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Plane graphs with large faces and small diameter

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Abstract

The face-degree of a face in a connected plane graph is the minimum length of a closed walk that spans all the vertices and edges of the boundary of the face. A plane graph is ρ-face-degree regular if every face has face-degree ρ. In this paper, the structure of 2-edge-connected plane graphs with large minimum face-degree is studied. We give an upper bound on the minimum face-degree of a plane graph with given radius, and characterise the graphs meeting this bound. We show that the girth and minimum face-degree of a plane graph coincide if any of these parameters is at least twice the diameter of the graph. Furthermore, we characterise all planar generalised polygons (bipartite graphs whose girth is twice their diameter). The well-studied degree/diameter problem consists of determining the maximum possible order of a graph given both its maximum degree and diameter. The structural results in this paper solve the degree/diameter problem for plane graphs of diameter D that are either $2D$-face-degree regular or $(2D+1)$-face-degree regular.

1 Background

The face-degree of a face in a connected plane graph is the minimum length of a closed walk that spans the subgraph bounding the face. Recall that the girth of a graph is the minimum length of any of its cycles and the diameter of a graph is the maximum distance between any two of its vertices. In this paper, we investigate plane graphs that are extremal with respect to their face-degree and diameter, and demonstrate the close relationship between girth and face-degree in these extremal graphs. This investigation is motivated by two well-studied topics in graph theory: Moore graphs and the degree/diameter problem. The degree/diameter problem

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consists of determining the maximum possible order n of a graph with diameter D and maximum degree Δ, and Moore graphs are those graphs that obtain the 'trivial upper bound' for the degree/diameter problem (this upper bound is known as the Moore bound). Miller and Širáň wrote a comprehensive survey of both topics in [13]. For plane graphs with $D = 2$ and $\Delta \geq 8$ in which every face is a triangle, Seyffarth has shown in [15] that $n \leq \frac{3}{2}\Delta + 1$. In [6], Dalfó, Huemer, and Salas demonstrated that plane graphs with $D = 2$ in which every face is a quadrangle satisfy $n \leq \Delta + 2$. They also solved the degree/diameter problem for plane graphs with $D = 3$ in which every face is a quadrangle, showing that $n \leq 3\Delta - 1$ for Δ odd and $n \leq 3\Delta - 2$ for Δ even.

Extremal plane graphs have also been considered outside the framework of the degree/diameter problem. The relationship among the radius, maximum face-degree, and order of a plane graph was investigated by Ali, Dankelmann, and Mukwembi in [1], where they determined that a 3-connected plane graph of radius r, order n, and maximum face-degree M satisfies $r \leq \frac{n+5M+4}{6}$. In [8], Dowden considered extremal plane graphs that do not contain either C_4 or C_5 as a subgraph, showing that a C_4-free planar graph of order n has at most $\frac{15}{7}(n-2)$ edges, and that a C_5-free planar graph has at most $\frac{12n-33}{5}$ edges. Lan, Shi, and Song considered the more general problem of bounding the maximum number of edges in a planar graph that does not contain some graph H as a subgraph in [12], and demonstrated a number of conditions under which an H-free planar graph of order n can obtain the trivial upper bound of $3n - 6$ edges.

2 Definitions

Most of the definitions and conventions we use can be found in Diestel's *Graph Theory* [7]. All graphs in this paper are finite and simple. We assume the reader has some familiarity with the topology of the plane.

Let G be a graph, let u and v be vertices of G, and let W be a walk in G. If a walk starts and ends at the same vertex and contains no edge repetitions, we call it a **circuit**. The **length** of the walk W, which we denote $\ell(W)$, is the total number of edges that appear in the walk, counting repeated edges. A $u - v$ **geodesic** is a $u - v$ path of minimum length. The **distance** between two vertices u and v in G, denoted $d_G(u, v)$, is the length of a $u - v$ geodesic in G. We will omit the subscript if the graph in question is clear from the context. The **girth** of G, denoted $g(G)$, is the minimum length of any cycle in G. The **eccentricity** of a vertex is the maximum distance between it and any other vertex of the graph. The **radius** and **diameter** of a graph are the maximum and minimum eccentricities of any vertex, respectively. A graph is **self-centered** if its radius and diameter are equal.

A **separator** of a connected graph is a subset of the vertex set whose removal disconnects the graph. A **separating cycle** is a cycle, the vertex set of which is a separator.

If $G = (V, E)$ is a connected graph with S and T subsets of V, then the distance between S and T is $d(S,T) = \min\{d(u,v) : u \in S, v \in T\}$. Given a vertex v, we write $d(v,S)$ for the distance $d(\{v\},S)$. It follows directly from these definitions that $d(S,T) = 0$ if and only if $S \cap T$ is non-empty, and that $d(S,T) > 1$ implies the induced subgraph $G[S \cup T]$ is disconnected.

Throughout the paper, we implicitly make use of the Jordan Curve Theorem. If X is an open subset of the plane, then a **region** of X is a maximal connected subset of X. A graph is **planar** if it can be embedded in the plane. An embedding of a planar graph is called a **plane graph**, and the regions of the complement of the plane graph are called **faces**. Different embeddings of the same planar graph can create plane graphs with different faces (see Figure 1), so we will work with a fixed embedding wherever ambiguity can arise.

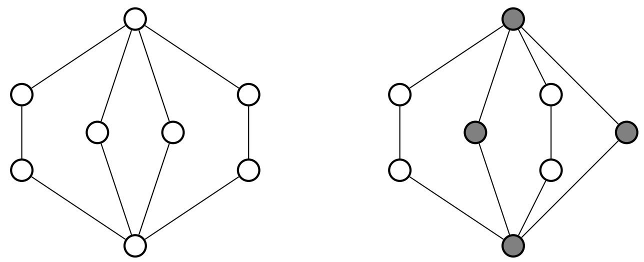

Figure 1: Two different embeddings of the same planar graph yield plane graphs with different faces. In the graph on the left, there is a face bounded by a 4-cycle, but every face of the graph on the right is bouned by a 5-cycle.

