The Groups A_7, A_8 and a Projective Plane of Order 16

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Dedicated to the memory of Alan Rahilly, 1947–1992

Abstract

Three results about the groups A_7, A_8, $L_3(2)$, $L_4(2)$ and the Lorimer-Rahilly plane are proved in a unified way.

My principal contact with Alan Rahilly was through a certain projective plane of order 16. He had found it as one of an infinite class of generalized Hall planes he constructed in his thesis at Sydney University [9], while I had found it as an isolated plane which I identified through the isomorphism between the groups A_8 and $L_4(2)$. [5]. It is now called the Lorimer-Rahilly plane. It has played a critical part in the classification of projective planes in that it has many interesting properties : for example, it is the only known plane of type $(6, m)$, as described by D.R. Hughes; it has type $(6, 2)$ and the only other value of m possible is 3 [6].

While Alan constructed an infinite class of planes, the one I constructed appears to be the most interesting of them. My original proof depended on a certain isomorphism between the groups A_8 and $L_4(2)$, but I gave a more direct one in [7]. However, much more turns out to be possible. By straight-forward elementary arguments it is possible, in a unified manner, to

1. prove that A_8 and $L_4(2)$ are isomorphic,

2. prove that A_7 has a 2-transitive representation of degree 15,

3. construct the Lorimer-Rahilly plane and present a large part of its collineation group.

The discovery that the groups A_8 and $L_4(2)$ are isomorphic seems to have been made by Jordan and it appears in his classic, "Traité des Substitutions et les Équations Algébriques" [4], where the proof is given in terms of the Galois group of a polynomial equation of degree 8. A group-theoretic proof was given by Moore [8], who showed, in the same paper, the fact about A_7 which appears here as Proposition 6. It was taken up again, by Dickson, and his proof appears in [1].

While none of these results is new, it seems appropriate to dedicate their unification to the memory of Alan Rahilly.

1. The groups $L_3(2)$, A_7, $L_4(2)$ and A_8.

The purpose of this section is to show that the groups A_8 and $L_4(2)$ are isomorphic and that A_7, as a subgroup of $L_4(2)$, has a 2-transitive representation of degree 15.

The story begins with the eight numbers $0, 1, 2, \ldots, 7$, taken as the vectors of a 3-dimensional vector space, V_3, over the field of order 2. To be definite, let us take the following as the sets of non-zero vectors of the 2-dimensional subspaces of V_3 : $\{1,2,3\}$, $\{3,4,5\}$, $\{5,6,1\}$, $\{1,7,4\}$, $\{3,7,6\}$, $\{5,7,2\}$, $\{2,4,6\}$. Otherwise described, these are the lines of a projective plane of order 2, which will be denoted by π_2.

In the sequel, A_8 will be taken as the alternating group on the vectors of V_3, A_7 will be the stabilizer of 0 in A_8, $L_3(2)$ will be the full linear group of V_3 and T will be the group of translations of V_3. Thus, $L_3(2)$ is a subgroup of A_7, T is a complement of A_7 in A_8 and $TL_3(2)$ is a subgroup of A_8 of index 15.

The key to this paper is a set, M, of permutations of A_8 defined from the 2-dimensional subspaces of V_3 (or from the lines of π_2):

$$M = \{1,\ (123),\ (132),\ (345),\ (354),\ (561),\ (516),\ (174),\ (147),\ (376),\ (367),\ (572),\ (527),\ (246),\ (264)\}.$$

The principal property of M is described in the first Proposition.

Proposition 1. *M is a set of representatives of the left cosets of $TL_3(2)$ in A_8 and of $L_3(2)$ in A_7.*

Proof. If m_1, m_2 are two different members of M, then $m_1^{-1}m_2$ is a 3-cycle or a 5-cycle. On the other hand, $TL_3(2)$ contains no such element. Thus, the members of M lie in different cosets of $TL_3(2)$ and, as this subgroup has index 15 in A_8, they form a set of coset representatives. As $M \subseteq A_7$, they also form a set of coset representatives of $L_3(2)$ in A_7.

