AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **78(2)** (2020), Pages 191–208

Contractible edges in subgraphs of 2-connected graphs

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Abstract

Contractible edges in spanning trees, longest paths and maximum matchings in 2-connected graphs non-isomorphic to K_3 are investigated. Every spanning tree and every longest path are shown to contain at least two contractible edges. All graphs with a spanning tree / a longest path containing exactly two contractible edges are characterized. Also, we prove that there always exists a longest path P which contains more than $|E(P)|/2$ contractible edges, and the bound is asymptotically optimal. Every maximum matching must contain a contractible edge and those graphs with a maximum matching having exactly one contractible edge are characterized. Finally, it is shown that there always exists a maximum matching M that contains at least $2(|M|+1)/3$ contractible edges, and the bound is optimal.

1 Introduction

The study of contractible edges started with the work of Tutte [20] who proved that every 3-connected graph non-isomorphic to K_4 contains a contractible edge. Further results on the number of contractible edges and non-contractible edges in terms of the order and size of a graph were obtained. Ando et al. [6] proved that every 3-connected graph G non-isomorphic to K_4 has at least $\frac{|V(G)|}{2}$ contractible edges and characterized all the extremal graphs (refer to McCuaig's paper [17] for further refinements). Ota [18] proved that every 3-connected graph G of order at least 19 has at least $\frac{2|E(G)|+12}{7}$ contractible edges and determined all the extremal graphs. Egawa et al. [9] showed that the number of non-contractible edges in a 3-connected graph G non-isomorphic to K_4 is at most $3|V(G)| - \lfloor \frac{3}{2}(\sqrt{24|V(G)|+25} - 5) \rfloor$.

The existence of contractible edges in certain types of subgraphs in 3-connected graphs was also investigated. For any 3-connected graphs of order at least seven,

ISSN: 2202-3518

Dean et al. [7] proved that for any two distinct vertices x, y, every longest x-y path contains at least two contractible edges and that every longest cycle contains at least three contractible edges. Later, Aldred et al. [1, 2] characterized all 3-connected graphs with a longest path containing exactly two contractible edges and Aldred et al. [3] characterized all 3-connected graphs having a longest cycle containing exactly three contractible edges. Ellingham et al. [10] proved that every non-hamiltonian 3-connected graph has at least six contractible edges in any longest cycle. For any 3-connected graph of order at least five, Fujita [12] proved that there exists a longest cycle C such that C contains at least $\frac{|V(C)|+9}{8}$ contractible edges, and later [13] improved the lower bound to $\frac{|V(C)|+7}{7}$. Maximum matchings were shown to contain a contractible edge by Aldred et al. [4]. They [5] also characterized all 3-connected graphs with a maximum matching containing precisely one contractible edge. Recently, Elmasry et al. [11] proved that every depth-first search tree in a 3-connected graph non-isomorphic to K_4 contains a contractible edge.

For 2-connected graphs, several analogous results on contractible and non-contractible edges were known in the more general context of matroids. Let M be a simple 2-connected matroid with rank $r(M)$. Oxley [19] showed that M has at least $r(M) + 1$ contractible elements. Wu [21] characterized the extremal matroids to be precisely the matroids arised from 2-connected outerplanar graphs. Kahn and Seymour [15] proved that if M has rank at least two, then M has at least $|E(M)| - r(M) + 2$ contractible elements, and characterized all the matroids where equality holds. When restricted to graphs, these correspond to maximally outerplanar graphs. In Section 3, we will provide graph-theoretical proofs of the above and related results.

Section 4 deals with contractible edges in spanning trees in 2-connected graphs. From the above result of Kahn and Seymour, every spanning tree must contain at least two contractible edges. Those graphs having a spanning tree containing exactly two contractible edges are characterized. In Section 5, we study contractible edges in longest cycles and longest paths. It is easy to see that every edge in a longest cycle is contractible, and the first and last edges in any longest path between two given vertices are contractible. Furthermore, we characterize all the graphs with a longest path containing exactly two contractible edges to be the square of a path. For 2-connected non-hamiltonian graphs, every longest path is shown to contain at least four contractible edges which is best possible. We also prove that for any 2-connected graph non-isomorphic to K_3, there exists a longest path P containing more than $|E(P)|/2$ contractible edges and this bound is asymptotically optimal. Lastly, in Section 6, every maximum matching is shown to contain a contractible edge. All 2-connected graphs with a maximum matching containing precisely one contractible edge are characterized. We also prove that for any 2-connected graph non-isomorphic to K_3, there exists a maximum matching M that contains at least $2(|M| + 1)/3$ contractible edges and the bound is optimal.

