Path Designs P(v,3,1) and P(v,4,1) with ovals. (*)

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Abstract. In this paper we give a complete solution of the problem of path designs P(v,3,1) and P(v,4,1) having an oval.

1. Introduction.

Let G and K be graphs with G simple; that is, G is a subgraph of K_v, the complete undirected graph on v vertices. A **G-design** of K is a pair (V,B), where V is the vertex set of K and B is an edge-disjoint decomposition of K into copies of the graph G. Usually we say that b is a block of the G-design if b∈B, and B is called the block-set.

Let L be a set of edges of the complete graph K. A **partial G-design** is the decomposition of K-L into copies of the graph G. The edge set L is called the **leave** of the partial G-design.

A **path design** $P(v,k,\lambda)$ [2] is a P_k-design of λK_v, where P_k is the simple path with k-1 edges (k vertices).

The condition $\lambda v(v-1)\equiv 0 \pmod{2(k-1)}$, $v\geq k$ is obviously necessary for the existence of a $P(v,k,\lambda)$. This condition is

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proved to be sufficient by M. Tarsi [9]. Therefore a $P(v,3,1)$ exists if and only if $v\equiv 0$ or $1 \pmod 4$, and a $P(v,4,1)$ exists if and only if $v\equiv 0$ or $1 \pmod 3$.

A **balanced G-design** [2,3] is a G-design such that each vertex belongs to exactly r copies of G. Obviously not every G-design is balanced. A (balanced) G-design of K_v is also called a **(balanced) G-design of order v**. A **handcuffed design** $H(v,k,\lambda)$ is a **balanced** path design.

Let (V,B) be a path design $P(v,k,\lambda)$, with point set V and block set B. A block $b\in B$ is called **secant** or **tangent** or **exterior** to a subset Ω of V if $|b\cap\Omega|=2$ or 1 or 0 respectively.

An **arc** in (V,B) is a subset Ω of points of V no three of which are on a block. Thus any block is either secant or tangent or exterior to Ω.

An **oval** in (V,B) is an arc Ω of V such that each point of Ω is on exactly one tangent.

Ovals were mainly investigated in projective planes. Recently a large body of research studies ovals in Steiner triple systems [1,5,6,7,10,11,12,13]. S. Milici [8] gives a complete solution of the existence problem of handcuffed designs $H(v,3,1)$ having an oval.

Obviously the same problem arises for path designs $P(v,k,1)$. In this paper we determine the possible cardinalities of an oval Ω in a $P(v,k,1)$ for $k=3,4$. In all these cases we construct a path design with an oval.

2. P(v,3,1) with ovals.

For every $v\equiv 0$ or $1 \pmod 4$, $v\geq 13$, let $R(v,3)$ be the set of the integers w satisfying one of the following conditions:

1) $2\leq w\leq \frac{2v-1}{3}$ if w is even;

2) $\frac{-5+\sqrt{25+8v}}{2} \le w \le \frac{2v-1}{3}$ if w is odd and $v-w \equiv 0$ or 1 (mod 4);

3) $\frac{-5+\sqrt{33+8v}}{2} \le w \le \frac{2v-1}{3}$ if w is odd and $v-w \equiv 2$ or 3 (mod 4).

Put $R(4,3) = R(5,3) = \{2\}$, $R(8,3) = \{2,3,4\}$, $R(9,3) = \{2,3,4,5\}$ and $R(12,3) = \{2,3,4,5,6,7\}$.

Theorem 1. Let Ω be an oval in a $P(v,3,1)$ (V,B), then $|\Omega| \in R(v,3)$.

Proof. Let $|\Omega| = w$. Let t be the number of tangent blocks meeting Ω in a vertex which is in the last or first position. Clearly $t \le w$ and the number of exterior blocks is $n = \frac{1}{2}\left[\binom{v-w}{2} - t\right]$. Since the number of secants is at least $\binom{w}{2}$, we obtain $n + w + \binom{w}{2} \le |B|$. Since $t \le w$ and $|B| = \frac{1}{2}\binom{v}{2}$ we obtain $w \le \frac{2v-1}{3}$ if $v \ge 13$ and $w \le \frac{2v-3}{3}$ if $v < 13$. Now suppose w is odd. Let $x \in V-\Omega$, then there is at least one edge $\{x,y\}$ with $y \in \Omega$ appearing either in a secant meeting x in an exterior position or in a tangent block. So we obtain $v-w \le \binom{w}{2} + t + 2(w-t)$. As $t \ge 0$ for $v-w \equiv 0$ or 1 (mod 4), and $t \ge 1$ for $v-w \equiv 2$ or 3 (mod 4) it is easy to complete the proof. ∎

Theorem 2. For every $v \equiv 0$ or 1 (mod 4), there exists a $P(v,3,1)$ containing an oval Ω of cardinality w if and only if $w \in R(v,3)$.