That proves Proposition 1.

Proposition 1 defines a representation of the group A_8 as a permutation group on M: if $g \in A_8$, define $g\colon M \to M$ by the equation

$$g(m)TL_3(2) = gmTL_3(2).$$

The next Proposition calculates this representation explicitly for some particular members of A_8.

Proposition 2.

1. *If $g = (abc)$ and (ade) are members of M and $\{a,b,c\} \neq \{a,d,e\}$, then*

$$g\big((ade)\big) = (ecf)$$

where f is the the third point on the line joining e and c.

2. *Let t be the translation of T which maps 0 onto a. If (abc), (bde) are members of M and $\{a,b,c\} \neq \{b,d,e\}$, then*

$$t\big((abc)\big) = (abc)$$
$$t\big((bde)\big) = (bed).$$

Proof.

1. When multiplied as permutations within A_7,

$$(efc)(abc)(ade) = (adfc)(be).$$

From the 3-cycles mentioned, the lines of the projective plane, π_2, can be reconstructed as $\{a,b,c\}$, $\{a,d,e\}$, $\{c,f,e\}$, $\{b,d,f\}$, $\{a,g,f\}$, $\{c,g,d\}$, $\{e,g,b\}$ and it is easy to see that these sets are mapped among themselves by $(adfc)(be)$; ie $(adfc)(be) \in L_3(2)$. Hence, $(abc)(ade)L_3(2) = (ecf)L_3(2)$: i.e.

$$g\big((ade)\big) = (ecf).$$

2. From the information given, it follows that $t = (0a)(bc)(dx)(ey)$ where x, y are the two remaining vectors of V_3. Now in terms of permutations

$$t(abc) = (abc)(0c)(ab)(dy)(ex)(xy)(de),$$
$$t(bde) = (bed)(0a)(bc)(dx)(ey)(cxy)(bde)$$

where $(0c)(ab)(dy)(ex)$, $(0a)(bc)(dx)(ey)$ are both members of T and $(xy)(de)$, $(cxy)(bde)$ are members of $L_3(2)$. Thus,

$$t(abc)TL_3(2) = (abc)TL_3(2)$$
$$t(bde)TL_3(2) = (bed)TL_3(2)$$

and the result follows.

The group A_8 has now been represented as a permutation group of degree 15 and the representations of some of its members have been calculated. The aim, now, is to show that A_8, in this representation, acts linearly on a 4-dimensional vector space over the field of order 2. To this end, put

$$M^+ = M \cup \{0\}$$

and define addition in M^+ as follows

Definition.

1. If (abc) is a member of M, then $\{0, 1, (abc), (acb)\}$ is to be an elementary abelian group.

2. If (abc), (ade) are members of M and $\{a, b, c\} \neq \{a, d, e\}$, then

$$(abc) + (ade) = (afg)$$

where $\{a, f, g\}$ is the third line of π_2 through a, and f is the third point on the line which passes through b and d.

Proposition 3.

1. *M^+ is a 4-dimensional vector space over the field of order 2.*

2. *The members of M and T, in their representations on M^+, are linear transformations.*

Proof.

1. As the proof consists solely of checking axioms in a number of different cases, it won't be given.

2. Consider a member, $g = (abc)$, of M. Then, the following can be taken as a basis of M^+: 1, (abc), (ade), (ecf), where f is the third point on the line through e and c. Then it follows from Proposition 2(1), that

$$\begin{aligned} g(1) &= (abc), \\ g\big((abc)\big) &= (acb) = 1 + (abc), \\ g\big((ade)\big) &= (ecf), \\ g\big((ecf)\big) &= (ebg) = (ade) + (ecf). \end{aligned}$$

Thus, the linear map on M^+ having the same action as g on those basis vectors has matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is routine to check that this linear map coincides with g on all vectors of M^+.

Now use the same method on a translation, t, of T, which maps 0 onto a. Taking a basis of the form 1, (abc), (adc), (bdf) leads, using Proposition 2(2), to t being a linear map with matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next, a small result about A_7 is required.

