ON THE CYCLOTOMIC IDENTITY AND RELATED PRODUCT EXPANSIONS

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Abstract.

The cyclotomic identity, that

$$\frac{1}{1-\alpha z} = \prod_{n=1}^{\infty} \left(\frac{1}{1-z^n}\right)^{M(\alpha,n)} ,$$

where $M(\alpha,n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right)\alpha^d$, and μ is the classical Möbius function, has several natural analogues. Polynomials in α of degree n related to $M(\alpha,n)$ in these identities share interesting properties with $M(\alpha,n)$. Many special cases are of combinatorial interest.

1. The Witt formula and a product expansion

The function of the two variables α and n given by $M(\alpha,n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right)\alpha^d$ arises naturally in many combinatorial problems. $M(\alpha,n)$ is a polynomial of degree n in α with rational coefficients which takes on integer values for integer arguments. It is sometimes called the necklace counting polynomial because it can be interpreted as enumerating non-periodic circular strings of n beads that can be strung from beads of at most α distinct colours. It is called the Witt formula when used to count the number of monic irreducible polynomials of degree n over $GF(\alpha)$ in the case when $\alpha = p^k$ for some prime p and some positive integer k. It also gives the dimension of the subalgebra generated by the homogeneous elements of degree n in the free Lie algebra over a set at α elements. Information about the parity of $M(\alpha,n)$ was obtained in [1]. References [4] and [5] offer the first combinatorial proof of the cyclotomic identity. We are interested here in several identities involving similar polynomials which arise in related identities. As a paradigm, we begin by giving a proof of the cyclotomic identity which relies on manipulations of formal power series.

Theorem 1 (The Cyclotomic Identity)

$$\frac{1}{1-\alpha z} = \prod_{n=1}^{\infty}\left(\frac{1}{1-z^n}\right)^{M(\alpha,n)} \quad \text{where} \quad M(\alpha,n) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\alpha^d .$$

Proof. From the identity as given, write

(1) $$\log(1-\alpha z) = \sum_{j=1}^{\infty} M(\alpha,j)\log(1-z^j) .$$

Series expansions give

(2) $$\sum_{n=1}^{\infty}\frac{\alpha^n z^n}{n} = \sum_{j=1}^{\infty} M(\alpha,j)\sum_{i=1}^{\infty}\frac{z^{ji}}{i} .$$

The change of variable $n = ij, d = j$ gives

(3) $$\sum_{n=1}^{\infty}\frac{\alpha^n z^n}{n} = \sum_{n=1}^{\infty}\sum_{d|n} M(\alpha,d)\frac{dz^n}{n} .$$

Equating coefficients of z^n gives

(4) $$\frac{\alpha^n}{n} = \sum_{d|n} M(\alpha,d)d/n$$

or

(5) $$\alpha^n = \sum_{d|n} dM(\alpha,d) .$$

Here we could write the recurrence

(6) $$M(\alpha,n) = \frac{1}{n}\left(\alpha^n - \sum_{\substack{d|n \\ d\neq n}} dM(\alpha,d)\right) ,$$

but Möbius inversion applies to give

(7) $$M(\alpha,n) = \frac{1}{n}\sum_{d|n} \mu(n/d)\alpha^d \ .$$

The variations we wish to consider involve signs in the factors of the product and the placement of the undetermined function of α and n as a coefficient rather than as an exponent. First, motivated by the analogy with generating functions for general partitions, we study the product that corresponds to the generating function for partitions into distinct parts.

2. A dual expansion to the Cyclotomic Identity

We study the nature of the exponents $N(\alpha,n)$ in the formal expansion

(A) $$1+\alpha z = \prod_{n=1}^{\infty}(1+z^n)^{N(\alpha,n)}$$

It is most convenient to express $N(\alpha,n)$ in terms of $M(\alpha,n)$.