For the following definitions, let $G = (V,E,F)$ be a connected plane graph (where F is the set of faces of G) and f a face of G. If v is a vertex of G, we let $d_G(v)$ denote the degree of v in G. An edge or vertex of G is **incident** with the face f if it is contained in the topological boundary of f. We denote $G[f]$ the subgraph consisting of the edges and vertices incident with f, and say that $G[f]$ **bounds** the face f. If some circuit or cycle bounds a face in a plane graph, we call it a **face-circuit** or **face-cycle**, respectively. The subgraph $G[f]$ bounding f can be traversed by a closed walk. The length of a shortest closed walk traversing $G[f]$ is the **face-degree** $\mathcal{E}(f)$ of f. We denote the **minimum face-degree** of G by $\mu(G) = \min\{\mathcal{E}(f) : f \in F\}$. We say G is ρ-**face-degree regular** if every face of the graph has face-degree ρ.

Let G be a 2-edge-connected plane graph of diameter D and minimum face-degree μ. A cycle C of G is a **short-cycle** if $\ell(C) < \mu$ (In Figure 1, the 4-cycle on the grey vertices is a short-cycle of the plane graph on the right).

Given a Jordan curve C in the plane (that is, C is the image of an injective, continuous map from the circle to the plane), we denote the bounded region of $\mathbb{R}^2 - C$ by $\mathrm{Int}(C)$, the unbounded region by $\mathrm{Ext}(C)$, and let $\mathrm{Int}[C] = \mathrm{Int}(C) \cup C$ and $\mathrm{Ext}[C] = \mathrm{Ext}(C) \cup C$. Note that any cycle of a plane graph induces a Jordan curve. If C is a cycle of a plane graph G, then $G[\mathrm{Int}[C]]$ is the subgraph of G that consists of all the edges and vertices contained in $\mathrm{Int}[C]$, and $G[\mathrm{Ext}[C]]$ is defined similarly as the subgraph consisting of all edges and vertices in $\mathrm{Ext}[C]$. If a cycle has vertices in both its interior and its exterior, we call it a **Jordan separating cycle**. Clearly a Jordan separating cycle is itself a separating cycle, but not every separating cycle of a plane graph is a Jordan separating cycle.

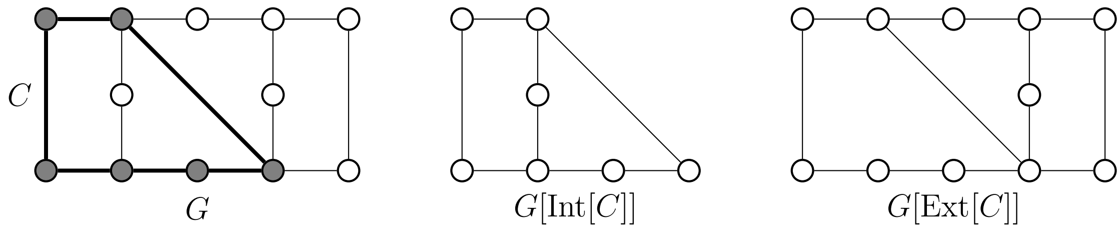

Figure 2: The bold cycle C in G, the subgraphs $G[\text{Int}[C]]$, and $G[\text{Ext}[C]]$. Note that C is a Jordan separating cycle.

3 Cycle length and minimum face-degree of plane graphs

We recall some results concerning the connectivity of plane graphs. The first is well-known (see, for example, Diestel's *Graph Theory* [7]).

Remark 3.1 A plane graph is 2-connected if and only if each face is bounded by a cycle.

The following result and its corollary are also known (a strengthening of both is given as an exercise in Bondy and Murty's *Graph Theory* [2]), although a literature proof is elusive.

Theorem 3.2 *A plane graph is 2-edge-connected if and only if every face of the graph is bounded by a circuit.*

Corollary 3.3 *Let f be a face of a plane graph G. If G is 2-edge-connected, then the face-degree of f is the number of edges in the subgraph $G[f]$.*

In a 2-edge-connected plane graph, the girth is bounded above by the minimum face-degree, as every face is bounded by either a cycle, or a circuit (and every circuit contains a cycle). However, the difference between the minimum face-degree and the girth can be arbitrarily large, as the two graphs in Figure 3 show. Given any positive integer $\rho \geq 3$, there is a ρ-face-degree regular 2-edge-connected graph containing a 4-cycle. Given any odd positive integer $\rho \geq 3$, there is a ρ-face-degree regular 2-edge-connected graph containing a 3-cycle.

However, if a 2-edge-connected plane graph does have a cycle of length strictly less than its minimum face-degree, that cycle must be a Jordan separating cycle.