2 Definitions

All basic graph-theoretical terminology can be found in Diestel [8]. Unless otherwise stated, all graphs $G = (V(G), E(G))$ considered in this paper are simple and finite. For any vertex x in G, denote the set of neighbors of x by $N_G(x)$ and the set of edges incident to x by $E_G(x)$. For any subset S of $V(G)$, define $N_G(S) := \bigcup_{x\in S} N_G(x) \setminus S$ to be the set of neighbors of S in $V(G) \setminus S$. Let A and B be two disjoint subsets of $V(G)$, define $E_G(A, B)$ to be the set of all edges between A and B. A graph is *acyclic* if it does not contain any cycle. The *square* of G, denoted by G^2, is the graph on $V(G)$ where two vertices are adjacent if and only if they have distance at most two in G. A *matching* is a set of independent edges and a *maximum matching* is a matching of largest cardinality. Let M be a matching in G. An *M-alternating* path is a path whose edges alternate between M and $E(G) \setminus M$. An M-alternating path is called *M-augmenting* if the first and last vertices of the path are not incident to any edges in M. Let H be a subgraph of G. An *H-path* is a non-trivial path meeting H exactly in its ends. Let Q be a path or a cycle. Two chords x_1x_2 and y_1y_2 of Q (x_1, x_2, y_1, y_2 are all distinct) are *overlapping* if x_1, y_1, x_2, y_2 appear in this order in Q.

A graph is *connected* if for any two of its vertices, there exists a path between them. A *component* of a graph is a maximally connected subgraph. Let G be a disconnected graph. The union of at least one but not all components of G is called a *fragment* of G. A graph G is *k-connected* ($k \geq 2$) if $|V(G)| > k$ and for every set $X \subseteq V(G)$ with $|X| < k$, $G - X$ is connected. Let G be a k-connected graph. An edge e of G is said to be *k-contractible* if the graph obtained by its contraction, G/e, is also k-connected. Otherwise, it is called *k-non-contractible*. Since this paper concerns only 2-connected graphs, we write 2-contractible as *contractible* and 2-non-contractible as *non-contractible*. Denote the set of contractible edges and non-contractible edges in G by $E_C(G)$ and $E_{NC}(G)$ respectively. Define $G_C := (V(G), E_C(G))$ to be the subgraph induced by all the contractible edges and $G_{NC} := (V(G), E_{NC}(G))$. For any vertex x in G, denote the set of contractible edges incident to x by $E_{GC}(x)$.

A graph is *outerplanar* if it can be embedded in the plane such that all the vertices lie on the boundary of one face. A graph is *maximally outerplanar* if it is outerplanar and the addition of any extra edge results in a non-outerplanar graph. It is well-known that a 2-connected outerplanar graph consists of a Hamilton cycle with non-overlapping chords.

3 Contractible and non-contractible edges in 2-connected graphs

Here we group together all the major results concerning contractible and non-contractible edges in 2-connected graphs. We start with two well-known facts about non-contractible edges in any 2-connected graph non-isomorphic to K_3, finite or infinite.

Lemma 3.1. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3. An edge e in G is non-contractible (G/e is not 2-connected) if and only if $G - V(e)$ is not connected.*

Lemma 3.2. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3. For every edge e of G, $G - e$ or G/e is 2-connected.*

By Lemmas 3.1 and 3.2, we can easily prove the following lemma which says that deleting a non-contractible edge preserves other non-contractible edges.

Lemma 3.3. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3. Let e and f be two distinct non-contractible edges of G. Then $G - e$ is 2-connected and f is non-contractible in $G - e$.*

Deleting a non-contractible edge does not affect contractible edges either as shown below.