Theorem 2. $1+\alpha z = \prod_{n=1}^{\infty}(1+z^n)^{N(\alpha,n)}$, where for $n = 2^k n', n'$ odd, $N(\alpha,n) = M(\alpha,n)$ if $k = 0$, else $N(\alpha,2^k n') = 1/2^k(2^{k-1}M(\alpha,n') - 2^{k-2}M(\alpha^2,n') - \ldots - M(\alpha^{2^{k-1}},n') - M(\alpha^{2^k},n'))$. *$N(\alpha,n)$ is a polynomial in α of degree n that assumes integer values for integer arguments.*

Proof. Proceeding through the steps of the Cyclotomic Identity proof, we obtain

(2A) $$\sum_{n=1}^{\infty}(-1)^n\frac{\alpha^n z^n}{n} = \sum_{j=1}^{\infty}N(\alpha,j)\sum_{i=1}^{\infty}(-1)^i\frac{z^{ji}}{i}$$

and

(5A) $$(-1)^n\alpha^n = \sum_{d|n}(-1)^{n/d}dN(\alpha,d)\ .$$

The next step in the derivation for $M(\alpha,n)$ does not obtain, however, since (5A) is not a formula to which Möbius inversion directly applies. Some properties about $N(\alpha,n)$ may be deduced from the recurrence

(6A) $$N(\alpha,n) = \frac{1}{n}\Big(\sum_{\substack{d|n\\ d\neq n}}(-1)^{n/d}dN(\alpha,d) - (-1)^n\alpha^n\Big)\ ,$$

but a better approach is to modify (5A) so that Möbius inversion is useful. Write $n = 2^k n'$ for n' odd. If $k = 0$ then (5A) reduces to (5) and $N(\alpha, n) = M(\alpha, n)$, so assume $k \geq 1$. Then (5A) says

$$\begin{aligned}
(-1)^n\alpha^n = \alpha^n = (\alpha^{2^k})^{n'} \\
= \sum_{d'|n'} (-1)^{n/d'} d' N(\alpha, d') + (-1)^{n/2d'} 2d' N(\alpha, 2d') + \ldots \\
+ (-1)^{n/2^{k-1}d'} 2^{k-1} d' N(\alpha, 2^{k-1} d') + (-1)^{n/2^k d'} 2^k d' N(\alpha, 2^k d') \\
= \sum_{d'|n'} d'(N(\alpha, d') + 2d' N(\alpha, 2d') + \ldots \\
+ 2^{k-1} d' N(\alpha, 2^{k-1} d') - 2^k d' N(\alpha, 2^k d') .
\end{aligned}$$

This can be written as

$$(\alpha^{2^k})^{n'} = \sum_{d'|n'} \left(\sum_{i=0}^{k-1} 2^i d' N(\alpha, 2^i d') - 2^k d' N(\alpha, 2^k d') \right) .$$

Now Möbius inversion applies to the expressions involving n', to give

$$\sum_{i=0}^{k-1} 2^i n' N(\alpha, 2^i n') - 2^k n' N(\alpha, 2^k n') = \sum_{d'|n'} \mu(n'/d')(\alpha^{2^k})^{d'} .$$

Since $2^k n' = n$, we have

$$\begin{aligned}
N(\alpha, n) &= \sum_{i=0}^{k-1} N(\alpha, 2^i n')/2^{k-i} - \frac{1}{n} \sum_{d'|n'} \mu(n'/d')(\alpha^{2^k})^{d'} \\
&= \sum_{i=0}^{k-1} N(\alpha, 2^i n')/2^{k-1} - \frac{1}{2^k} M(\alpha^{2^k}, n') .
\end{aligned} \tag{7A}$$

With the last step iterated, we have the formula we wanted, expressing $N(\alpha, n)$ in terms of $M(\alpha^{2^i}, 2^i n')$ for $0 \leq i \leq k$.

By (6A), $N(\alpha, n)$ is a polynomial in α of degree n. We establish that $N(\alpha, n)$ assumes integer values for integer arguments by induction on n. First, observe that $N(\alpha, 1) = \alpha$. Now suppose that $N(\alpha, 1)$, $N(\alpha, 2), \ldots, N(\alpha, k)$ are integers. Then $\prod_{n=1}^{k} (1 + z^n)^{N(\alpha,n)}$ is a polynomial with integer coefficients, chosen to make the partial product $1 + \alpha z + 0z^2 + 0z^3 + \ldots + 0z^k +$ other terms, where the other terms may have non-zero coefficients. Suppose z^{k+1} has the integer coefficient c_{k+1}.