Lemma 3.4 *Every short-cycle of a plane graph is a Jordan separating cycle.*

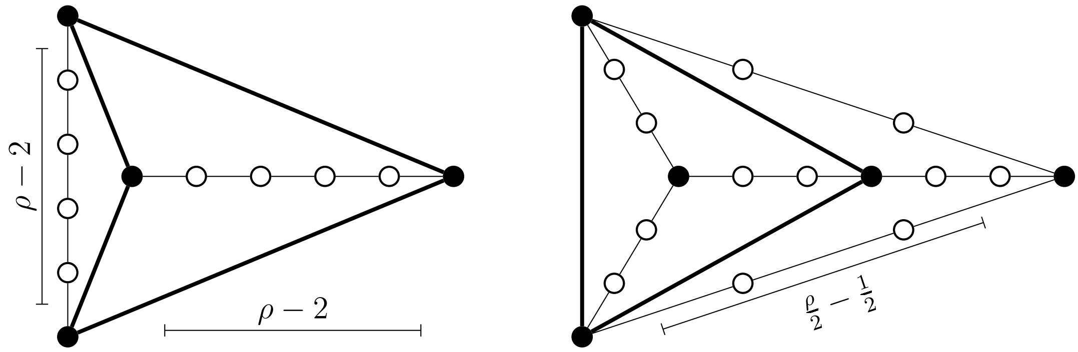

Figure 3: Left: Let all paths of white internal vertices, starting and ending at a black vertex, have length $(\rho - 2)$. This creates a ρ-face-degree-regular graph with girth ≤ 4 for any $\rho \geq 3$. Right: Let all paths between black vertices having only white internal vertices have length $(\frac{\rho}{2} - \frac{1}{2})$. Then, we obtain a ρ-face-degree-regular graph with girth 3 for any odd ρ.

Proof: Let G be a plane graph and let $C = v_1, v_1, \ldots, v_k, v_1$ be a short-cycle of length $k < \mu$ in G. Certainly, C cannot be a face-cycle as every face-cycle has length μ or greater.

We claim that $\text{Int}(C)$ must contain at least one vertex. Assume to the contrary that it does not, and consider the induced subgraph $H = G[\text{Int}[C]]$. All the faces of H, except the external face bounded by C, are also faces of G and, hence, all faces of H except the external face must have degree at least μ. Since $V(H) = V(C)$ and C is a subgraph of H, the subgraph H is 2-connected. By Remark 3.1, every face of H is bounded by a cycle. But there are only k vertices in H with which to construct a cycle. Thus, every face of H, including all the interior faces, which are faces of G, is bounded by a cycle of length at most k. This contradicts the fact that every face of G has degree at least μ.

The same argument shows that the exterior of C must also contain a vertex, so C is a Jordan separating cycle. □

Before presenting the main result of this section, we will need some extra machinery. Given a connected graph G, with any spanning tree T of G, there is a (possibly empty) collection of cycles of G called **fundamental cycles** (with respect to T). A fundamental cycle is a cycle of G formed by the addition of a single edge of $E(G) - E(T)$ to T. Given an edge e of $E(G) - E(T)$, denote by C_e the fundamental cycle induced in $T + e$. Note that if G has any cycles at all, it has at least one fundamental cycle. Further discussions of fundamental cycles in plane graphs can be found in both Bondy and Murty's *Graph Theory* [2] and Mohar and Thomassen's *Graphs on Surfaces* [14]. A **radius-preserving spanning tree** of G is a spanning tree T of G such that both T and G have the same radius. We will use the following well-known lemma, which follows from the discussion on breadth-first-search in [2].

Lemma 3.5 [2] *Every connected graph has a radius-preserving spanning tree.*

The following simple lemma has almost certainly appeared in the literature before, but we give a proof here for completeness.

Lemma 3.6 *Let be G a connected graph and T a spanning tree of G with radius r. If C_e is a fundamental cycle of G with respect to T, then the length of C_e is at most $2r + 1$.*

PROOF: Every path in T is a geodesic. Thus, any path in T has length at most $2r$. Every fundamental cycle is formed by the addition of a single edge to a path in T and, hence, it has length at most $2r + 1$. □

It is well-known and easy to see that if a graph has diameter D and girth g, then the diameter bounds the girth from above by the inequality $g \leq 2D + 1$. The next result shows that the same constraint holds if we replace the girth by the minimum face-degree.

Theorem 3.7 *If G is a 2-edge-connected plane graph of radius r, then $\mu(G) \leq 2r+1$. This bound is sharp.*

PROOF: Assume for the sake of contradiction that $G = (V, E, F)$ is a 2-edge-connected plane graph with minimum face-degree μ and radius r, and $\mu > 2r + 1$. By Lemma 3.5, the graph has a spanning tree T of radius r. By Lemma 3.6, every fundamental cycle of G with respect to T has length at most $2r + 1$. By Lemma 3.4, every fundamental cycle with respect to T is a Jordan separating cycle.

Choose an edge uv in $E(G) - E(T)$ that minimises the number of vertices in $\text{Int}(C_{uv})$. Since T is spanning and connected, and G is planar, any vertex in the interior of C_{uv} is connected to C_{uv} by some path of T in $\text{Int}[C_{uv}]$. Since the only cycle in $T + uv$ is C_{uv}, there is some vertex in the interior of C_{uv}, say x, such that $d_{T+uv}(x) = 1$ (see part (1) of Figure 4).

Since G is 2-edge-connected, the vertex x has degree at least two in G. Thus, there is some edge xy in $E(G) - E(T)$ that lies inside $\text{Int}[C_{uv}]$ (see part (2) of Figure 4).

As the induced subgraph $(T+uv)[\text{Int}[C_{uv}]]$ is connected, the addition of xy to $T+uv$ divides the interior of C_{uv} into two regions - exactly one of which has the edge uv on its boundary (see part (3) of Figure 4).

The region not containing the edge uv on its boundary contains only edges of T, and the edge xy on its boundary and, hence, it is bounded by a fundamental cycle. Thus, $T + xy$ contains a fundamental cycle C_{xy} that has fewer vertices in its interior than C_{uv} does, contradicting the minimality of C_{uv}.

