 1$ fixed points, and so on inductively, until the $\mathcal{A}$-permutations in $V(H_\rho)$ have no fixed points (J_ρ included) and are at distance ρ from I_ρ, so they satisfy (1), too. □

4.4 Cauchy's Two-Line Notation for Permutations

Another way to look at J_ρ is in Cauchy's two-line notation:

$$J_\rho = \binom{\xi_\rho}{\eta_\rho} = \binom{123}{312}, \binom{1234567}{3475612}, \binom{123456789abcdef}{3478bcfde9a5612}, \binom{123456789abcdefghijklmnopqrstuv}{3478bcfgjknorsvtupqlmhide9a5612}, \text{ etc.},$$

where $\rho = 2, 3, 4, 5$, etc., respectively. The upper level of J_ρ is $\xi_\rho = 12\cdots(2^\rho-2)(2^\rho-1)$ and the lower level η_ρ follows the following constructive pattern. The pairs in the list:

$$L := 12,\ 34,\ 56,\ \ldots,\ (2i-1)(2i),\ \ldots,\ (2^\rho-3)(2^\rho-2),$$

are placed below the $2^{\rho-1}-1$ position pairs $(2i-1)(2i)$ of points of $\mathbf{P}_2^{\rho-1}$ in ξ_ρ different from $2^\rho - 1$ (placed last; see **(iv)** below) in the level η_ρ according to the following rules:

- **(i)** place the starting pair $(2i-1)(2i)$ of L in the rightmost pair of still-empty positions of level η_ρ and erase it from L, that is: set $L := L \setminus \{(2i-1)(2i)\}$;
- **(ii)** place the starting pair $(2i-1)(2i)$ of L in the leftmost pair of still-empty positions of level η_ρ and erase it from L, that is: set $L := L \setminus \{(2i-1)(2i)\}$;
- **(iii)** repeat **(i)** and **(ii)** alternately until L is empty;
- **(iv)** place $m_0 = 2^\rho - 1$ in the (still empty) $(2^{\rho-1}-1)$-th position of level η_ρ.

Now, level η_ρ looks like:

$$3478\ldots(4i-1)(4i)\ldots(2^\rho-5)(2^\rho-4)(2^\rho-1)(2^\rho-3)(2^\rho-2)\ldots(4i+1)(4i+2)\ldots5612,$$

and can also be expressed by means of a function f given by:

$$\begin{array}{lll@{\qquad}lll}
f(2i) & = 4i, & (i=1,\ldots,2^{\rho-2}-1); & f(2^\rho-2i+1) & = 4i+2, & (i=1,\ldots,2^{\rho-2}); \\
f(2i-1) & = 4i-1, & (i=1,\ldots,2^{\rho-2}-1); & f(2^\rho-2i) & = 4i+1, & (i=1,\ldots,2^{\rho-2}); \\
f(2^{\rho-1}-1) & = 2^\rho-1. & & & &
\end{array}$$

4.5 The Remotest Vertex Types from the Identity Permutation

The leftmost $(2^{\rho-1}-1)$ symbols in the lower level η_ρ of Subsection 4.4 form a $(\rho-2)$-subspace ζ_ρ of $\mathbf{P}_2^{\rho-1}$. Let $z(j) = p\,(\zeta_\rho, f(j)) \in V(H_\rho)$, with fixed-point set ζ_ρ and pivot $f(j) \in \zeta_\rho$, where $j = 1, \ldots, 2^{\rho-1}-1$.

Observation 4.2. The composite permutations $w_\rho(j) = J_\rho.z(j)$, for $j = 2, 4, 6, \ldots, 2^{\rho-1}-2$, are at distance ρ from I_ρ, yielding pairwise different types. Subsequent powers of these permutations $w_\rho(j)$ are not necessarily different; they are at distance ρ from I_ρ and must be inspected in order to check for any remaining pairwise different types.

Let us illustrate Observation 4.2 for $\rho = 3$, 4 and 5. First, $w_3(2) = J_3.z(2) = (1372456).437(15)(26) = (1376524)$, a 7-cycle with ds-cycle (2413765) switched 2 positions to the right with respect to (1376524), fact that we indicate by defining type $\tau_3(w_3(2)) = (7_2)$, the sub-index 2 meaning the number of positions the 7-cycle (1376524) is displaced cyclically to the right to yield its ds-cycle. Summarizing:

$w_3(2)$	(1376524)
ds–level	(2413765)
type	(7_2)

However, $\tau_3(w_3(2)) = \tau_3((w_3(2))^2) = \cdots = \tau_3((w_3(2))^5) = \tau_3((w_3(2))^6) = (7_2)$, but $(w_3(2))^7$ is the identity permutation, whose type is $\tau_3(w_3(2)) = (1)$. So, taking powers of $w_3(2)$ does not contribute any new types.