Lemma 3.4. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3. Let e be a non-contractible edge of G and f be a contractible edge of G. Then $G - e$ is 2-connected and f is contractible in $G - e$.*

Proof. Let $e = xy$. By Lemma 3.2, $G - e$ is 2-connected. Suppose f is non-contractible in $G - e$. By Lemma 3.1, $G - e - V(f)$ is not connected. Since f is contractible in G, $G - V(f)$ is connected by Lemma 3.1, and e connects the two components of $G - e - V(f)$. This means $x, y \notin V(f)$ and every x-y path in $G - e$ intersects $V(f)$. Let C be the component of $G - x - y$ containing f Then $G - C - e$ has an x-y path not intersecting $V(f)$, a contradiction. □

The existence of contractible edges in certain finite edge sets follows directly from Lemmas 3.2 and 3.3.

Lemma 3.5. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3 and F be a finite subset of $E(G)$.*

(a) *If $G - F$ is disconnected, then F contains at least two contractible edges.*

(b) *If $G - F$ is connected but not 2-connected, then F contains at least one contractible edge.*

Lemma 3.6. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3, and xy be a non-contractible edge in G. Consider a component C of $G - x - y$. If $|E_G(x, C)|$ is finite, then $E_G(x, C)$ contains a contractible edge.*

Proof. Obviously, $G - E_G(x, C)$ is connected. Since y is a cutvertex of $G - E_G(x, C)$, $E_G(x, C)$ contains a contractible edge by Lemma 3.5. □

Lemma 3.5 implies that for any 2-connected graph non-isomorphic to K_3, every vertex is incident to at least two contractible edges. Hence, the number of contractible edges is at least the number of vertices. The 2-connected graphs satisfying the lower bound were characterized by Wu [21] to be outerplanar graphs. Since Wu's work concerns simple 2-connected matroids, we give a graph-theoretical proof below. This requires the following theorem which says that the subgraph induced by all the contractible edges is spanning and 2-connected.

Theorem 3.1. *Let G be a 2-connected graph non-isomorphic to K_3. Then $G_C := (V(G), E_C(G))$ is 2-connected.*

Proof. By Lemmas 3.2, 3.3 and 3.4, we can repeatedly delete all the non-contractible edges while preserving the original contractible edges so that the resulting graph G_C is 2-connected. □

Theorem 3.2 (Wu [21])**.** *Every 2-connected graph G non-isomorphic to K_3 has at least $|V(G)|$ contractible edges. The equality holds if and only if G is outerplanar.*

Proof. By Lemma 3.5, every vertex is incident to at least two contractible edges. Therefore, the number of contractible edges is at least $|V(G)|$.

Suppose G is outerplanar. Since G is 2-connected, G consists of a Hamilton cycle with non-overlapping chords. The edges in the Hamilton cycle are the only contractible edges and the equality holds. Suppose the equality holds. From above, we have $|E_{GC}(x)| \geq 2$. Now, $|V(G)| = |E_C(G)| = \frac{1}{2}\sum_{x\in V(G)} |E_{GC}(x)| \geq \frac{1}{2}\sum_{x\in V(G)} 2 = |V(G)|$. Therefore, every vertex of G is incident to exactly two contractible edges. By Theorem 3.1, G_C is a Hamilton cycle of G. All edges of G outside G_C are chords of G_C and are non-contractible. By Lemma 3.1, no chords of G_C are overlapping. Hence, G is outerplanar. □

There is also a similar result for the upper bound of the number of non-contractible edges in a 2-connected graph. As noted in the Introduction, this was already proved by Kahn and Seymour [15] for matroids. For 2-connected graphs, we will adopt Kriesell [16]'s arguments, and make use of the following two lemmas on how deleting or contracting a fragment affects the contractability of the remaining edges.

Lemma 3.7. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3. Let xy be a non-contractible edge in G and C be a fragment of $G - x - y$. Then*

(1) *$G - C$ is 2-connected.*

(2) *Every non-contractible edge in $G - C$ is non-contractible in G.*

(3) *Let $e \in E(G - C) \setminus xy$. If e is non-contractible in G, then e is non-contractible in $G - C$.*

Proof. (1) and (2) are obvious.