The bound is sharp as the cycle C_{2k+1} has radius and diameter k, and both faces of the cycle have face-degree $2k + 1$. □

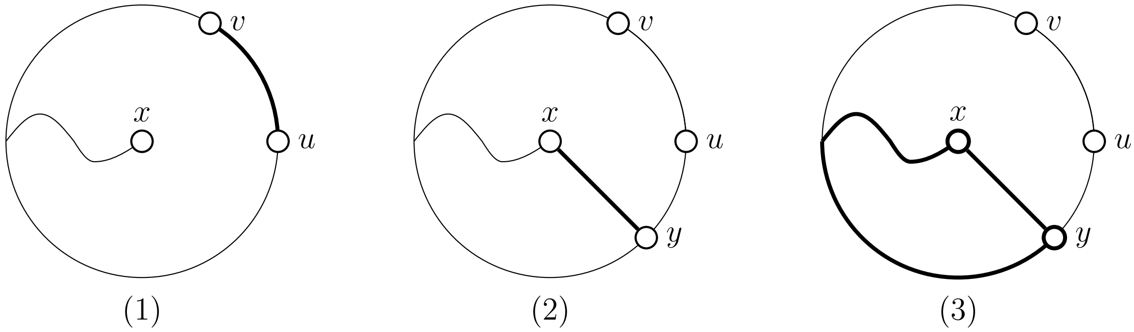

Figure 4: (1) The fundamental cycle C_{uv} in $T + uv$, with end vertex x in its interior. The edge uv is bold. (2) The end vertex in $T + uv$ is adjacent to some vertex y in $\text{Int}[C_{uv}]$. The vertex y is not necessarily part of the cycle itself and may lie in $\text{Int}(C)$. The edge xy is bold. (3) One of the regions induced by adding the edge xy to $T+uv$ is bounded by the fundamental cycle C_{xy}, which is bold.

4 Extremal graphs for Theorem 3.7

In this section, we show that the odd cycle C_{2D+1} is the only graph that is 2-edge-connected, has diameter D and has minimum face-degree $2D + 1$.

Lemma 4.1 *Let G be a 2-edge-connected plane graph with diameter D and minimum face-degree μ. If $\mu = 2D + 1$, then G is self-centered.*

Proof: Assume to the contrary that G is a 2-edge-connected plane graph with minimum face-degree $\mu = 2D + 1$, diameter D and radius $r < D$. By Lemma 3.5, the graph G has a radius-preserving spanning tree of radius at most $D-1$. Thus, by Theorem 3.7, the minimum face-degree satisfies $\mu \leq 2D-1 < 2D+1$, a contradiction. □

We need two lemmas; the first was originally proven by Buckley [3], and a proof can be found in a paper by Jarry and Laugier [10]. The second was proven by Harary and Norman [5], and a proof is given in Buckley and Harary's *Distance in Graphs* [4].

Lemma 4.2 [3, 10] *If $G = (V, E)$ is a 2-connected graph of diameter D, then*

$$\left\lceil \frac{(|V|-2)D-1}{D-1} \right\rceil \leq |E|.$$

Lemma 4.3 [5, 4] *The center of a graph is contained within a single maximal non-separable subgraph.*

Theorem 4.4 *If G is a 2-edge-connected plane graph with diameter D and minimum face-degree $2D + 1$, then G is the odd cycle C_{2D+1}.*

Proof: Let $G = (V, E)$ be a 2-edge-connected plane graph of diameter D and minimum face-degree $2D + 1$. The graph G is self-centered by Lemma 4.1. Thus, by Lemma 4.3, the graph G lies entirely within a single maximal non-separable subgraph (of itself), and so it is 2-connected. By Lemma 4.2, we obtain the following inequalities:

$$\frac{(|V|-2)D-1}{D-1} \leq \left\lceil \frac{(|V|-2)D-1}{D-1} \right\rceil \leq |E|. \tag{4.1}$$

Recall that the symbol $\mathcal{E}(f)$ denotes the face-degree of the face f. Noting that the minimum number of edges bounding any face is μ, we see that

$$\mu|F| \leq \sum_{f \in F(G)} \mathcal{E}(f) = 2|E|.$$

From this, we deduce

$$|F| \leq \frac{2}{\mu}|E|. \tag{4.2}$$

Substituting inequality (4.2) into the equation for the Euler characteristic of G, that is, $|V| - |E| + |F| = 2$, we get another inequality:

$$|E| + 2 \leq |V| + \frac{2}{\mu}|E|.$$

Hence,

$$|E| \leq \frac{\mu(|V|-2)}{\mu-2}. \tag{4.3}$$

Combining the inequalities (4.1) and (4.3), and substituting $\mu = 2D + 1$, we show that G must satisfy the following inequality:

$$\frac{(|V|-2)D-1}{D-1} \leq \frac{(|V|-2)(2D+1)}{2D-1}.$$

With some rearrangement, we finally bound the order of G:

$$|V| \leq 2D + 1.$$

Since G is 2-connected, every face of G is bounded by a cycle per Remark 3.1. Since $\mu = 2D+1$, the graph G contains the cycle C_{2D+1} as a subgraph. Since $|V| \leq 2D+1$, we conclude that G is the cycle C_{2D+1}. □

5 Planar generalised polygons

A graph of diameter D is a **generalised polygon** if it is bipartite and has girth $2D$. The structure of generalised polygons is explored in Godsil and Royle's *Algebraic Graph Theory* [9]. In this section, we characterise planar generalised polygons. Much like the cycle C_{2D+1} is the only 2-edge-connected planar graph that has diameter D and is $(2D+1)$-face-degree regular (Theorem 4.4), planar generalised polygons are exactly the 2-edge-connected planar graphs that have diameter D and are $2D$-face-degree regular (Corollary 5.15). Thus, the results of this section demonstrate that planar generalised polygons are a useful class of 'nearly extremal' planar graphs.