For $\rho \geq 3$, a generalization of the type τ_ρ takes place in which *pivot domination of a transposition* generalizes to *cycle domination of a cycle*, to be shown parenthesized as in Subsection 4.3 (with additional examples in Subsection 4.6). A special case will happen when a c_1-cycle C_1 *dominates* a c_2-cycle C_2 which in turn *dominates* a c_3-cycle C_3, and so on, until a c_x-cycle C_x in the sequel *dominates* the first cycle C_1, so that a *domination cycle* $(C_1, C_2, C_3, \ldots, C_x)$ is created. In such a special case, the *type* of the resulting permutation, or permutation factor, will be denoted $(c_1(c_2(c_3(\ldots(c_x(_y)\ldots)))))$, where y, appearing as a sub-index between the innermost parentheses, is obtained by aligning C_1, C_2, …, C_x and their respective ds-cycles D_1, D_2, …, D_x, so that each dominated ds-cycle D_{i+1} is presented in the same order as the dominating cycle C_i, for $i = 1, \ldots, x$. In this disposition, y is given by the shift of the ds-cycle of C_1 with respect to its dominating cycle C_x. For example, the values of $w_4(j)$ and the corresponding types $\tau_4(w_4(j))$, for $j = 2, 4, 6$, are now given as follows:

j	2	4	6
$w_4(j)$	$(5be)(2489ad)(137f6c)$	$(2485b)(137fa)(cde96)$	$(137feda5b6c9248)$
ds–level	$(e5b)(6c137f)(2489ad)$	$(6cde9)(2485b)(137fa)$	$(248137feda5b6c9)$
type	$(3_1)(6(6(_2)))$	$(5(5(5(_1))))$	(15_3)

Powers of $w_4(2)$ yield new types:

i	2	3
$(w_4(2))^i$	$(5eb)(28a)(49d)(176)(3fc)$	$5be(8d)(36)(29)(7c)(1f)(4a)$
ds–level	$(b5e)(a28)(d49)(617)(c3f)$	$5be(55)(55)(bb)(bb)(ee)(ee)$
type	$(3_1)^5$	$(1((2)^2))(1((2)^2))(1((2)^2))=(1((2)^2))^3$

The first type, $(3_1)^5$ here, still represents a permutation at maximum distance (= 4) from I_4. However, the second type, $(1((2)^2))^3$, has distance 2 from I_4. Subsequent powers of $w_4(j)$ $(j = 2, 4, 6)$ do not yield new types of elements of $V(H_4)$.

We present the $\mathcal{A}$-permutations $w_5(2i)$ $(1 \leq i \leq 6)$ and their types:

$$
\begin{array}{lcl|l}
w_5(2) & = & (137fv6co9ju5bnmlit248gpakhqdres) & \tau_5(w_5(2)) = (31_{13}); \\
w_5(4) & = & (137fvaktesdr248glu9jihmp6co5bnq) & \tau_5(w_5(4)) = (31_{19}); \\
w_5(6) & = & (137fves9jmtakp248ghilq5bnudr6co) & \tau_5(w_5(6)) = (31_{17}); \\
w_5(8) & = & (137fvi9jak5bn248gdrqpuhesl6cotm) & \tau_5(w_5(8)) = (31_{18}); \\
w_5(10) & = & (137fvm5bn6copqtidrul248g9jeshak) & \tau_5(w_5(10)) = (31_{11}); \\
w_5(12) & = & (137fvqh6colestupm9j248g5bnakdri) & \tau_5(w_5(12)) = (31_{12}).
\end{array}
$$

No new types are obtained from $w_5(2i)$ by considering subsequent powers.

4.6 Nearer Vertex Types from the Identity Permutation

Observation 4.3. For $j = 1, 3, \dots, 2^{\rho-1} - 1$, the elements $w_\rho(j) = J_\rho.z(j)$ of $V(H_\rho)$ are at distance $\rho - 1$ from I_ρ and provide pairwise different new types. Subsequent powers of $w_\rho(j)$ are not necessarily different, are at distances $< \rho - 1$ from I_ρ, and must be checked in order to detect any remaining different new types.