Proof. Let $w \in R(v,3)$. Suppose at first either $w \equiv 2$ (mod 4), or $w \equiv 1$ (mod 4) and $v \equiv 0$ (mod 4), or $w \equiv 3$ (mod 4) and $v \equiv 1$ (mod 4). If $w \le \frac{2v-3}{3}$, then $(v-w)w \ge \binom{w}{2}$ and it is possible to choose opportunely the elements $x,y \in \Omega$ and $a,b \in V-\Omega$ to form w-1 tangents $\{a,x,b\}$, one tangent $\{x,a,b\}$, $\binom{w}{2}$ secants $\{x,y,a\}$ and $[w(v-w) - \binom{w}{2} - 2w + 1]$ secants $\{x,a,y\}$, so that the remaining edges form a connected graph G. If

$w=\frac{2v-1}{3}$, then form the secants $\{x,y,a\}$ and the tangents $\{x,a,b\}$ by choosing the edges $\{a,b\}$ in such a way that the remaining edges of the complete graph on $V-\Omega$ form a connected graph G.

Now let either $w\equiv 0 \pmod 4$, or $w\equiv 1 \pmod 4$ and $v\equiv 1 \pmod 4$, or $w\equiv 3 \pmod 4$ and $v\equiv 0 \pmod 4$. It is easy to form the tangents $\{a,x,b\}$ and the secants $\{x,y,a\}$ and, if it is necessary, $\{x,a,y\}$ in such a way that the remaining edges of the complete graph on $V-\Omega$ form a connected graph G.

Since the number of the edges of the connected graph G is even, we can decompose G into paths of length 2 (see [4]) to form the exterior blocks.

Theorem 1 completes the proof. ∎

3. P(v,4,1) with ovals.

For every $v\equiv 0$ or $1 \pmod 3)>4$, let $R(v,4)$ be the set of all integers w verifying the inequalities $2\le w\le r(v,4)=\frac{-3+\sqrt{12v^2-12v+9}}{6}$.

Theorem 3. Let (V,B) be a P(v,4,1) containing an oval Ω of cardinality w. Then $w\in R(v,4)$.

Proof. Count the secants that cover the edges in the oval and the tangents. This number is less than or equal to $|B|$. ∎

Corollary 1. There is not a P(4,4,1) containing an oval.

Lemma 1. For every $v\equiv 0$ or $1 \pmod 3)>4$ there is a P(v,4,1) (V,B) containing an oval Ω of cardinality 2.

Proof. Let $\Omega=\{x,y\}$. For v=6, put $V=\Omega\cup\{1,2,3,4\}$ and

$B=\{xy41, 2x13, 1y24, 1234, y3x4\}$. For $v=7$, put $V=\Omega\cup\{1,2,3,4,5\}$ and $B=\{x451, 2y13, xy41, 1234, 3524, 2x3y, 1x5y\}$.

Let (S,T_1) be a $P(v,4,1)$ containing the oval Ω, and let $S=\Omega\cup\{1,2,\ldots,v-2\}$. Put $V=S\cup\{\infty_1,\infty_2,\infty_3\}$, $T_2=\{x\infty_1\infty_2y,\ x\infty_2\infty_3y,\ x\infty_3\infty_1y\}$ and either $T_3=\{\infty_1 i\infty_2(i+\frac{v-2}{2}),\ \infty_1(i+\frac{v-2}{2})\infty_3 i\ :\ i=1,2,\ldots,\frac{v-2}{2}\}$ if v is even, or $T_3=\{\infty_1 i\infty_2(i+\frac{v-5}{2}),\ \infty_1(i+\frac{v-5}{2})\infty_3 i\ :\ i=1,2,\ldots,\frac{v-5}{2}\}\cup\{(v-4)\infty_1(v-2)\infty_2,\ (v-3)\infty_2(v-4)\infty_3,\ (v-2)\infty_3(v-3)\infty_1\}$ if v is odd. Let $B=T_1\cup T_2\cup T_3$. It is easy to see that (V,B) is a $P(v+3,4,1)$ containing the oval Ω. ■