As Figure 3 illustrates, we cannot normally use face-degrees to bound the girth of a graph from below. However, we show that if the face-degrees are all sufficiently high, then the girth is bounded below by the minimum face-degree. First, we need some lemmas. In the 1870's, Kempe [11] observed that if every vertex of a plane graph has even degree, then the faces of the graph may be 2-coloured such that no two faces of the same colour share an edge. Hence, by plane duality, we obtain the next well-known remark.

Remark 5.1 If every face of a plane graph has even face-degree, then the graph is bipartite.

The next two lemmas are familiar to many graph theorists.

Lemma 5.2 *Let $G = (V, E)$ be a connected graph, and $S \subset V$ a separator of G. If two vertices u and v of G are in different components of $G - S$, then $d(u, v) \geq d(u, S) + d(v, S)$.*

PROOF: Let P be a $u - v$ geodesic. As S separates u and v, there exists a vertex s in $S \cap P$, so the edges of P can be partitioned into a $u - s$ path $P[u, s]$ and an $s - v$ path $P[s, v]$. Hence, we obtain the following sequence of inequalities:

$$d(u, v) = \ell(P) = \ell(P[u, s]) + \ell(P[s, v]) \geq d(u, S) + d(v, S).$$

□

Lemma 5.3 *If $G = (V, E)$ is a connected graph of diameter D, and there are subsets A, B, and S of V such that $\{A, S, B\}$ is a partition of V and $d(A, B) > 1$, then $\max_{v \in A}\{d(v, S)\} \leq \lfloor \frac{D}{2} \rfloor$ or $\max_{v \in B}\{d(v, S)\} \leq \lfloor \frac{D}{2} \rfloor$.*

PROOF: Assume to the contrary that there exist vertices u in A and v in B such that $d(u, S) > \lfloor \frac{D}{2} \rfloor$ and $d(v, S) > \lfloor \frac{D}{2} \rfloor$. The set S separates u and v so, by Lemma 5.2, we have the following inequalities:

$$d(u, v) \geq d(u, S) + d(v, S) \geq 2\left(\left\lfloor \frac{D}{2} \right\rfloor + 1\right) > D.$$

□

Remark 5.4 If B and C are two cycles of a plane graph such that C lies in the interior of B, then all vertices v in the interior of C satisfy $d(v, C) \leq d(v, B)$.

The previous remark follows from the fact that any $v - B$ geodesic must contain some vertex of C.

The next result shows that if the minimum face-degree of a plane graph with diameter D is large enough, then the graph contains no short cycles.

Theorem 5.5 *Let G be a 2-edge-connected plane graph of diameter D. If $\mu(G) = 2D$, then $g(G) = 2D$.*

PROOF: Assume, for the sake of contradiction, that $g < 2D$, and let B be a short-cycle in G. By Lemma 3.4, the cycle B is a Jordan separating cycle. We know by Lemma 5.3 that, without loss of generality, all vertices v in the interior of B satisfy $d(v, B) \leq \lfloor \frac{D}{2} \rfloor$. Choose C to be an interior-minimal short-cycle in $\text{Int}[B]$, so, choose C such that there does not exist a short-cycle C' having $\text{Int}(C') \subset \text{Int}(C)$ (it is possible that $C = B$). Clearly, C is itself a Jordan separating cycle and, by Remark 5.4, if v is a vertex in the interior of C, then $d(v, C) \leq \lfloor \frac{D}{2} \rfloor$.

Among all vertices in the interior of C, let v be one at maximum distance from C. By Lemma 3.4, such a vertex v must exist. Let P be a $v - C$ geodesic, and let u be the vertex of $C \cap P$. Since G is 2-edge-connected, we have that $d(v) \geq 2$. Thus, there is a vertex v' in $\text{Int}[C] - P$ that is adjacent to v.

Let P' be a $v' - C$ geodesic and u' the vertex of $C \cap P'$ (it is possible that $u' = v'$). If u and u' are distinct, then the cycle C can be divided into two $u - u'$ paths. Let Q denote the shorter of these two paths and note that $\ell(Q) \leq D - 1$. If $u = u'$, then let Q be the trivial path containing only the vertex u. Whether or not $u = u'$, it is possible that there are other vertices common to both P and P'. The maximality of v and the choice of v' ensures that the closed walk on $P \cup Q \cup P' \cup \{vv'\}$ contains a cycle C'. There are two cases to consider.

Case 1: The diameter D is odd.
The paths P and P' both have length at most $\lfloor \frac{D}{2} \rfloor = \frac{D-1}{2}$, so C' must satisfy the following inequalities:

$$\begin{aligned} \ell(C') &\leq \ell(P \cup Q \cup P' \cup \{vv'\}) \\ &\leq \frac{D-1}{2} + (D-1) + \frac{D-1}{2} + 1 \\ &< 2D. \end{aligned}$$

Thus, C' is a short-cycle contained entirely in the interior of C, contradicting the minimality of C and completing the proof in the case that D is odd.

Case 2: The diameter D is even.
We claim that $d(v, C) = d(v', C) = \frac{D}{2}$, and that $\ell(Q) = D - 1$. Certainly, these

values are all upper bounds. Assume to the contrary that $d(v, C) < \frac{D}{2}$, or that $d(v', C) < \frac{D}{2}$, or that $\ell(Q) < D - 1$. In any of these cases, we get that:

$$\begin{aligned}\ell(C') &\leq \ell(P \cup Q \cup P' \cup \{vv'\}) \\ &< \frac{D}{2} + (D - 1) + \frac{D}{2} + 1 = 2D.\end{aligned}$$

This contradicts the minimality of C and, thus, proves the claim.