Then $N(\alpha, k+1)$ is chosen to make the coefficient of z^{k+1} in $\prod_{n=1}^{k+1}(1+z^n)^{N(\alpha,n)}$ to be zero. Hence $N(\alpha, k+1) = -c_{k+1}$, an integer.

3. General product identities, and other representations for $N(\alpha, n)$

We begin with a result whose proof belies its importance. It gives a simple relationship between the factorization of $A(z)$ in the form $\prod_{n=1}^{\infty}(1+z^n)^{b_n}$ and $\prod_{n=1}^{\infty}(1-z^n)^{b_n}$.

Theorem 3. Given the factorization $A(z) = \prod_{n=1}^{\infty}(1+z^n)^{b_n}$, we can deduce that $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n}$ where

$$d_{2n-1} = b_{2n-1}, \qquad d_{2n} = b_{2n} - b_n, (n \geq 1)\,. \tag{8}$$

Conversely, if $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n}$ is known, then $A(z) = \prod_{n=1}^{\infty}(1+z^n)^{b_n}$ where, for each n,

$$b_n = \sum_{i=0}^{k} d_{2^i n'}, \quad \text{where} \quad n = 2^k n', k \geq 0, \ n' \ \text{odd}. \tag{9}$$

Proof. From

$$\left(\prod_{n=1}^{\infty}(1+z^n)^{b_n}\right)\left(\prod_{n=1}^{\infty}(1-z^n)^{b_n}\right) = \prod_{n=1}^{\infty}(1-z^{2n})^{b_n}$$

we deduce

$$\prod_{n=1}^{\infty}(1+z^n)^{b_n} = \prod_{n=1}^{\infty}(1-z^{2n})^{b_n - b_{2n}} \prod_{n=1}^{\infty}(1-z^{2n-1})^{-b_{2n-1}}. \tag{10}$$

It follows that $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n}$, where

$$\begin{aligned} d_{2n-1} &= b_{2n-1} \quad \text{and} \\ d_{2n} &= b_{2n} - b_n \qquad n \geq 1. \end{aligned} \tag{11}$$

The converse follows by induction on the exponent of the highest power of 2 dividing n.

Applying (9) to the Cyclotomic Identity, we obtain the following result.

Corollary 4. $\frac{1}{1-\alpha z} = \prod_{n=1}^{\infty}(1+z^n)^{\hat{N}(\alpha,n)}$, where

$$\hat{N}(\alpha,n) = M(\alpha,n), \quad n \quad \text{odd}$$

(12)

$$\hat{N}(\alpha,n) = \sum_{r=0}^{k} M(\alpha,2^r n'), n = 2^k n', k \geq 0, \; n' \quad \text{odd}\,.$$

From (8) we obtain

Corollary 5. $1+\alpha z = \prod_{n=1}^{\infty}(1+z^n)^{N(\alpha,n)} = \prod_{n=1}^{\infty}(1-z^n)^{-\hat{M}(\alpha,n)}$, where

(13)

$$\hat{M}(\alpha,n) = N(\alpha,n), \qquad n \quad \text{odd}$$
$$\hat{M}(\alpha,n) = N(\alpha,n) - N(\alpha,n/2), \qquad n \quad \text{even.}$$

Note that we can rewrite the equation in Corollary 4 as

$$1+\alpha z = \prod_{n=1}^{\infty}(1+z^n)^{-\hat{N}(-\alpha,n)},$$

whence we obtain another representation of $N(\alpha,n)$:

Corollary 6. $N(\alpha,n) = -\hat{N}(-\alpha,n) = -\sum_{r=0}^{k} M(-\alpha,2^r n')$, where $n = 2^k n'$, $k \geq 0$, n' odd.

Several other general theorems related to these expansions are given below.

Theorem 7. If $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n}$ then

$$\frac{1}{A(-z)} = \prod_{n=1}^{\infty}(1+z^n)^{c_n} \qquad \text{where}$$

$$c_n = d_n \,, \; n \;\; \text{odd}$$

$$c_{2n} = -\sum_{r=0}^{k} d_{2^{r+1}n'}, n = 2^k n' \,, \; n' \;\; \text{odd.}$$

Proof. If $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n}$ we want to describe the coefficients $\{c_n\}$ of $\frac{1}{A(-z)} = \prod_{n=1}^{\infty}(1+z^n)^{c_n}$.