Put

$$\alpha(v,w)=\begin{cases} 0 \text{ if either } (v\equiv 0 \pmod 3 \text{ and } w\equiv 0 \text{ or } 1 \pmod 3) \\ \qquad\qquad \text{or } (v\equiv 1 \pmod 3 \text{ and } w\equiv 0 \pmod 3) \\ 1 \text{ if } v\equiv 0 \pmod 3 \text{ and } w\equiv 2 \pmod 3 \\ 2 \text{ if } v\equiv 1 \pmod 3 \text{ and } w\equiv 1 \text{ or } 2 \pmod 3 \end{cases}$$

and

$$\chi(v,w)=\begin{cases} \alpha(v,w) \text{ if } w\neq\frac{v-2}{2} \\ \alpha(v,w)+3\left\lceil\frac{w-2\alpha}{6}\right\rceil \text{ if } w=\frac{v-2}{2} \end{cases}$$

Lemma 2. For every $v\equiv 0$ or $1 \pmod 3)>4$ and $w<\frac{v-1}{2}$, there is a partial $P(v-w,4,1)$ whose leave contains $w+\chi(v,w)$ edges.

Proof. Suppose at first $w\neq\frac{v-2}{2}$ and $v\equiv 0 \pmod 3$. If $w\equiv 0$ or $2 \pmod 3$, then there is a $P(v-w,4,1)$ (S,T). Deleting respectively from T either $\frac{w}{3}$ or $\frac{w+1}{3}$ blocks, we obtain the proof. Suppose $w=1+3k$, then $v-w\equiv 2 \pmod 3$. Let $S_1=\{1,2,3,4,5\}$, $T_1=\{1234, 4513, 4253\}$ and $L=\{24\}$. Clearly (S_1,T_1) is a partial $P(5,4,1)$ whose leave is L. Let (S_2,T_2) be a partial $P(v-w-3,4,1)$ whose leave is an edge. Using the construction given in Lemma 1 embed (S_2,T_2) in a partial $P(v-w,4,1)$ (S,T) having an edge as leave. Deleting from T exactly k blocks we complete the proof. If

$w\neq\frac{v-2}{2}$ and $v\equiv 1 \pmod 3$, then we can prove the lemma in a similar way.

For $w=\frac{v-2}{2}$ construct a partial $P(v-w,4,1)$ as above, then delete exactly $\left\lceil\frac{w-2\alpha}{6}\right\rceil$ blocks. ∎

Similarly to Lemma 2, it is possible to prove the following lemma.

Lemma 3. For every $v\equiv 0$ or $1 \pmod 3)>4$ and $\frac{v-1}{2}\leq w\leq r(v,4)$, there is a partial $P(v-w,4,1)$ whose leave contains $w(2w-v+2)$ edges.

Lemma 4. For every $v\equiv 0$ or $1 \pmod 3)\geq 7$ and for every w such that $\frac{v-1}{2}\leq w\leq r(v,4)$, there is a $P(v,4,1)$ (V,B) containing an oval Ω of cardinality w.

Proof. Let (S,T) be the partial path design constructed in Lemma 3. Say L is its leave. Let $\Omega=\{x_0,x_1,\ldots,x_{w-1}\}$ be a w-set such that $\Omega\cap S=\emptyset$. Our purpose is to embed (S,T) in a $P(v,4,1)$ $(S\cup\Omega,B)$ such that Ω is an oval and $T\subseteq B$ is the set of the exterior blocks. If $\frac{v-1}{2}<w$ then construct, for every $j=1,2,\ldots,2w-v+1$, the following set of ordered pairs of elements of Ω, $P_j=\{(x_i x_{j+i}) : i=0,1,\ldots,w-1\}$. Since $2w-v+1<\frac{w-1}{2}$, it is $(x_{i_1} x_{j+i_1})\neq(x_{j+i_2} x_{i_2})$ for every $i_1,i_2\in\{0,1,\ldots w-1\}$. Using all the edges $\{bc\}\in L$ and (if $\frac{v-1}{2}<w$) all the ordered pairs $(xy)\in P_j$, $j=1,2,\ldots,2w-v+1$, it is easy to form the paths $\{x_i bc\}$ and the secants $\{xybc\}$ without introducing any repeated edge. Let E_1 be the set of the $w(2w-v+2)$ edges between the points in the oval and those outside the oval contained in these paths and secants. Obviously every $x_i\in\Omega$ is in exactly $2w-v+2$ edges of E_1. Since $w\leq r(v,4)$, it is easy to see that