We furthermore claim that u is the only vertex of C such that $d(u, v) \leq \frac{D}{2}$, and that u' is the only vertex of C such that $d(u', v') \leq \frac{D}{2}$. Assume to the contrary that there exists a vertex u^* in $C - u$ such that $d(u^*, v) \leq \frac{D}{2}$. Let P^* be a $v - u^*$ geodesic, and let Q^* be a $u - u^*$ geodesic in the cycle C. The closed walk $P \cup Q^* \cup P^*$ contains a cycle of length at most $\frac{D}{2} + \frac{D}{2} + (D - 1) = 2D - 1$, contradicting the minimality of C. The case in which u' is not the only vertex of C with $d(u', v') \leq \frac{D}{2}$ follows similarly, completing the proof of the second claim.

For a cycle C^* of G that does not contain v, define the **v-exterior** of C^*, denoted $v\text{Ext}(C^*)$, to be the region of $\mathbb{R}^2 - C^*$ that does not contain the vertex v. We also define $v\text{Ext}[C^*] = v\text{Ext}(C^*) \cup C^*$. Let $\mathfrak{S}$ be the set of all short-cycles of G that are contained in $\text{Ext}[C]$. Since C is in $\mathfrak{S}$ (with $v\text{Ext}(C) = \text{Ext}(C)$), this set $\mathfrak{S}$ is nonempty. Choose a short-cycle A in $\mathfrak{S}$ that is v-exterior minimal, that is, choose A in $\mathfrak{S}$ such that there does not exist any short-cycle A' in $\mathfrak{S}$ having $v\text{Ext}(A') \subset v\text{Ext}(A)$.

Because A is a short-cycle, the v-exterior of A contains some vertex of G by Lemma 3.4. Let w be any vertex in $v\text{Ext}(A)$. Since the cycle C separates w from v, and $d(v, C) = \frac{D}{2}$, we must have that $d(w, C) \leq \frac{D}{2}$. The cycle A either is itself C or separates w from C so, by Remark 5.4, we have that $d(w, A) \leq \frac{D}{2}$. Repeat the entire first part of the proof, replacing the cycle C with A, and the region $\text{Int}(C)$ with the region $v\text{Ext}(A)$, to show the existence of four distinct vertices x, x', y, and y' (analogous to u, u', v, and v', respectively) in $v\text{Ext}[A]$ that satisfy the following conditions:

(1) both x and x' lie on A, and a shortest $x - x'$ path in A has length $D - 1$.

(2) $d(x, y) = \frac{D}{2}$, and every vertex w in $A - x$ satisfies $d(y, w) > \frac{D}{2}$.

(3) $d(x', y') = \frac{D}{2}$, and every vertex w in $A - x'$ satisfies $d(y', w) > \frac{D}{2}$.

Note that both the cycles C and A (which are possibly the same) separate v from both y and y'. To be possible that $d(v, y) \leq D$, it must be the case that $u = x$, since u is the unique vertex of C such that $d(u, v) \leq \frac{D}{2}$, and x is the unique vertex of A such that $d(x, y) \leq \frac{D}{2}$. Similarly, to have that $d(v, y') \leq D$, it must be the case that $u = x'$. Because x and x' must be distinct, this yields a contradiction, completing the proof. □

Corollary 5.6 *Let G be a 2-edge-connected plane graph of diameter D. If either $g(G) \geq 2D$ or $\mu(G) \geq 2D$, then $g(G) = \mu(G)$.*

PROOF: This corollary follows from Theorems 5.5 and 4.4, as well as the fact that $g(G) \leq \mu(G)$ in a 2-edge-connected plane graph. □

Theorems 3.7 and 4.4 demonstrate that if a 2-edge-connected plane graph G with diameter D has $\mu(G) \geq 2D + 1$, then G is the cycle $2D + 1$ and, thus, has girth $2D + 1$. As such, there exists a function f of the graph diameter D such that if $\mu(G) \geq f(D)$, then G contains no short cycle, and Theorem 5.5 illustrates that $f(D) \leq 2D$. The next Theorem demonstrates that the result given by Theorem 5.5 cannot be improved.

Theorem 5.7 *For each integer $D \geq 3$, there exists a 2-edge-connected plane graph G_D of diameter D such that $\mu(G_D) = 2D - 1$ and $g(G_D) = 2D - 2$.*

PROOF: Let G_D be the graph consisting of two vertices u and v, and four internally disjoint $u - v$ paths. Let two paths have length D, while the other two have length $D-1$. Embed G_D in the plane such that every face is bounded by one path of length D, and one path of length $D - 1$ (see Figure 5).

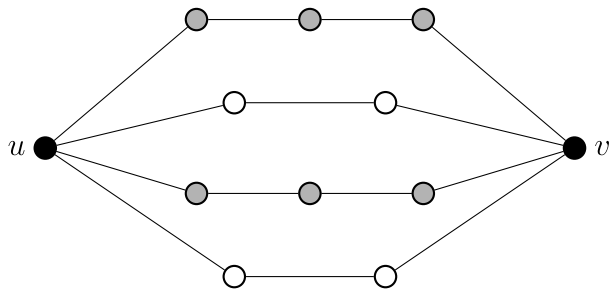

Figure 5: The graph G_4 of diameter 4, described in the proof of Theorem 5.7. Observe that $\mu(G_4) = 7$ but $g(G_4) = 6$.

Noting that any two vertices of G_D lie on a cycle of length at most $2D$, we see that the diameter of G_D is at most D. On the cycle formed by the two paths of length D, there exist two vertices at distance D, so the diameter of G_D is exactly D. Each face of G_D is bounded by a cycle formed by the union of a path of length D and a path of length $D - 1$, so every face of G_D has face-degree $2D - 1$. The cycle C formed by the union of the two paths of length $D - 1$ has length $2D - 2$ and it is, thus, a short-cycle. It is easy to see that C is the shortest cycle of G_D and, hence, $g(G_D) = 2D - 2$. □

The next series of lemmas and remarks (that are likely well-known to those working with generalised polygons) culminate in a characterisation of planar generalised polygons.

