Note on the Parameters of Balanced Designs

W D Wallis

Southern Illinois University, Carbondale, IL 62901, USA

Dedicated to the memory of Alan Rahilly, 1947 – 1992

Abstract

Suppose a balanced design exists which has constant replication number r, and suppose its parameters v, b, r and λ are such that a balanced incomplete block design could exist. Then the design has constant block size.

In discussing block designs we use the usual notations and terminology of the combinatorial literature (see, for example, [1], [2]) - "balance" will always mean pairwise balance and blocks do not contain repeated elements. It is well-known that a balanced design need not have constant block-size or constant replication number. However, it is easy to show ([2, p.22]) that in any balanced design with constant block-size, the replication number r must also be constant. The converse is not true; there exist (r, λ)-designs, with constant λ and constant replication number r, which have more than one size of block - an example is the (3,1)-design on six treatments with blocks 015, 024, 123, 345, 03, 14, 25. Our object here is to prove the following partial converse.

Theorem. If the positive integers v, b, r and λ are such that

$$bk = vr \qquad (1)$$

$$\lambda(v-1) = r(k-1) \qquad (2)$$

for some integer k, and if there is an (r, λ)-design on v treatments with b blocks, then the design has constant block-size k (whence it is a balanced incomplete block design).

Proof. Suppose such an (r, λ)-design exists, with b_1 blocks of size k_1, b_2 blocks of size k_2, and so on. Counting blocks, elements of blocks and ordered pairs of elements of blocks, we get

$$\sum b_i = b \qquad (3)$$

$$\sum b_i k_i = vr \qquad (4)$$

$$\sum b_i k_i (k_i - 1) = \lambda v(v-1) \qquad (5)$$

respectively; substituting (4) in (5),

$$\sum b_i k_i^2 = vr + \lambda v(v-1). \qquad (6)$$

If κ is the mean value of the k_i, then $b\kappa = vr$ from (3) and (4); so, using (6),

$$\begin{aligned}\sum b_i(k_i - \kappa)^2 &= \sum b_i k_i^2 - b\kappa^2 \\ &= vr + \lambda v(v-1) - vr\kappa \\ &= \lambda v(v-1) - vr(\kappa - 1). \qquad (7)\end{aligned}$$

From (1), $\kappa = k$. So, substituting from (2) into (7),

$$\begin{aligned}\sum b_i(k_i - \kappa)^2 &= \sum b_i(k_i - k)^2 \\ &= \lambda v(v-1) - vr(\kappa - 1) \\ &= \lambda v(v-1) - v\,\lambda(v-1) \quad = \quad 0.\end{aligned}$$

So $k_i = \kappa$ for each i.

References

[1] A P Street and D J Street, *Combinatorics of Experimental Design*, Clarendon Press, Oxford, 1987.

[2] W D Wallis, *Combinatorial Designs*, Marcel Dekker, New York, 1988.

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