Let us illustrate Observation 4.3 for $\rho = 3, 4, 5$. First, we have:

j	1	3
$w_3(j)$	5(246)(713)	6(24)(1375)
$ds\!-\!level$	5(624)(624)	6(66)(2424)
$type$	$(1(3_1(3)))$	$(1(2(4)))$

The square of $w_3(1)$ still preserves its type. However, $(w_3(3))^2 = 624(17)(35) = p(624, 6)$. Thus, $\tau_3((w_3(3))^2) = (1((2)^2))$. Also, it can be seen that $w_4(2i+1)$ has types

$$(1(2(4((4)^2)))), \quad (7_3(7)), \quad (7_5(7)), \quad (1(2))(3_1((3)(6))),$$

for $i = 0, 1, 2, 3$, respectively.

By taking the squares of these permutations, we get that $(w_4(3))^2$ and $(w_4(5))^2$ preserve the respective types of $w_4(3)$ and $w_4(5)$, while the types of $(w_4(1))^2$ and $(w_4(7))^2$ are

$$(1((2)^2))^3 \quad \text{and} \quad (3_1((3)^3)),$$

respectively, the first one seen already in Subsection 4.5. Finally, it can be seen that $w_5(2i+1)$ has types

$$\begin{array}{llll} (5((5)(5((5)(5(5)(_1)))))), & (1(2))(7_4(7(14))), & (1(2(4)))(3(3(6(12)))), & (15_3(15)), \\ (1(2))(7_2(7(14))), & (3(3))(6((6)(6(6)))), & (15_{11}(15)), & (1(2(4(8)4(8))), \end{array}$$

for $i = 0, \dots, 7$, respectively.

4.7 Coset Representatives

We define the *types* $\tau'_\rho = \tau_\rho(v)$ for some vertices $v \in V(H_\rho)$ mentioned above as follows:

$$\begin{array}{ll|ll} \tau'_2 & =(1(2)), & \tau'_6 & =(1(2(4((4(8(16)))^2)))), \\ \tau'_3 & =(1(2(4))), & \tau'_7 & =(1(2(4((4(8(16((16)^2)))^2)))))), \\ \tau'_4 & =(1(2(4((4)^2)))), & \tau'_8 & =(1(2(4((4(8(16((16(32))^2)))^2)))))), \\ \tau'_5 & =(1(2(4((4(8))^2)))) & \tau'_9 & =(1(2(4((4(8(16((16(32(64)))^2)))^2)))))), \\ \dots & =\dots\dots & \dots & =\dots\dots \\ \tau'_{3s-1} & =(1(2(\dots(2^{2s-1})\dots))), & \tau'_{3s+1} & =(1(2(\dots(2^{2s-1}(2^{2s}((2^{2s})^2)))\dots))), \\ \tau'_{3s} & =(1(2(\dots(2^{2s-1}(2^{2s}))\dots))), & \tau'_{3s+2} & =(1(2(\dots(2^{2s-1}(2^{2s}((2^{2s}(2^{2s+1}))^2)))\dots))), \end{array}$$

where $s > 0$. Just enough representatives of all the cosets of $V(H_\rho)$ mod $V(H_{\rho-1})$ distribute in the following five categories (**a**)-(**e**), where (**b**) and (**d**) admit two subcategories indexed α and β each, and (Q, a)-permutations $p(Q, a)$ are as in Subsection 4.1:

(**a**) the identity permutation I_ρ;

($\mathbf{b}_\alpha$) the permutations $p(\mathbf{P}_2^{\rho-2}, a)$, where $a \in \mathbf{P}_2^{\rho-2}$; for example:

ρ=3	123(45)(67)	ρ=4	1234567(89)(ab)(cd)(ef)	5123467(8d)(9c)(af)(be)
	231(46)(57)		2134567(8a)(9b)(ce)(df)	6123457(8e)(9f)(ac)(bd)
	312(47)(56)		3124567(8b)(9a)(cf)(de)	7123456(8f)(9e)(ad)(bc)
			4123567(8c)(9d)(ad)(bf)	

($\mathbf{b}_\beta$) those $p(Q, a)$ for which Q is a $(\rho-2)$-subspace containing $a = m_0 = 2^\rho - 1$; for example:

ρ=3	716(25)(34)	ρ=4	f123cde(4b)(5a)(69)(78)	f2578ad(1e)(3c)(4b)(69)
	725(16)(34)		f145abe(2d)(3c)(69)(78)	f3478bc(1e)(2d)(5a)(69)
	734(16)(25)		f16789e(2d)(3c)(4b)(5a)	f3569ac(1e)(2d)(4b)(78)
			f2469bd(1e)(3c)(5a)(78)	

($\mathbf{c}$) those $p(Q, a)$ for which $Q \subset \mathbf{P}_2^{\rho-1}$ is a $(\rho-2)$-subspace containing $a = m_0 - x$, where $x \in \mathbf{P}_2^{\rho-2}$; for example:

ρ=3	415(26)(37)	ρ=4	81239ab(4c)(5d)(6e)(7f)	81459cd(2a)(3b)(6e)(7f)
	514(27)(36)		91238ab(4d)(5c)(6f)(7e)	91458cd(2b)(3a)(6f)(7e)
	624(17)(35)		a12389b(4e)(5f)(6c)(7d)	c14589d(2e)(3f)(6a)(7b)
	426(15)(37)		b12389a(4f)(5e)(6d)(7c)	d14589c(2f)(3e)(6b)(7a)
	536(14)(27)		...	...
	635(17)(24)		...	...