$v-w \geq 2w-v+4$. Then for every path $\{x_i bc\}$ it is possible to find an element $a \in S$ such that $\{ax_i\} \notin E_1$ and $a \notin \{b,c\}$. Then $\{ax_i bc\} \in B$ is the tangent to Ω in the point x_i. Let E_2 be the set of the w edges $\{ax_i\}$. Let Γ be the set of the edges of the complete graph on Ω not appearing in the above secants. For every $\{xy\} \in \Gamma$ form the quadruple $\{dxyt\}$ with $d,t \in S$ such that $\{dx\},\{yt\} \notin E_1 \cup E_2$. If $d \neq t$ then put $\{dxyt\}$ in B. If $d=t$ then let $\{axbc\}$ be the tangent to Ω in the point x. Clearly it is $a \neq d$ and $b \neq d$. If $c \neq d$ then replace $\{axbc\}$ with $\{dxbc\}$ and put $\{axyd\}$ in B. If $c=d$, then replace $\{axbc\}=\{axbd\}$ with $\{axdb\}$ and put $\{bxyd\}$ in B. ∎

Lemma 5. For every $v \equiv 0$ or $1 \pmod 3 \geq 7$ and for every w such that $3 \leq w < \frac{v-1}{2}$, there is a $P(v,4,1)$ (V,B) containing an oval Ω of cardinality w.

Proof. Let (S,T) be the partial $P(v-w,4,1)$ constructed in Lemma 2. Say $L=\{\{b_j c_j\} : j=0,1,\ldots,w+\chi-1\}$ is its leave. Let $\Omega=\{x_0,x_1,\ldots,x_{w-1}\}$ be a w-set such that $\Omega \cap S=\emptyset$. Our purpose is to embed (S,T) in a $P(v,4,1)$ $(S \cup \Omega, B)$ such that Ω is an oval and $T \subseteq B$ is the set of the exterior blocks. All we need to do is define the tangents and the secants. In the following we'll sketch how to form these blocks.

Step 1. Form the paths $\{x_i b_i c_i\}$, $i=0,1,\ldots,w-1$. Let $E_1=\{\{x_i b_i\} : i=0,1,\ldots,w-1\}$.

Step 2. If $\chi=0$ then put $E_2=\emptyset$ and go to step 3. If $\chi>0$ then form the blocks $\{x_{j-w} y_{j-w} b_j c_j\}$, $j=w,w+1,\ldots,w+\chi-1$, in such a way that: $(b_j c_j) \in L$; y_{j-w} is an element of Ω such that $\{y_{j-w} b_j\} \notin E_1$ and $|\{y_{j-w} : j=w,w+1,\ldots,w+\chi-1\}|=\chi$. For $j=w,w+1,\ldots,w+\chi-1$ put $\{x_{j-w} y_{j-w} b_j c_j\}$ in B. Let $E_2=\{\{y_{j-w} b_j\} : j=w,w+1,\ldots,w+\chi-1\}$.

Step 3. Let $\Omega_0=\{x_0,x_1,\ldots,x_{\chi-1}\}$ (if $\chi=0$ then $\Omega_0=\emptyset$) and

$\Omega_1=\{x_\chi, x_{\chi+1}, \ldots, x_{w-1}\}$.

We need to construct a set D of (not necessarily different) ordered pairs (xy), $x\neq y$, of elements of Ω satisfying the following conditions: $|D|=\frac{w(v-2w-1)+\chi}{3}$; for every $(xy)\in D$ the first element x has weight 1 and the second element y has weight 2; the sum of all the weights of any fixed element of Ω_i is v-2w-i.

In order to form the set D, we have to consider the following two cases:

1) If v-2w-1 is odd then form the pairs of D in such a way that: every element of Ω_1 appears an odd number of times (at least 1) in the first position x; the number of elements of Ω_0 in the first position is either 0 or even. Since $w-\chi\leq\frac{w(v-2w-1)+\chi}{3}$, then we can form these pairs.

