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# On the lettericity of paths

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**Abstract**

Verifying a conjecture of Petkovšec, we prove that the lettericity of an $n$-vertex path is precisely $\left\lfloor \frac{n+4}{3} \right\rfloor$.

## 1 Introduction

The concept of lettericity was introduced in 2002 by Petkovšec [2]. We begin by presenting his definitions. Let $\Sigma$ be a finite alphabet, and consider $D \subseteq \Sigma^2$, which we call the *decoder*. Then for a word $w = w_1 w_2 \dots w_n$ with each $w_i \in \Sigma$, the *letter graph* of $w$ is the graph $\Gamma_D(w)$ with $V(\Gamma_D(w)) = \{1, 2, \dots, n\}$ and for indices $i < j$, $(i, j) \in E(\Gamma_D(w))$ if and only if $(w_i, w_j) \in D$.

If $\Sigma$ is an alphabet of size $k$, we say that $\Gamma_D(w)$ is a $k$-letter graph. For some graph $G$, the minimum $k$ such that a $G$ is a $k$-letter graph is known as the *lettericity* of $G$, denoted $\ell(G)$. Note that every finite graph is the letter graph of some word over some alphabet, and in particular the lettericity of a graph $G$ is at most $|V(G)|$.

Petkovšec determined bounds or precise values for the lettericity of a number of different families of graphs, most notably threshold graphs, cycles, and paths. We focus our attention on paths, proving a conjecture of Pekovšec's and giving a precise value for their lettericity. Before we begin our proof, however, we first introduce a few pieces of additional notation.

Given a letter graph $\Gamma_D(w)$ and some letter $a \in \Sigma$, we then say that *a encodes* the set of vertices that correspond to some instance of $a$ in the word. In particular, these vertices must form a clique if $(a, a) \in D$, and an anticlique otherwise. Further, given a graph $G$ such that $G = \Gamma_D(w)$, we say that $(D, w)$ is a *lettering* of $G$, and in particular an *$r$-lettering* if $w$ uses an alphabet of size $r$.

## 2 Lemmas

We now establish a few lemmas necessary for the proof of our theorem. We begin with a simple but useful property of letter graphs.

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**Lemma 1.** *If a letter graph $\Gamma_D(w)$ has some pair of vertices with indices $i$ and $k$ such that $i < k$ and $w_i = w_k$, and this pair is distinguished by some third vertex $j$ (that is, $j$ is adjacent to exactly one of $i$ and $k$), then $i < j < k$.*

*Proof.* If it were the case that $j < i < k$ or that $i < k < j$, then the vertex $j$ of $\Gamma_D(w)$ is adjacent to either both of the vertices $i$ and $k$ or neither of them, depending on whether $(w_j, w_i) \in D$, in the first case, and $(w_i, w_j) \in D$ in the second. Thus $i < j < k$. □

With this established, we now move on to examining matchings. Petkovšec noted that $\ell(rK_2) = r$, and this was explicitly proven by Alecu, Lozin and De Werra [1]. We will reprove this in a different way.

**Lemma 2.** *In any lettering of $rK_2$, no letter encodes more than two vertices.*

*Proof.* Suppose there exists some lettering $(D, w)$ of $rK_2$ with some letter $a$ that encodes at least three vertices of $\Gamma_D(w)$, say $i$, $j$, and $k$ with $i < j < k$. Our graph contains no cliques of size greater than 2, so these vertices form an anticlique. Each of these vertices is incident with a distinct edge, so there must be some vertex, say $x$, which is adjacent to $j$ but not $i$ or $k$. Then, by Lemma 1 it must be that $i < x < j$, but also that $j < x < k$. This is a clear contradiction, so no such lettering exists. □

This lemma establishes $r$ as a lower bound for the lettericity of $rK_2$. To establish the upper bound, we examine any word $w$ over the alphabet $\Sigma = \{1, 2, \ldots, r\}$ in which each letter occurs exactly twice, with the decoder $D = \{(1, 1), (2, 2), \ldots, (r, r)\}$, so that the vertices of each letter form a clique of size two. Then $(D, w)$ is an $r$-lettering of $rK_2$, and we can show further that each $r$-lettering of $rK_2$ must be of a similar type.