From
$$A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n},$$
$$\frac{1}{A(-z)} = \prod_{\substack{n=1 \\ n\ odd}}^{\infty}(1+z^n)^{d_n} \prod_{n=1}^{\infty}(1-z^{2n})^{d_{2n}},$$

so $c_n = d_n$ for n odd.

By Theorem 3, if $B(z) = \prod_{n=1}^{\infty}(1-z^{2n})^{d_{2n}}$ then $B(z) = \prod_{n=1}^{\infty}(1+z^{2n})^{c_{2n}}$ where

$$c_{2n} = -\sum_{r=0}^{k} d_{2^{r+1}n'}, \quad n = 2^k n'\ ,\ n' \quad \text{odd}.$$

A related fact is that if $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n}$ then $\frac{1}{A(-z)} = \prod_{n=1}^{\infty}(1-z^n)^{e_n}$, where $e_n = -d_n$ if n is odd, $e_n = d_n + d_{n/2}$ if $n \equiv 2 \bmod 4$, and $e_n = d_n$ if $n \equiv 0 \bmod 4$.

Corollary 8. For $A(z) = \frac{1}{1-\alpha z} = \prod_{n=1}^{\infty}(1-z^n)^{-M(\alpha,n)}$, we have

$$1+\alpha z = \prod_{n=1}^{\infty}(1+z^n)^{N(\alpha,n)}$$

where
$$N(\alpha,n) = M(\alpha,n) \quad n \quad \text{odd}$$
$$N(\alpha,2n) = -\sum_{r=0}^{k} M(\alpha, 2^{r+1}n') \quad , \quad n = 2^k n'\ ,\ n' \quad \text{odd}.$$

This is another representation of N in terms of M. From the identity used in Theorem 3 we obtain

Theorem 9. If $A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-b_n}$ then

$$\prod_{n=1}^{\infty}(1+z^n)^{b_n} = A(z)/A(z^2)\,.$$

Applying Theorem 9 to the cyclotomic identity, we obtain

Corollary 10.

$$\prod_{n=1}^{\infty}(1+z^n)^{M(\alpha,n)} = \frac{1-\alpha z^2}{1-\alpha z}.$$

$$\prod_{n=1}^{\infty}(1+z^n)^{\hat{M}(\alpha,n)} = \frac{1+\alpha z}{1+\alpha z^2}.$$

A dual result to Theorem 9 is also available.

Theorem 11. If $A(z) = \prod_{n=1}^{\infty}(1+z^n)^{b_n}$ then

$$\prod_{n=1}^{\infty}(1-z^n)^{-b_n} = \prod_{k=0}^{\infty} A(z^{2^k}).$$

The result corresponding to Corollary 10 is

Corollary 12.

$$\prod_{n=1}^{\infty}(1-z^n)^{-N(\alpha,n)} = \prod_{k=0}^{\infty}(1+\alpha z^{2^k}).$$

$$\prod_{n=1}^{\infty}(1-z^n)^{-\hat{N}(\alpha,n)} = \prod_{k=0}^{\infty}(1-\alpha z^{2^k})^{-1}.$$

We remark that Theorem 3 can be extended by viewing the factors in the infinite product as more general partial sums of geometric series. Thus if

$$A(z) = \prod_{n=1}^{\infty}(1+z^n+z^{2n}+\ldots+z^{(m-1)n})^{b_n},$$

we have the alternate representation

$$A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_m(n)},$$

where $d_m(n) = b_n$ if m does not divide n, and $d_m(n) = b_n - b_{n/m}$ otherwise. The converse is that for

$$A(z) = \prod_{n=1}^{\infty}(1-z^n)^{-d_n},$$

$$A(z) = \prod_{n=1}^{\infty}\left(\sum_{j=0}^{m-1} z^{jn}\right)^{b_m(n)},$$

where

$$b_m(n) = \sum_{i=0}^{k} d_{m^i n'}$$

for $n = m^k n', m$ does not divide n'.