Lemma 5.8 *Generalised polygons are self-centered.*

Proof: Assume to the contrary that G is a generalised polygon with radius r, diameter D, and that $r < D$. Let T_r be a radius-preserving spanning tree of G. Since G has girth $2D$, it must contain some cycle, and so there exists at least one fundamental cycle of G with respect to T_r. By Lemma 3.6, this fundamental cycle has length at most $2r + 1$, which is strictly less than $2D$, contradicting that G is a generalised polygon. □

The next remark follows from Lemmas 5.8 and 4.3.

Remark 5.9 Generalised polygons are 2-connected.

To characterise planar generalised polygons, we will make use of two lemmas from *Algebraic Graph Theory* by Godsil and Royle [9].

Lemma 5.10 [9] *Let G be a graph of diameter D and girth $2D$, and let u and v be vertices of G. If $d(u, v) = k < D$, then there is a unique $u - v$ path of length k in G.*

Lemma 5.11 [9] *Let G be a graph of diameter D and girth $2D$, and let u and v be vertices of G. If $d(u, v) = D$, then $d(u) = d(v)$.*

We will also need the following two simple propositions.

Proposition 5.12 *Let G be a graph of diameter D and girth $2D$, and let H be a subgraph of G. If $d_H(u, v) = D$, then $d_G(u, v) = D$.*

Proof: Assume to the contrary that $d_G(u, v) < D$. Let P be a $u - v$ geodesic in H and Q a $u - v$ geodesic in G. The closed walk $P \cup Q$ has length less than $2D$ and contains some cycle, contradicting that $g(G) = 2D$. □

The next proposition follows from the well-known fact that if G is a bipartite graph and v is a vertex of G, then the vertices at odd and even distance from v form partite sets of G. Nevertheless, we include a short proof for completeness.

Proposition 5.13 *Let G be a generalised polygon of diameter D, and let u, v, and w be vertices of G. If u and v are adjacent, and $d(u, w) = D$, then $d(v, w) = D - 1$.*

Proof: Certainly $d(v, w) > D-2$, so it is suffices to show that $d(v, w) < D$. Assume to the contrary that $d(v, w) = D$, and let P be a $w - u$ geodesic, and Q a $w - v$ geodesic. Let x be the vertex of $P \cap Q$ that is furthest from the vertex w (it is possible that $x = w$). The union $P[x, u] \cup Q[x, v] \cup \{uv\}$ is an odd cycle, contradicting the fact that G is bipartite. □

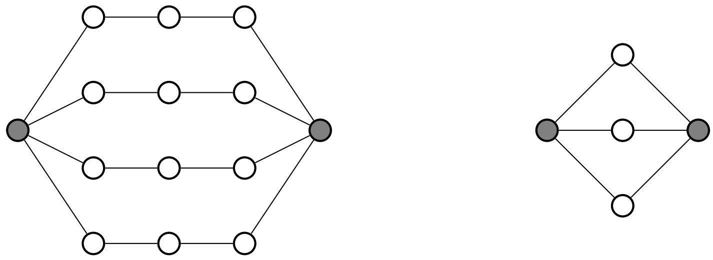

Figure 6: On the left is the unique planar generalised polygon with maximum degree and diameter both equal to 4. On the right is the unique planar generalised polygon with maximum degree 3 and diameter 2. The vertices of maximum degree are grey.

The next result, Theorem 5.14, demonstrates that for each pair (Δ, D) of integers with $\Delta \geq 2$ and $D \geq 2$, there exists a unique planar generalised polygon with maximum degree Δ and diameter D.

Theorem 5.14 *If G is a planar generalised polygon of maximum degree $\Delta \geq 2$ and diameter $D \geq 2$, then G consists of two vertices of degree Δ joined by Δ internally disjoint paths of length D.*

PROOF: Let G be a planar generalised polygon of maximum degree Δ and diameter D, with a fixed embedding as a plane graph. We may assume that $\Delta > 2$, as the cycle C_{2D} is the only generalised polygon with $\Delta = 2$ and diameter D. Let u be a vertex of degree Δ. By Corollary 5.8, there is a vertex v such that $d(u, v) = D$. By Lemma 5.11, the vertex v satisfies $d(v) = \Delta$.

Label the vertices of $N(u) = \{a_1, a_2, \ldots, a_\Delta\}$ such that any pair a_i and a_{i+1} of vertices (subscripts are taken mod Δ) are on the boundary of the same face. By Proposition 5.13, we have that $d(a_i, v) = D - 1$ for all i in $\{1, 2, \ldots, \Delta\}$. By Lemma 5.10, there is a unique $a_i - v$ path of length $D - 1$ for all i. Each $a_i - v$ path of length $D - 1$ can be extended to a $u - v$ path of length D. Let P_i be the extended $u - v$ path containing the vertex a_i, and let b_i be the vertex of P_i that is adjacent to v.

The paths P_i and P_j are internally disjoint whenever $i \neq j$. Were they not, the union $P_i \cup P_j$ would contain a cycle of length less than $2D$.

We now have that G contains as a subgraph Δ internally disjoint $u - v$ paths of length D (the paths $P_1, P_2, \ldots, P_\Delta$). Let H be the subgraph containing only the union of all the P_i's, and denote by C_i the cycle of length $2D$ on $P_i \cup P_{i+1}$ (subscripts are taken mod Δ). The graph H divides the plane into Δ regions, each bounded by a cycle C_i.