2) If v-2w-1 is even then we can proceed as in the above case exchanging Ω_0 with Ω_1. Since v is odd and $v\geq7$, it is $\chi\leq\frac{w(v-2w-1)+\chi}{3}$. So it is possible to form these pairs.

Since every element of Ω meets at most two edges of $E_1\cup E_2$, it is possible to construct $\frac{w(v-2w-1)+\chi}{3}$ blocks $\{xayb\}\in B$ such that $a,b\in S$, $(xy)\in D$ and no any edge of $E_1\cup E_2$ appears in one of these blocks. Let E_3 be the set of the $w(v-2w-1)+\chi$ edges between the points in the oval and those outside the oval contained in these blocks.

Step 4. For every path $\{xbc\}$ constructed in step 1 form the block $\{axbc\}$ in such a way that $a\in S$, $a\neq c$ and $\{ax\}\notin E_1\cup E_2\cup E_3$. Note that any element of Ω_i, i=0,1, meets exactly one edge E_1, at most one edge of E_2 and at most v-2w-i edges of E_3. Since v-2w-i+2<v-w, then it is possible to find an element $a\in S$ useful to form these tangents. Let E_4 be the set of all the edges $\{ax\}$.

Step 5. Let E be the set of all the edges between the points in Ω and those outside Ω, and let Γ be the set of the edges of the complete graph on Ω not appearing in the blocks of Step 2. For every $\{xy\}\in\Gamma$ form the quadruple $\{axyb\}$ with $a,b\in S$ such that $\{ax\},\{yb\}\in E\setminus(\bigcup_{i=1}^{4}E_i)$. It is easy to see that the quadruples $\{axyb\}$ cover all the edges of $E\setminus(\bigcup_{i=1}^{4}E_i)$. If $a\neq b$ then put $\{axyb\}$ in B. If $a=b$ then take the tangent to Ω in the point x constructed in step 4, say $\{cxdt\}$. Clearly it is $c\neq a$ and $d\neq a$. If $t\neq a$ then replace $\{cxdt\}$ with $\{axdt\}$ and put $\{cxya\}$ in B. If $t=a$, then replace $\{cxdt\}=\{cxda\}$ with $\{cxad\}$ and put $\{dxya\}$ in B. ∎

By Theorem 3 and Lemmas 1, 4 and 5 we obtain the following theorem.

Theorem 4. For every $v\equiv 0$ or $1 \pmod 3 > 4$, there exists a $P(v,4,1)$ containing an oval Ω of cardinality w if and only if $w\in R(v,4)$.

Remark. It is easy to see that the ovals Ω constructed in Lemma 4 satisfy the following property:

(s) Let b be a secant block meeting Ω in the points x and y, then the path b contains the edge $\{xy\}$.

The same property (s) holds also for the ovals in a $P(v,3,1)$ constructed in Theorem 2 and having cardinality either $\frac{2v-1}{3}$ (for $v\equiv 5$ or $8 \pmod{12}$) or $\frac{2v-3}{3}$ (for $v\equiv 0$ or $9 \pmod{12}$) .

Obviously, the property (s) is satisfied by every oval in a projective plane or, more generally, in a design. Thus one could be interested into looking for the ovals in a $P(v,k,1)$, $k\geq 3$, satisfying the property (s). Let Ω be such an oval, and let $|\Omega|=w$. Since the tangents and the secants cover all the edges between the points in Ω and those outside Ω, it is easy to see that either $w+\binom{w}{2}\leq w(v-w)\leq 2w+\binom{w}{2}$ if $k=3$, or $w+\binom{w}{2}\leq w(v-w)\leq 2w+2\binom{w}{2}$ if $k\geq 4$. These inequalities imply that either

$$\frac{2v-1}{3} \le w \le \frac{2v-3}{3} \qquad \text{if } k=3$$

or (1)

$$\frac{v-1}{2} \le w \le \frac{2v-3}{3} \qquad \text{if } k \ge 4.$$

Moreover, similarly to Theorem 3, we can prove that $w \le r(v,k) = \frac{1-k+\sqrt{4(k-1)v^2-4(k-1)v+(k-1)^2}}{2(k-1)}$. Since $\frac{v-1}{2} \le r(v,k)$ only for $k=3$ or 4, we obtain the folowing result:

A $P(v,k,1)$ containing an oval Ω of cardinality w and satisying the property (s) exists if and only if $k=3$ or 4, and w verifies the necessary conditions (1).

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