This approach gives, in the spirit of Theorem 9, that

$$A(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-b_n}$$

implies

$$\prod_{n=1}^{\infty} \left(\sum_{j=0}^{m-1} z^{jn} \right) = \frac{A(z)}{A(z^m)}$$

Another approach to generalizing these identities involves working with factors that are more general polynomials on both sides. For example, we can write

$$\prod_{n=1}^{\infty} \left(\sum_{k=0}^{2n} (-1)^k z^{kn} \right)^{b_n} = \prod_{n=1}^{\infty} \left(\sum_{k=0}^{2n} z^{kn} \right)^{-d_n},$$

where $d_{2n-1} = b_{2n-1}$, and $d_{2n} = b_{2n} - b_n$.

4. Applications to product expansions and generating functions

Pólya and Szegö [6,p 126] note two product expansions related to the exponential function.

$$\prod_{n=1}^{\infty} (1 - z^n)^{\mu(n)/n} = e^{-z} \tag{14}$$

$$\prod_{n=1}^{\infty} (1 - z^n)^{\phi(n)/n} = e^{-z/1-z}\ . \tag{15}$$

Several related expansions are available from results in section 3. Applying Theorem 9 to (14), we obtain

Corollary 13. $\displaystyle\prod_{n=1}^{\infty} (1 + z^n)^{\mu(n)/n} = e^{z-z^2}\ .$

Theorem 9 applied to (15) gives

Corollary 14. $\prod_{n=1}^{\infty}(1+z^n)^{\phi(n)/n} = e^{z/(1-z^2)}$.

Theorem 3 applied to (14) and (15) give

Corollary 15. $e^z = \prod_{n=1}^{\infty}(1+z^n)^{b_n}$, where $b_{2n-1} = \dfrac{\mu(2n-1)}{2n-1}$, and for $n = 2^k n'$, n' odd, $k \geq 1$,

$$b_n = \sum_{r=0}^{k} \frac{\mu(2^r n')}{2^r n'} = \frac{\mu(n')}{n'} + \frac{\mu(2n')}{2n'} = \frac{\mu(n')}{n'} - \frac{\mu(n')}{2n'} = \frac{\mu(n')}{2n'} .$$

Corollary 16.

$$e^{z/(1-z)} = \prod_{n=1}^{\infty}(1+z^n)^{(k+2)\phi(n')/(2n')}, \quad n = 2^k n' \ , \ n' \text{ odd.}$$

Proof. In applying Theorem 7, note

$$b_n = \sum_{r=0}^{k} \frac{\phi(2^r n')}{2^r n'} = \frac{\phi(n')}{n'} \sum_{r=0}^{k} \frac{\phi(2^r)}{2^r}$$
$$= \frac{\phi(n')}{n'}\left(1 + \sum_{r=1}^{k} \frac{1}{2}\right) = \frac{(k+2)\phi(n')}{2n'} .$$

Equation (8) in Theorem 3 applied to the generating function for the number of partitions of n into distinct parts, $q(n)$,

$$\prod_{n=1}^{\infty}(1+z^n) = \sum_{n=0}^{\infty} q(n) z^n$$

gives the product representation of the generating function for partitions of n into odd parts. Equation (9) in Theorem 3 applied to $\prod_{n=1}^{\infty}(1-z^n)^{-1}$ gives

Corollary 17.

(16) $$1 + \sum_{n=1}^{\infty} p(n) z^n = \prod_{n=1}^{\infty}(1+z^n)^{a(n)}$$

where $a(n) = k+1$, the number of trailing zeros in the binary expansion of $2n$.

A combinatorial proof of Corollary 17 is available as well. This approach involves interpreting the right hand side of (16) as the generating function for partitions of n into distinct parts chosen from the set

$$\{1, 2, 3, \ldots, 2_2, 4_2, 6_2, \ldots, 4_3, 8_3, 12_3, \ldots, 8_4, \ldots,\}$$

where the subscripts indicate that different copies of the same integer are distinguishable in the partitions that are generated.

Another interesting identity from (9) in Theorem 3 is

Corollary 18. $$\prod_{n=0}^{\infty}(1-z^{2^n})^{-1} = \prod_{n=0}^{\infty}(1+z^{2^n})^{n+1}.$$

Corollary 18 also has a combinatorial interpretation in terms of partitions.

Two identities involving the generating function for plane partitions are also available from Theorem 3.