What remains is to show that $G = H$. Since $g(G) = 2D$ any two vertices of H lie on a cycle of length $2D$, no edge can be added between two vertices of H, so it suffices to show that $V(G) = V(H)$. Thus, we assume to the contrary that G contains a vertex not in H. Since G is connected and $d(u) = d(v) = \Delta$, an internal vertex of some P_i has a neighbour in $G - H$. Let x be the internal vertex of P_i and let y be its neighbour in $G - H$. Without loss of generality, the vertex y is in the region bounded by C_i. Since $\Delta \geq 3$, there is some path P_j of H such that $P_j \cap C_i = \{u, v\}$, and the internal vertices of P_j are not in the same region of $\mathbb{R}^2 - C_i$ as the vertex y.

Let z be the vertex of P_j that satisfies $d_H(x, z) = D$. We know by Proposition 5.12 that $d_G(x, z) = D$, and from Proposition 5.13 that $d_G(y, z) = D - 1$. Thus, there is a $y - z$ geodesic Q of length $D - 1$ in G, which does not contain x, but it must contain some other vertex of C_i. Let w be the vertex of $Q \cap C_i$ that is closest to y, and note that since w and z must be distinct, the path segment $Q[y, w]$ has length at most $D - 2$. Since x and w are distinct, the cycle C_i is divided into two internally disjoint $x - w$ paths. Let R be the shorter of these two paths, and note that $\ell(R) \leq D$. The union of paths $R \cup Q[y, w] \cup \{xy\}$ forms a cycle that has length at most $D + (D - 2) + 1 < 2D$, a contradiction since $g(G) = 2D$. □

Corollary 5.15 *A plane graph of diameter D is a generalised polygon if and only if it is 2-edge-connected and $2D$-face-degree regular.*

Proof: By Theorem 5.14 and Remark 5.9, it is clear that a plane graph that is a generalised polygon is 2-edge-connected and $2D$-face-degree regular. The converse follows from Theorem 5.5 and Remark 5.1. □

6 The degree/diameter problem for face-degree regular plane graphs

We obtain some new results on the degree/diameter problem for face-degree regular graphs as corollaries of the structural results obtained thus far.

Corollary 6.1 *If G is a 2-edge-connected plane graph of diameter D and order n in which every face has degree $2D + 1$, then $n = 2D + 1$.*

Proof: This follows from Theorem 4.4. □

Corollary 6.2 *If G is a 2-edge-connected plane graph of diameter D, maximum degree Δ and order n in which every face has degree $2D$, then $n = \Delta(D - 1) + 2$.*

Proof: This follows from Corollary 5.15 and Theorem 5.14. □

Note that in Corollary 6.2, we cannot replace the condition that G is $2D$-face-degree regular with the condition that G has minimum face-degree $2D$. To show this, we create a plane graph $G(\Delta, D)$ with maximum degree $\Delta \geq 3$, diameter $D \geq 2$, minimum face-degree $2D$, and order $\Delta(D - 1) + 3$ as follows. Let u, v, and w be three vertices, and let u and v be adjacent. Create two internally disjoint paths of length D between v and w, and $\Delta - 2$ internally disjoint paths of length D between u and w. This completes the construction of $G(\Delta, D)$ (see Figure 7).

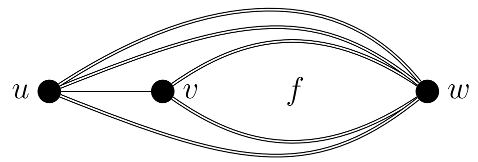

Figure 7: Consider every double-line in the diagram to be a path of length D. The above graph is $G(5, D)$, with diameter D and order $5(D-1)+3$. Observe that the face f has degree $2D$.

7 Further questions

Question 7.1 Corollary 5.6 gives a sufficient condition for the minimum face-degree and girth of a 2-edge-connected plane graph to be equal. Can other sufficient/necessary conditions be found for these two parameters to be equal?

Question 7.2 If a plane graph has minimum face-degree μ and girth g, what can be said about the quantity $\mu - g$?

We raise the following conjectures as possible answers to Question 7.2.

Conjecture 7.3 Consider a 2-edge-connected plane graph with minimum face-degree μ, girth g, and diameter D, and let $k \geq 0$ be an integer. If $\mu \geq 2D - k$, then $\mu - g \leq k$.

Conjecture 7.4 A 2-edge-connected plane graph with diameter D, girth g and minimum face-degree μ satisfies

$$D \geq \frac{2\mu - g - 1}{2}.$$

Notice that the first conjecture yields Corollary 5.6 by setting $k = 0$. When $\mu = g$, the second conjecture gives $\frac{\mu-1}{2} \leq D$, which is a weakening of Theorem 3.7. As such, both conjectures certainly hold when μ is sufficiently large.

Question 7.5 We resolved the degree/diameter problem for 2-edge-connected, $2D$-face-degree regular graphs of diameter D by Corollary 6.2. Can we obtain a similar bound for 2-edge-connected plane graphs with minimum face-degree $2D$?

In light of the example in Figure 7, the author has the following conjecture regarding this question.

Conjecture 7.6 If G is a 2-edge-connected plane graph of diameter D, maximum degree Δ, order n and minimum face-degree $2D$, then $n \leq \Delta(D-1)+3$, and this bound is sharp.

Question 7.7 The degree/diameter problem in planar graphs that have every face bounded by a cycle (or circuit) of length ρ has only been studied in depth for $\rho = 3$ and $\rho = 4$. What bounds can be found for the case where ρ is an arbitrary integer?

The author believes that proof by Dalfó, Huemer and Salas of Theorem 10 in [6] may be adapted to solve the degree/diameter problem in ρ-face-degree regular graphs when ρ is even.

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