($\mathbf{d}_\alpha$) an $\mathcal{A}$-permutation α_ρ of type τ'_ρ selected as follows, for each $(\rho-3)$-subspace Z_ρ of $\mathbf{P}_2^{\rho-2}$ and each $x_\rho \in (\overline{\mathbf{P}_2^{\rho-2}} \setminus \overline{Z_\rho})$, where $\overline{Z_\rho} = \{m_0 - x : x \in Z_\rho\}$, for $\emptyset \subset Y \subset \mathbf{P}_2^{\rho-1}$: make the fixed point of α_ρ to be the smallest point y_ρ in $\overline{Z_\rho}$; take the 2-cycle of α_ρ with $ds = y_\rho$, containing x_ρ and dominating a 4-cycle containing m_0; if applicable, take the subsequent pairs, quadruples, $\dots 2^s$-tuples $\dots$ of intervening 4-cycles, 8-cycles, $\dots, 2^{s+1}$-cycles, $\dots$, respectively, to have the first 2^{s+1}-cycle ending at the smallest available point of Z_ρ, for $s = 1, 2$, etc.; for example:

Z_3	$\overline{Z_3}$	x_3	α_3	Z_4	$\overline{Z_4}$	x_4	α_4
1	6	4	6(42)(7315)	123	edc	8	c(84)(f73b)(a521)(69ed)
1	6	5	6(53)(7214)	123	edc	9	c(95)(f63a)(b421)(78ed)
2	5	4	5(41)(7326)	123	edc	a	c(a6)(f539)(8721)(4bed)
2	5	6	5(63)(7124)	123	edc	b	c(b7)(f438)(9621)(5aed)
3	4	5	4(51)(7236)	145	eba	8	a(82)(f75d)(c341)(69eb)
3	4	6	4(62)(7135)	...	...	...	...

($\mathbf{d}_\beta$) The inverse permutations of those α_ρ just defined in subcategory ($\mathbf{d}_\alpha$);

($\mathbf{e}$) a total of $(2^{\rho-1} - 1)(2^{\rho-2} - 1)$ $\mathcal{A}$-permutations ξ of type τ'_ρ with fixed point in $\mathbf{P}_2^{\rho-2}$, 2-cycle containing m_0 and leftmost dominating 4-cycle η starting at the smallest available point for the first $2^{\rho-3}$ of these ξ if $\rho \geq 3$; at the next smallest available point for the first $2^{\rho-4}$ of the remaining ξ not yet used in η if $\rho \geq 4$, etc.; remaining dominated 4-cycles, 8-cycles, etc., if applicable, varying with the next available smallest points; for example:

ρ=3	1(76)(2435)	ρ=4	1(fe)(2d3c)(46b8)(57a9)	2(fd)(1e3c)(45b8)(679a)
	2(75)(1436)		1(fe)(2d3c)(649a)(758b)	2(fd)(1e3c)(54a9)(768b)
	3(74)(1526)		1(fe)(4b5a)(26d8)(37c9)	2(fd)(4b69)(15e8)(37ca)
			...	...

The representatives of the cosets of $V(H_\rho)$ mod $V(H_{\rho-1})$ presented above will be called the *selected coset representatives* of $V(H_\rho)$.

Proposition 4.4. *Assume $\rho \leq 5$. The $\mathcal{A}$-permutations v in a fixed category $x \in \{(\mathbf{a}), \dots, (\mathbf{e}))\}$ are in one-to-one correspondence with the cosets mod $V(H_{\rho-1})$ they determine in $V(H_\rho)$. Moreover, any of these cosets has the same number $N_\rho(x)$ of $\mathcal{A}$-permutations in each type $\tau_\rho(v)$, where v varies in the category x. Thus, the distribution of types in a coset of $V(H_\rho)$ mod $V(H_{\rho-1})$ generated by the $\mathcal{A}$-permutations in x depends solely on x.*