Corollary 19. $$\prod_{n=1}^{\infty}(1-z^n)^{-n} = \prod_{n=1}^{\infty}(1+z^n)^{2n-n'},$$
where n' is the largest odd divisor of n.

Corollary 20. $$\prod_{n=1}^{\infty}(1+z^n)^n = \prod_{n=1}^{\infty}(1-z^{2n-1})^{-(2n-1)}(1-z^{2n})^{-n}.$$

The last identity we mention also follows from Theorem 3.

Corollary 21. $$\prod_{n=1}^{\infty}(1+z^n)^{1/n} = \prod_{n=1}^{\infty}(1-z^n)^{(-1)^n/n}.$$

5. <u>Power products and the cyclotomic identity</u> Following the development in section 2, we consider the following formulas, with the goal of determining a suitable function of α and n to give a formal identity.

(B) $$\left(\frac{1}{1-z}\right)^{\alpha} = \prod_{n=1}^{\infty}\frac{1}{1-D(\alpha,n)z^n}$$

(C) $$(1+z)^{\alpha} = \prod_{n=1}^{\infty}(1+C(\alpha,n)z^n).$$

Expansion (B) may also be manipulated using formal power series. From

(1B) $$\alpha \log(1-z) = \sum_{n=1}^{\infty} \log(1 - D(\alpha,n)z^n)$$

we obtain

(3B) $$\alpha \sum_{n=1}^{\infty} z^n/n = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{D(\alpha,j)^i z^{ij}}{i} = \sum_{n=1}^{\infty}\sum_{d|n} \frac{dD(\alpha,d)^{n/d} z^n}{n} .$$

Equating coefficients of z^n,

(5B) $$\alpha = \sum_{d|n} dD(\alpha,d)^{n/d}$$

or

(6B) $$D(\alpha,n) = \frac{1}{n}(\alpha - \sum_{\substack{d|n \\ d\neq n}} dD(\alpha,d)^{n/d}) .$$

This relationship, and an argument analogous to the induction in Theorem 2, is enough to establish an important property of $D(\alpha,n)$.

Theorem 22. *$D(\alpha,n)$ is a polynomial in α of degree n that takes on integer values for integer arguments.*

In this case, though, there seems to be no hope of using Möbius inversion to gain information about the nature of $D(\alpha,n)$. A table of values of $D(\alpha,n)$ is provided for $1 \leq n \leq 16$.

Table 1. *Values of $D(\alpha,n)$,* $1 \leq n \leq 16$

$D(\alpha,1) = \alpha$

$D(\alpha,2) = \frac{1}{2}(\alpha - \alpha^2)$

$D(\alpha,3) = \frac{1}{3}(\alpha - \alpha^3)$

$D(\alpha,4) = \frac{1}{8}(2\alpha - \alpha^2 + 2\alpha^3 - 3\alpha^4)$

$D(\alpha,5) = \frac{1}{5}(\alpha - \alpha^5)$

$D(\alpha,6) = \frac{1}{72}(12\alpha - 4\alpha^2 - 3\alpha^3 + 17\alpha^4 - 9\alpha^5 - 13\alpha^6)$

$D(\alpha,7) = \frac{1}{7}(\alpha - \alpha^7)$

$D(\alpha,8) = \frac{1}{128}(16\alpha - 4\alpha^2 + 4\alpha^3 - 11\alpha^4 + 24\alpha^5 - 22\alpha^6 + 20\alpha^7 - 27\alpha^8)$

$D(\alpha,9) = \frac{1}{81}(90\alpha - \alpha^3 + 3\alpha^5 - 3\alpha^7 - 8\alpha^9)$

$D(\alpha,10) = \frac{1}{800}(80\alpha - 16\alpha^2 - 5\alpha^5 + 57\alpha^6 - 50\alpha^7 + 50\alpha^8 - 25\alpha^9 - 91\alpha^{10})$

$D(\alpha,11) = \frac{1}{11}(\alpha - \alpha^{11})$

$D(\alpha,12) = \frac{1}{13824}(1152\alpha - 192\alpha^2 + 56\alpha^3 + 140\alpha^4 - 846\alpha^5 + 1141\alpha^6 + 78\alpha^7 -$
$555\alpha^8 + 142\alpha^9 + 679\alpha^{10} - 582\alpha^{11} - 1213\alpha^{12})$