**Lemma 3.** *In every $r$-lettering of $rK_2$, each letter encodes the two vertices of a $K_2$.*

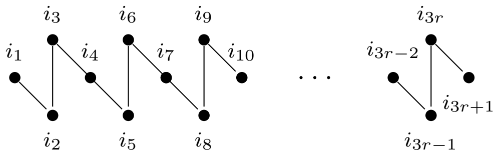
*Proof.* That each letter encodes exactly two vertices follows easily from Lemma 2. Now suppose $rK_2$ has some other $r$-lettering, and choose $a$ to be the earliest occurring letter that encodes an anticlique. In particular, suppose it first occurs at index $i$. Then vertex $i$ is adjacent to some vertex encoded by a different letter, say $b$. Then $b$ also encodes an anticlique, and by our choice of $a$, both of the vertices it encodes must lie after $i$ in the word. They then must both be adjacent to $i$; since $rK_2$ has no vertices of degree two, no such $r$-lettering exists. □

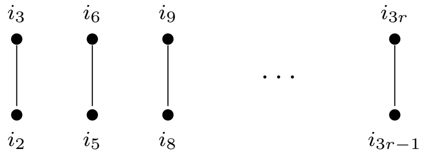
## 3 Theorem and Proof

We now prove our main result.

**Theorem 4.** *For $n \geq 3$, the lettericity of $P_n$ is $\left\lfloor \frac{n+4}{3} \right\rfloor$.*

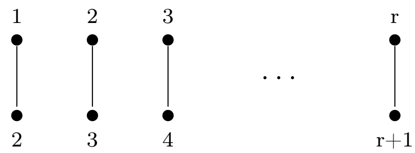
*Proof.* We begin with the lower bound; it suffices to examine a path $P_n$ with $n = 3r + 1$, which our theorem claims has lettericity $r + 1$. Label the vertices of $P_n$ as $i_1, i_2, \ldots, i_{3r+1}$ so that its edge set is $E(P_n) = \{(i_i, i_2), (i_2, i_3) \ldots (i_{3r}.i_{3r+1})\}$, and consider its subgraph $P_n[i_2, i_3, i_5, i_6, \ldots, i_{3r-1}, i_{3r}] = rK_2$, as shown below.



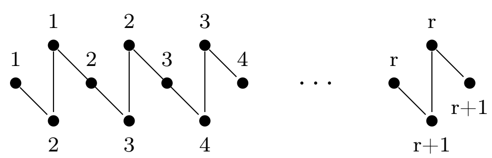

Suppose, for the sake of contradiction, that $P_n$ has some $r$-lettering $(D, w)$. Then $rK_2$ is a letter graph for some subword of $w$, which must still require an alphabet of size $r$. By Lemma 3, this is only possible if each letter is assigned to a distinct adjacent pair. The vertices encoded by each letter thus form cliques; they then do so in $\Gamma_D(w)$ as well. As $\Gamma_D(w)$ contains no cliques of size larger than 2, no such lettering exists, and so $\ell(P_n) \geq r + 1$.

The upper bound has already been established by Petkovšek, but here we show how this bound is obtained from an $r + 1$-lettering of $rK_2$. Take an ordering of the adjacent pairs in $rK_2$, and take the lettering of $rK_2$ which assigns to the $i$th adjacent pair the letters $i, i + 1$. Since we have $r$ pairs, this requires $r + 1$ letters in total.




The graph above is the letter graph of the word $21324354 \ldots r(r-1)(r+1)r$ with the decoder $D = \{(2, 1), (3, 2), \ldots (r + 1, r)\}$.

We now add $r - 1$ new vertices, giving the $j$th new vertex the label $j + 1$ and connecting it to the vertex in the $j$th pair labelled $j$ and the vertex in the $j + 1$st pair labeled $j + 2$. Finally, we add a vertex labeled 1 adjacent to the vertex in the first pair labeled 2 and a vertex labeled $r + 1$ adjacent to the vertex in the last pair labeled $r$.




This new graph, shown above, is the letter graph of the word $21321432543 \ldots (r+1)r(r-1)(r+1)r$ with the same decoder $D = \{(2, 1), (3, 2), \ldots (r + 1, r)\}$. This gives us a path on $3r + 1$ vertices; to obtain a path on $3r$ vertices we remove the first instance of 1 in our word, and to obtain a path on $3r - 1$ we additionally remove the last instance of $r + 1$. $\square$

## References

[1] B. Alecu, V. V. Lozin and D. de Werra, The micro-world of cographs, In *Combinatorial Algorithms*, (Eds.: L. Gąsieniec, R. Klasing and T. Radzik), *Lec. Notes in Comp. Sci.* Vol. 12126, Springer, Cham, Switzerland, 2020, pp. 30–42.

[2] M. Petkovšek, Letter graphs and well-quasi-order by induced subgraphs, *Discrete Math.* 244 (1-3) (2002), 375–388.

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