Proof. The selection of categories ($\mathbf{a}$)-($\mathbf{e}$) produces specific representatives of distinct classes of $V(H_\rho)$ mod $V(H_{\rho-1})$ for $\rho \leq 5$, as the symbol $m_0 = 2^\rho - 1$ is placed once in each adequate position, while the remaining entries and difference symbols (ds s) are set to yield all existing cases, covering each coset just once. A concise account of involved details is found in Subsection 4.8 below. The representatives in each category are equivalent with respect to the structure of the cosets of $\mathbf{P}_2^{\rho-1}$ mod $\mathbf{P}_2^{\rho-2}$ that yield the classes of $V(H_\rho)$ mod $V(H_{\rho-1})$. So, each of these cosets has the same number of representatives, in particular in each type $\tau_\rho(v)$, where v is in category $x \in \{(\mathbf{a}), \dots, (\mathbf{e})\}$. $\square$

Question 4.5. *Does the statement of* Proposition 4.4 *hold for $\rho > 5$.*

4.8 Vertex Super-Types

In order to present a reasonably concise table of the calculations involved in Proposition 4.4, the *super-type* $\gamma_\rho(v)$ of an $\mathcal{A}$-permutation v of $V(H_\rho)$ is given by expressing from left to right the parenthesized cycle lengths of the type $\tau_\rho(v)$ in non-decreasing order (no dominating parentheses or sub-indices now) with the cycle-length multiplicities $\mu > 1$ written via external superscripts. Such a table, presented as Table I below, is subdivided into four sub-tables. Each such sub-table, for $\rho = 2, 3, 4, 5$, contains from left to right:

(1) a column citing the different existing super-types $\gamma_\rho(v)$, starting with the identity permutation: $\gamma_\rho(I_\rho) = \gamma_\rho(1 \dots (2^\rho - 1)) = \gamma_\rho(1 \dots m_0) = (1)$;

(2) a column for the common distance d of the $\mathcal{A}$-permutations of each of these $\gamma_\rho(v)$ to I_ρ according to Theorem 4.1;

(3) a column that enumerates the vertices v in each category $x \in \{(\mathbf{a}), \dots, (\mathbf{e})\}$ (a total of five columns); and

(4) a column Σ_{row} explained after display (3) below.

After a common header row in Table I, an auxiliary top row in each sub-table indicates the number $N_\rho(x)$ of Proposition 4.4 in each category x; each row below it, but for the last row, contains in column x the number $row_{\gamma_\rho}(x)$ of selected coset representatives of $V(H_\rho)$ in $x \in \{(\text{Ә}),\dots,()\}$ with a specific super-type $\gamma_\rho(v)$; the final column Σ_{row} contains in row_{γ_ρ} the scalar product of the 5-vectors

$$(row_{\gamma_\rho}(\mathbf{a}), row_{\gamma_\rho}(\mathbf{b}), \dots, row_{\gamma_\rho}(\mathbf{e})) \quad \text{and} \quad (N_\rho(\mathbf{a}), N_\rho(\mathbf{b}), \dots, N_\rho(\mathbf{e})). \tag{3}$$

Finally, the order of H_ρ is given by the sum of the values of the column Σ_{row}. This order of H_ρ is placed in Table I at the lower-right corner, for each $\rho = 2, 3, 4, 5$. The doubling provided by the embeddings $\Psi_\rho : V(H_\rho) \to V(H_\rho)$ (Subsection 4.1) happens in several places in the sub-tables. If we indicate by ψ_ρ the map induced by Ψ_ρ at the level of super-types, then we have: $\psi_3((2)) = (2)^2$, $\psi_3((3)) = (3)^2$, etc. In fact, all the super-types of $V(H_\rho)$ appear squared in $V(H_\rho)$.

Arising from the sub-tables, cycle lengths of super-types $\gamma'_\rho = \gamma_\rho(v)$ are shown below corresponding to the types $\tau'_\rho = \tau_\rho(v)$ in Subsection 4.7 and expressed as products of prime powers between parentheses to distinguish obtained exponents of prime decompositions in the $\tau_\rho(v)$ from the new external multiplicity superscripts. Indeed, the $\mathcal{A}$-permutations of type τ'_ρ in the first paragraph of Subsection 4.7 yield super-types γ'_ρ in Table 1.

Proposition 4.6. *Let $2 < \rho$, let $V_\rho = \Pi_{i=1}^{\rho-2}(2^{i-1}(2^i - 1))$ and let $N'_\rho(x)$ be the number of selected coset representatives of $V(H_\rho)$ mod $V(H_{\rho-1})$ with super-type γ'_ρ in category $x \in \{(\mathbf{a}),\dots,(\mathbf{e})\}$. Then, for $\rho \le 5$, it holds that:*

1. $N'_\rho(\mathbf{a}) = 0$;

2. $N'_\rho(\mathbf{b}) = N'_\rho(\mathbf{c}) = N'_\rho(\mathbf{e}) = 2^{\rho-2}V_\rho$;

3. $N'_\rho(\mathbf{d}) = (2^{\rho-2} - 1)V_\rho$.