(3) Since $e \neq xy$, without loss of generality, assume $x \notin V(e)$. Let D be the component of $G - V(e)$ containing $C \cup x$. Note that $C \subsetneq D$ and $G - D - V(e) \neq \emptyset$. Since $G - V(e) - C$ is not connected, e is non-contractible in $G - C$. □

Lemma 3.8. *Let G be a 2-connected finite or infinite graph non-isomorphic to K_3. Let xy be a non-contractible edge in G and C be a fragment of $G - x - y$. Let H be the graph obtained from $G - C$ by adding a vertex a, and edges ax and ay. Then*

(1) *H is 2-connected.*

(2) *Every non-contractible edge in H is non-contractible in G.*

(3) *Let $e \in E(H)$. If e is a non-contractible edge in G, then e is non-contractible in H.*

Proof. First, note that $\deg_H(a) = 2$, and ax and ay are contractible in H by Lemma 3.2.

(1) By Lemma 3.7(1), $G - C = H - a$ is 2-connected. This implies that no vertex in H is a cutvertex and hence H is 2-connected.

(2) Let e be a non-contractible edge in H. Then $e \neq ax, ay$. The result is true if $e = xy$. Suppose $e \in E(H - a) \setminus xy$. By applying Lemma 3.7(3) to H, xy and a, e is non-contractible in $H - a = G - C$. By Lemma 3.7(2), e is non-contractible in G.

(3) Since e is a non-contractible edge in G, $e \neq ax, ay$. Hence, $e \in E(H - a)$. The result is true if $e = xy$. Suppose $e \in E(H-a) \setminus xy = E(G-C) \setminus xy$. By Lemma 3.7(3), e is non-contractible in $G - C = H - a$. By applying Lemma 3.7(2) to H, xy and a, e is non-contractible in H. □

Theorem 3.3 (Kahn and Seymour [15]). *Every 2-connected graph G non-isomorphic to K_3 has at most $|V(G)| - 3$ non-contractible edges. The equality holds if and only if G is maximally outerplanar.*

Proof. The first statement was already proved by Kriesell [16]. We prove the second statement using the same inductive arguments. The 'if' part is obvious. For the 'only if' part, the result is true when $|V(G)| = 4$. Suppose $|V(G)| > 4$. Consider a non-contractible edge xy in G. Let C_1 be a component of $G-x-y$ and $C_2 := G-C_1-x-y$. Suppose $C_1 = a$. Then $\deg_G(a) = 2$, and ax and ay are contractible in G by Lemma 3.2. By Lemma 3.7(1), $G - a$ is 2-connected. By Lemma 3.7(2) and (3), $E_{NC}(G) = E_{NC}(G - a)$ if xy is non-contractible in $G - a$, or $E_{NC}(G) = E_{NC}(G - a) \cup \{xy\}$ if xy is contractible in $G - a$. The first case cannot occur because $G - a$ has at most $|V(G)| - 4$ non-contractible edges by the first part of the theorem. Hence, xy is a contractible edge in $G - a$ and $G - a$ has exactly $|V(G)| - 4$ non-contractible edges. By the induction hypothesis, $G - a$ is maximally outerplanar and so is G. Suppose $|V(C_1)| > 1$ and $|V(C_2)| > 1$. For $i = 1, 2$, let G_i be the graph obtained from $G - C_i$ by adding a vertex a_i, and edges a_ix and a_iy. By Lemma 3.8(1), G_i is 2-connected. By the first part of the theorem, G_i has at most $|V(G_i)| - 3$ non-contractible edges. By Lemma 3.8(2) and (3), $E_{NC}(G) = E_{NC}(G_1) \cup E_{NC}(G_2)$. Note that $E_{NC}(G_1) \cap E_{NC}(G_2) = \{xy\}$. Since $|V(G)| - 3 = |E_{NC}(G)| = |E_{NC}(G_1)| + |E_{NC}(G_2)| - 1 \leq |V(G_1)| - 3 + |V(G_2)| - 3 - 1 = |V(C_1)| + |V(C_2)| - 1 = |V(G)| - 3$, each G_i has exactly $|V(G_i)| - 3$ non-contractible edges. By the induction hypothesis, each G_i is maximally outerplanar and so is G. □

By combining Theorems 3.2 and 3.3, we obtain a lower bound for the number of contractible edges and an upper bound for the number of non-contractible edges in terms of the size of a graph.