Proposition 4. *The permutations of M generate A_7.*

Proof. Let (pqr) be any 3-cycle of $L_3(2)$. If x is the third point of the line of π_2 containing q, r and y is the third point of the line containing p, x, then

$$(pqr) = (yxp)(xqr)(ypx)$$

represents (pqr) as a product of members of M. As A_7 is generated by its 3-cycles, it is also generated by the 3-cycles of M.

Sufficient has now been proved to establish the principal results of this section

Proposition 5. *The groups A_8 and $L_4(2)$ are isomorphic.*

Proof. The section has established a representation of A_8 as a group of permutations of the set M and, thus, of the 4-dimensional vector space M^+ over the field of order 2. By Propositions 3 and 4, a set of generators of A_8 acts linearly on M^+ in this presentation: hence, so does A_8. As A_8 is simple, the representation is faithful and, as A_8 and $L_4(2)$ have the same orders, they are isomorphic.

Proposition 6. *As a subgroup of $L_4(2)$, A_7 acts sharply transitively on linearly independent triples of vectors of the 4-dimensional vector space over the field of order 2. In particular, A_7 acts 2-transitively on the 15 non-zero vectors of this vector space.*

Proof. Work with the representation of A_8 as $L_4(2)$ that has just been presented, with M^+ being the vector space. As M is a set of left coset representatives of $L_3(2)$ in A_7, A_7 acts transitively on M and, clearly, $L_3(2)$ is the stabilizer of 1. From the definition of addition in M^+, its members 1, (abc), (ade) are linearly independent if and only if $\{a,b,c\} \neq \{a,d,e\}$. Examination of the action of $L_3(2)$ on V_3 shows that $L_3(2)$ acts sharply transitively on ordered pairs $\big((abc),(ade)\big)$ with $\{a,b,c\} \neq \{a,d,e\}$. That proves the Proposition.

2. The plane of order 16.

In this section, the Lorimer-Rahilly plane will be constructed, using the method of [7], and some partial information about its group of collineations will be given.

The set M^+ is a 4-dimensional vector space over the field of order 2. A multiplication is defined in M^+ by the rules:

1. $0m = m0 = 0$ for all m in M^+;
2. If (abc) is a 3-cycle of M^+ then $\{1, (abc), (acb)\}$ is a cyclic group of order 3;
3. If (abc), (ade) are 3-cycles of M^+ and $\{a, b, c\} \neq \{a, d, e\}$, then

$$(abc)(ade) = (ecf).$$

It is a consequence of Proposition 3 that multiplication on the left in M^+ is a linear transformation: i.e. the left distributive law holds in M^+:

$$m_1(m_2 + m_3) = m_1m_2 + m_1m_3$$

Moreover, by Proposition 1, right cancellation holds: if $m_1m_3 = m_2m_3$ then $m_1 = m_2$ or $m_3 = 0$.

This is enough to ensure that the following familiar construction leads to an affine plane of order 16. Take, as the points of the plane, all ordered pairs (x, y) with $x, y \in M^+$. For each pair (m, c), the set of points (x, y) with $y = mx + c$, is a line. For each b in M^+, the set of all points (b, y) is a line. The Lorimer-Rahilly plane, as a projective plane, is constructed from this affine plane by adjoining a line at infinity in the classical way.

Because of the way the plane has been constructed, it is a translation plane: if a, b are two fixed members of M^+, the translation

$$(x, y) \rightarrow (x + a, y + b)$$

of the points of the plane induces a collineation of it. Also, because of its construction, other collineations are induced by the members of $L_3(2)$: if ϕ is a member of $L_3(2)$, then the mapping

$$(x, y) \rightarrow (\phi(x), \phi(y))$$

where $\phi(0) = 0$, $\phi(1) = 1$, $\phi(abc) = (\phi(a)\phi(b)\phi(c))$, also induces a collineation of the plane.

If A is the group of translations, then all these collineations together form a subgroup $AL_3(2)$ of index 6 in the full group of collineations of the plane. For more details see [5].

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