Proof. The statement follows by enumeration of the selected coset representatives of $V(H_\rho)$ mod $V(H_{\rho-1})$ with super-type γ'_ρ in categories $(\mathbf{a})$-$(\mathbf{e})$ from their values in the sub-tables, for $\rho = 2, 3, 4, 5, \dots$ □

Corollary 4.7. *A partition of $V(H_\rho)$ mod $V(H_{\rho-1})$ is obtained by means of the selected pairwise disjoint coset representatives of $V(H_\rho)$ composing categories $(\mathbf{a})$-$(\mathbf{e})$, for $\rho \le 5$.*

Proof. For $\rho > 2$, the corollary follows from Proposition 4.4 with distribution as in Proposition 4.6 for the vertices of type τ'_ρ, or super-type γ'_ρ. It is easy to see that the statement also holds for $\rho = 2$. □

TABLE I

$\gamma_\rho(v)$=	d	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	Σ_{row}
$\gamma_2(v)$=	$N_2(x)$=	1	2	–	–	–	3
(1)	0	1	–	–	–	–	1
(2)	1	1	1	–	–	–	3
(3)	2	–	1	–	–	–	2
	Σ_{col}=	2	2	–	–	–	6
$\gamma_3(v)$=	$N_3(x)$=	1	6	6	12	3	28
(1)	0	1	–	–	–	–	1
$(2)^2$	1	3	2	1	–	–	21
$(3)^2$	2	2	2	1	3	–	56
$(2)(4)$	2	–	2	2	1	2	42
(7)	3	–	–	2	2	4	48
	Σ_{col}=	6	6	6	6	6	168
$\gamma_4(v)$=	$N_4(x)$=	1	14	28	56	21	120
(1)	0	1	–	–	–	–	1
$(2)^4$	1	21	4	1	–	–	105
$(2)^6$	2	–	6	3	–	2	210
$(2)^2(4)^2$	2	42	30	12	6	6	1260
$(3)^4$	2	56	24	6	10	–	1120
$(2)(4)^3$	3	–	24	24	18	24	2520
$(2)(3)^2(6)$	3	–	32	26	30	24	3360
$(7)^2$	3	48	48	48	48	48	5760
$(3)(6)^2$	4	–	–	12	18	16	1680
$(5)^3$	4	–	–	12	12	16	1344
(15)	4	–	–	24	24	32	2688
$(3)^5$	4	–	–	–	2	–	112
	Σ_{col}=	168	168	168	168	168	20160
$\gamma_5(v)$=	$N_5(x)$=	1	30	120	240	105	496
(1)	0	1	–	–	–	–	1
$(2)^8$	1	105	8	1	–	–	465
$(2)^{12}$	2	210	84	21	–	12	6510
$(2)^4(4)^4$	2	1260	308	56	28	20	26040
$(3)^8$	2	1120	224	28	36	–	19840
$(2)^2(4)^6$	3	2520	1848	672	504	504	312480
$(2)^2(3)^4(6)^2$	3	3360	2464	812	756	756	416640
$(7)^4$	3	5760	2688	896	896	640	476160
$(2)^6(4)^4$	3	–	504	210	84	168	78120
$(3)^2(6)^4$	4	1680	1680	1512	1848	1488	833280
$(5)^6$	4	1344	1344	1344	1344	1344	666624
$(15)^2$	4	2688	2688	2688	2688	2688	1333248
$(3)^{10}$	4	112	112	56	168	48	55552
$(2)(7)^2(14)$	4	–	3072	3072	2688	3072	1428480
$(2)(4)^3(8)^2$	4	–	1344	1344	1176	1344	624960
$(2)(3)^2(4)(6)(12)$	4	–	1792	1624	1736	1600	833280
$(3)(7)(21)$	5	–	–	1792	2176	2048	952320
(31)	5	–	–	4032	4032	4608	1935360
	Σ_{col}=	20160	20160	20160	20160	20160	9999360

$$
\begin{array}{ll|ll}
\gamma'_2 & =(2), & \gamma'_6 & =(2)(4)^3(8)^2(16)^2, \\
\gamma'_3 & =(2)(4), & \gamma'_7 & =(2)(4)^3(8)^2(16)^6, \\
\gamma'_4 & =(2)(4)^3, & \gamma'_8 & =(2)(4)^3(8)^2(16)^6(32)^4, \\
\gamma'_5 & =(2)(4)^3(8)^2, & \gamma'_9 & =(2)(4)^3(8)^2(16)^6(32)^4(64)^4, \\
 & & \dots & =\dots\dots \\
\gamma'_{s+1} & =(\gamma'_s)(2^{2s})^s, & \gamma'_{s+3} & =(\gamma'_s)(2^{2s})^{3s}(2^{2s+1})^{2s}, \\
\gamma'_{s+2} & =(\gamma'_s)(2^{2s})^{3s}, & \gamma'_{s+4} & =(\gamma'_s)(2^{2s})^{3s}(2^{2s+1})^{2s}(2^{2s+2})^{2s},
\end{array}
$$

for $s \equiv 2 \bmod 4$.