Theorem 3.4. *Every 2-connected graph G non-isomorphic to K_3 has at least $\frac{|E(G)|+3}{2}$ contractible edges and at most $\frac{|E(G)|-3}{2}$ non-contractible edges. In both cases, the equality holds if and only if G is maximally outerplanar.*

Proof. By Theorems 3.2 and 3.3, $|V(G)| \leq |E_C(G)|$ and $|E_{NC}(G)| \leq |V(G)| - 3 \leq |E_C(G)|-3$. Therefore, $2|E_{NC}(G)|+3 \leq |E(G)| = |E_C(G)|+|E_{NC}(G)| \leq 2|E_C(G)|-3$. We have $|E_C(G)| \geq \frac{|E(G)|+3}{2}$ and $|E_{NC}(G)| \leq \frac{|E(G)|-3}{2}$. In both cases, the equality holds if and only if $|E_{NC}(G)| = |V(G)| - 3 = |E_C(G)| - 3$ which is equivalent to G being maximally outerplanar by Theorem 3.3. □

4 Contractible edges in spanning trees

Another question that can be asked about contractible edges in a 2-connected finite graph is: How many contractible edges are there in certain types of subgraphs? By Theorem 3.1, every 2-connected graph non-isomorphic to K_3 contains a spanning tree consisting of contractible edges only. Theorem 3.3 implies that every spanning tree contains at least two contractible edges. Below we characterize all 2-connected graphs having a spanning tree containing exactly two contractible edges.

Theorem 4.1. *Let G be a 2-connected graph non-isomorphic to K_3. Then every spanning tree of G contains at least two contractible edges. Moreover, G has a spanning tree containing exactly two contractible edges if and only if G is maximally outerplanar and G_{NC} is acyclic.*

Proof. Consider a spanning tree T of G. By Theorem 3.3, $|E_{NC}(G)| \leq |V(G)| - 3$. Hence, $|E(T)\cap E_C(G)| = |E(T)|-|E(T)\cap E_{NC}(G)| \geq |E(T)|-|E_{NC}(G)| \geq (|V(G)|-1) - (|V(G) - 3) = 2$. Suppose T contains exactly two contractible edges. Then $|E(T) \cap E_{NC}(G)| = |E_{NC}(G)| = |V(G)| - 3$. By Theorem 3.3, G is maximally outerplanar. Also, G_{NC} is acyclic as $E_{NC}(G) \subseteq E(T)$.

Suppose G is maximally outerplanar and G_{NC} is acyclic. If the subgraph induced by all the non-contractible edges $G[E_{NC}(G)]$ is not connected, then a chord can be added to join two closest components of $G[E_{NC}(G)]$ without destroying the outplanarity, a contradiction. Hence, $G[E_{NC}(G)]$ is connected. Since G_{NC} is acyclic, $G[E_{NC}(G)]$ is acyclic and hence is a tree of order $|E_{NC}(G)| + 1 = |V(G)| - 2$ by Theorem 3.3. Now, $G[E_{NC}(G)]$ can be extended to a spanning tree of G containing exactly two contractible edges. □

Suppose l is the minimum number of contractible edges a spanning tree of G can contain. It is easy to show that there exists a spanning tree containing exactly k contractible edges for $l \leq k \leq |V(G)| - 1$.

Theorem 4.2. *Let G be a 2-connected graph non-isomorphic to K_3 and l be the minimum number of contractible edges a spanning tree of G contains. Then, for $l \leq k \leq |V(G)| - 1$, G has a spanning tree containing exactly k contractible edges.*

Proof. If $l = |V(G)| - 1$, then the result follows from Theorem 3.1. Assume $l < |V(G)| - 1$. Suppose we have proved that G has a spanning tree T containing exactly k contractible edges ($k < |V(G)| - 1$). Let xy be a non-contractible edge in T. Denote the subtree of $T - xy$ containing x by T_x and that containing y by T_y. By Lemma 3.5(a), $E_G(T_x, T_y)$ contains a contractible edge, say uv. Then $T - xy + uv$ is a spanning tree containing exactly $k + 1$ contractible edges. By induction, the theorem follows. □

5 Contractible edges in longest cycles and paths

Inspired by the results of Dean et al. [7] and Aldred et al. [3], we also study contractible edges in longest paths and longest cycles in 2-connected graphs.