Corollary 4.7 can be proved alternatively by means of the $\mathcal{A}$-permutation $(J_{\rho-1})^2$ (obtained via the doubling of $J_{\rho-1}$ in $V(H_\rho)$, Subsection 4.1) and the coset representatives of $V(H_\rho)$ selected with the type of $(J_{\rho-1})^2$, yielding alternative super-types $\gamma_2'' = (3)^2$, $\gamma_3'' = (7)^2$, $\gamma_4'' = (15)^2$, $\gamma_5'' = ((3)(7)(21))^2$, ... In this case, by defining $N_\rho''(x)$ as $N_\rho'(x)$ was in Proposition 4.6, but with γ_ρ'' instead of γ_ρ', we get uniformly that $N_\rho''(x) = 2^{\rho-2}V_\rho$, where $x \in \{(\mathbf{a}), \ldots, (\mathbf{e})\}$. This covers all the classes of $V(H_\rho)$ mod $V(H_{\rho-1})$ and reconfirms the statement.

Proposition 4.8. *With the notation of* Proposition 4.6, $|V(H_\rho)| = V_{\rho+2}$ *at least for* $\rho \le 5$. *Moreover, the following properties of the graphs* G_r^σ *hold for* $\sigma \ge 1$, $\rho \ge 2$ *and at least for* $\rho \le 5$*:*

(A) $|V(G_r^\sigma)|$ *is as claimed in* Conjecture 3.4*;*

(B) G_r^σ *is* $sm_0(t-1)$*-regular;*

(C) *The diameter of* G_r^σ *is* $\le 2r-2$.

Thus, order, degree and diameter of G_r^σ *are respectively:* $O(2^{(r-1)^2})$, $O(2^{r-1})$ *and* $O(r-1)$.

Proof. Item **(C)** is an immediate corollary of Theorem 4.1. Item **(B)** can be deduced from Definition 3.1. Recall that $N_\rho(x)$ is the number of cosets (Proposition 4.4) in each category $x \in \{(\mathbf{a}), \ldots, (\mathbf{e})\}$. Counting cosets obtained via doubling (Subsection 4.1) in each category shows that at least for $\rho \le 5$:

- $N_\rho(\mathbf{a}) = 1$;
- $N_\rho(\mathbf{b}) = 2(2^{\rho-1} - 1)$;
- $N_\rho(\mathbf{c}) = 2^{\rho-2}(2^{\rho-1} - 1)$;
- $N_\rho(\mathbf{d}) = 2N_\rho(c)$;
- $N_\rho(\mathbf{e}) = (2^{\rho-2} - 1)(2^{\rho-1} - 1)$.

Each coset in these categories contains exactly $|V(H_{\rho-1})|$ $\mathcal{A}$-permutations. Thus, $|V(H_\rho)| = V_{\rho+2}$. Since G_r^σ is the disjoint union of $\binom{r}{\sigma}_2$ copies of $T_{st,t}$, item **(A)** follows. Finally, we get that $|V(G_r^\sigma)| = O(2^{(r-1)^2})$, since $(2^r - 1) \le \binom{r}{\sigma}_2$ and $|V(G_r^1)| = (2^r - 1)\Pi_{i=2}^{r-1}(2^{i-1}(2^i - 1)) = O(2^{r-1}4^{1+2+3+\cdots+(r-2)}) = O(2^{r-1+(r-2)(r-1)})$, which is $O(2^{(r-1)^2})$. ☐

Question 4.9. *Do the statements of* Corollary 4.7 *and* Proposition 4.8 *hold for* $\rho > 5$*?*

4.9 Menger-Graph Parameters

By Subsection 3.1, an automorphism $\phi \in \mathcal{A}(G_r^\sigma)$ can be presented as $\omega = \phi^\omega.\psi^\omega$, where ϕ^ω is a permutation of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ and ψ^ω is a permutation of the non-initial entries of ordered pencils that are vertices of G_r^σ, e.g., a permutation of the indices k in entries A_k of such ordered pencils, where $0 < k \le m_0 = 2^\rho - 1$. The subgroup of $\mathcal{N}_r^\sigma = \mathcal{A}(N_{G_r^\sigma}(v_r^\sigma))$ that fixes u_r^σ is formed by the $2^{\rho-1}$ automorphisms ω in item **(A)** of Subsection 3.1 with $\pi \in \mathbf{P}_2^{\rho-1}$ as the third lexicographically smallest such point, namely point $3 \times 2^{\sigma-1}$.

Theorem 4.10. *Assume $r \leq 8$ and $\rho \leq 5$. If $t < 2s$, then G_r^σ is a connected $s(t-1)m_0$-regular $\{K_{2s}, T_{st,t}\}_{\ell_0,\ell_1}^{m_0,m_1}$-H graph, but if $r > 3$ then G_r^σ is not $\{K_{2s}, T_{st,t}\}_{\ell_0,\ell_1}^{m_0,m_1}$-UH. Furthermore, G_r^σ is the Menger graph of a configuration $(|V(G_r^\sigma)|_{m_0}, (\ell_0)_{2s})$ whose points and lines are the vertices and copies of K_{2s} in G_r^σ, respectively. If $\sigma = 1$, then* **(a)** *$|V(G_r^\sigma)|_{m_0} = (\ell_0)_{2s}$;* **(b)** *such configuration is self-dual; and* **(c)** *as a Menger graph, G_r^σ coincides with the corresponding dual Menger graph. In addition, $G_r^\sigma = \mathcal{G}_r^\sigma$ if and only if $r - \sigma = 2$. If $t \geq 2s$, then* **(i)** *G_r^σ is $\{K_{2s} \not\subset T_{st,t}, T_{st,t}\}$-H and the remaining properties above hold by taking only $K_{2s} \not\subset T_{st,t}$;* **(ii)** *if $r - \sigma = 2$ then G_r^σ is $\{K_4 \not\subset T_{4t,t}\}$-UH.*

Proof. For two induced copies Z_0, Z_1 of K_{2s} (respectively, $T_{st,t}$) in G_r^σ and arcs $(v_0, w_0), (v_1, w_1)$ in Z_0, Z_1, respectively, there exist automorphisms Φ_0, Φ_1 of G_r^σ with $\Phi_i(v_r^\sigma) = v_i, \Phi_i(u_r^\sigma) = w_i$ that send $N_{G_r^\sigma}(v_i) \cap Z_i$ onto $N_{G_r^\sigma}(v_r^\sigma) \cap Z'$, for $i \in \{0, 1\}$, where Z' is the lexicographically smallest copy of K_{2s} (respectively, $T_{st,t}$) in G_r^σ, namely $Z' = [U]_r^\sigma$ with U as the third lexicographically smallest $(r-1, \sigma-1)$-ordered pencil of $\mathbf{P}_2^{r-1}$, sharing with $[(\mathbf{P}_2^\sigma)_1]_r^\sigma$ just u_r^σ (respectively, $Z' = [(\mathbf{P}_2^\sigma)_1]_r^\sigma$). As a result, and because of Proposition 3.7 and Proposition 4.8 resulting from the mentioned computations leading to the connectedness of G_r^σ at least for $r \leq 8$ and $\rho \leq 5$, the composition $\Phi_2\Phi_1^{-1}$ in $\mathcal{A}(G_r^\sigma)$ takes Z_0 onto Z_1 and (v_0, w_0) onto (v_1, w_1). Taking this into account, and that G_r^σ satisfies conditions **(i)**-**(iv)** in Subsection 1.1, implies that G_r^σ is a $\{K_{2s}, T_{st,t}\}_{\ell_0,\ell_1}^{m_0,m_1}$-H graph. Recall from [3] that G_3^1 is $\{K_4, T_{6,3}\}$-UH. It holds that $\mathcal{G}_r^\sigma = G_r^\sigma$ whenever $\rho = 2$; this G_r^σ is K_4-UH by an argument similar to that of [3]. From Remarks 3.5-3.6, for $(r, \sigma) \neq (3, 1)$ there are automorphisms of the copy of $T_{st,t}$ in G_r^σ containing the edge $v_r^\sigma u_r^\sigma$ that fixes both v_r^σ and u_r^σ but this is non-extensible to an automorphism of G_r^σ. A similar conclusion holds for K_{2s}, provided $\sigma \neq r - 2$, for which $K_{2s} = K_4$. The assertions involving configurations in the statement arise as a result of the translation of the generated construction in terms of configurations of points and lines and their Menger graphs (Section 2). □

Question 4.11. *Does the statement of* Theorem 4.10 *hold for all pairs (r, ρ) with either $r > 8$ or $\rho > 5$?*

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