Lemma 5.1. *Let G be a 2-connected graph non-isomorphic to K_3, and x, y be two distinct vertices in G. Suppose $P := x_1x_2 \ldots x_n$ is a longest x-y path in G ($x = x_1$ and $y = x_n$). If x_ix_{i+1} is non-contractible, then $i \notin \{1, n-1\}$, $G - x_i - x_{i+1}$ has exactly two components, one of which contains x_1Px_{i-1} and the other contains $x_{i+2}Px_n$, and there is no x_1Px_{i-1}-$x_{i+2}Px_n$ path in $G - x_i - x_{i+1}$. In particular, x_1x_2 and $x_{n-1}x_n$ are contractible.*

Proof. We need only to show that every component C of $G - x_i - x_{i+1}$ intersects P. Suppose $C \cap P = \emptyset$. Let y_i be a neighbor of x_i in C, y_{i+1} be a neighbor of x_{i+1} in C, and Q be a y_i-y_{i+1} path in C. Then $x_1Px_iy_iQy_{i+1}x_{i+1}Px_n$ is an x-y path longer than P which is impossible. Note that if $i = 1$ or $n - 1$, then there is always a component of $G - x_i - x_{i+1}$ not intersecting P. Summing up, $i \notin \{1, n-1\}$ (or equivalently, x_1x_2 and $x_{n-1}x_n$ are contractible), $G - x_i - x_{i+1}$ has exactly two components, one of which contains x_1Px_{i-1} and the other contains $x_{i+2}Px_n$, and there is no x_1Px_{i-1}-$x_{i+2}Px_n$ path in $G - x_i - x_{i+1}$. □

The existence of contractible edges in longest path and longest cycle follows immediately from Lemma 5.1.

Theorem 5.1. *Let G be a 2-connected graph non-isomorphic to K_3. Then the first and the last edges in a longest path in G are contractible, and all edges in a longest cycle in G are contractible.*

Proof. The first part follows from Lemma 5.1. Let C be a longest cycle in G. Suppose C contains a non-contractible edge xy. Let z be a neighbor of y in C other than x. Then $C - yz$ is a longest y-z path in G. By Lemma 5.1, yx is contractible, a contradiction. □

As a natural step, we characterize all 2-connected graphs having a longest path containing exactly two contractible edges.

Theorem 5.2. *Let G be a 2-connected graph non-isomorphic to K_3. Then G has a longest path containing exactly two contractible edges if and only if G is the square of a path.*

Proof. First, notice that every 2-connected finite or infinite graph G non-isomorphic to K_3 has a path of length at least three. To see that, consider a cycle C in G. If C has at least four vertices, then the result follows. Otherwise, C is a triangle. Since $G \not\cong K_3$, there exists a vertex in $G - C$ adjacent to C and we can find a path of length at least three.

The 'if' part is obvious. For the 'only if' part, suppose $P := x_1x_2 \dots x_n$ is a longest path in G containing exactly two contractible edges. By Lemma 5.1, x_1x_2 and $x_{n-1}x_n$ are the only contractible edges in P. As discussed above, $n \geq 4$. For $k = 1, 2, \dots, n-3$, define P_k to be the subpath $x_1x_2 \dots x_k$ of P and C_k to be the component of $G - x_{k+1} - x_{k+2}$ containing P_k.

If $V(C_1) \neq x_1$, then there exists a vertex in C_1, say y_1, adjacent to x_1. By applying Lemma 5.1 to P and x_2x_3, $x_4Px_n \nsubseteq C_1$ and $y_1x_1x_2Px_n$ is a longer path than P, a contradiction. Therefore, $V(C_1) = x_1 = V(P_1)$, $N_G(P_1) = \{x_2, x_3\}$ and $G[P_3] = P_3^2$.

Suppose we have proved that for $i = 1, 2, \dots, k$, (1) $V(C_i) = V(P_i)$, (2) $N_G(P_i) = \{x_{i+1}, x_{i+2}\}$ and (3) $G[P_{i+2}] = P_{i+2}^2$. Consider C_{k+1}. Suppose $V(C_{k+1}) \neq V(P_{k+1})$. Let y be a vertex in $V(C_{k+1}) \setminus V(P_{k+1})$ adjacent to P_{k+1}. By (2), y is adjacent to x_{k+1}. If k is odd, $yx_{k+1}x_{k-1} \dots x_2x_1x_3 \dots x_kx_{k+2}Px_n$ is a longer path than P, a contradiction. If k is even, $yx_{k+1}x_{k-1} \dots x_1x_2x_4 \dots x_kx_{k+2}Px_n$ is a longer path than P, a contradiction. Therefore, $V(C_{k+1}) = V(P_{k+1})$ and $N_G(P_{k+1}) = \{x_{k+2}, x_{k+3}\}$. By (2), x_{k+1} is the only neighbor of x_{k+3} in C_{k+1} and $G[P_{k+3}] = P_{k+3}^2$. By induction, $G = P^2$. □

Since the square of a path is Hamiltonian, the above theorem implies that every longest path in a 2-connected non-hamiltonian graph contains at least three contractible edges. In fact, the correct lower bound is four. This is best possible as demonstrated by $K_{2,n}$ where $n \geq 3$.

Theorem 5.3. *Let G be a 2-connected non-Hamiltonian graph. Then every longest path contains at least four contractible edges.*

Proof. Suppose $P := x_1x_2 \dots x_n$ is a longest path in G containing exactly three contractible edges. By Lemma 5.1, x_1x_2 and $x_{n-1}x_n$ are contractible. Let x_kx_{k+1} be the third contractible edge in P. By arguing as in the proof of Theorem 5.2, we have

$$N_G(x_1) = \{x_2, x_3\}, N_G(x_2) = \{x_1, x_3, x_4\}, N_G(x_3) = \{x_1, x_2, x_4, x_5\}, \dots,$$
$$N_G(x_{k-2}) = \{x_{k-4}, x_{k-3}, x_{k-1}, x_k\}, N_G(x_{k+3}) = \{x_{k+1}, x_{k+2}, x_{k+4}, x_{k+5}\}, \dots,$$
$$N_G(x_{n-2}) = \{x_{n-4}, x_{n-3}, x_{n-1}, x_n\}, N_G(x_{n-1}) = \{x_{n-3}, x_{n-2}, x_n\},$$
$$N_G(x_n) = \{x_{n-2}, x_{n-1}\}.$$

By the maximality of P, $N_G(x_{k-1}) \subseteq P$ and $N_G(x_{k+2}) \subseteq P$. Since x_kx_{k+1} is contractible, $G - x_k - x_{k+1}$ is connected, and x_{k-1} and x_{k+2} are adjacent. Again by the maximality of P, $N_G(x_k) \subseteq P$ and $N_G(x_{k+1}) \subseteq P$. Now, $V(G) = V(P)$ and G is Hamiltonian, a contradiction. □

Theorem 5.1 tells us that every longest path has at least two contractible edges but is it possible to find a longest path that contains many contractible edges? The following theorem provides an affirmative answer.

Theorem 5.4. *Let G be a 2-connected graph non-isomorphic to K_3 and P be a longest path in G containing as many contractible edges as possible. Then P has more than $|E(P)|/2$ contractible edges.*

Proof. Let $P := x_1x_2\dots x_n$. By Lemma 5.1, x_1x_2 and $x_{n-1}x_n$ are contractible, and the result is true if $|E(P)| = 3$. Therefore, we can assume $|E(P)| \geq 4$.

Claim 5.1. *The first four and last four edges of P are contractible.*

Proof. Suppose x_2x_3 is non-contractible. By the maximality of P and by applying Lemma 5.1 to x_2x_3 (refer to the proof of Theorem 5.2), $N_G(x_1) = \{x_2, x_3\}$. Then x_1x_3 is a contractible edge by Lemma 3.2 and $x_2x_1x_3Px_n$ has more contractible edges than P, a contradiction. Therefore, x_2x_3 is contractible.

Suppose x_3x_4 is non-contractible. Then by the maximality of P and by applying Lemma 5.1 to x_3x_4, $N_G(x_1) \subseteq \{x_2, x_3, x_4\}$. Since x_2x_3 is contractible, $G - x_2 - x_3$ is connected and x_1 is adjacent to x_4. Suppose x_1x_4 is non-contractible. By Lemma 3.6, there exists a contractible edge incident to x_1, say x_1y, such that $y \notin \{x_2, x_3, x_4\}$ which is impossible. Therefore, x_1x_4 is contractible and $x_3x_2x_1x_4Px_n$ has more contractible edges than P, a contradiction. Therefore, x_3x_4 is contractible.

Suppose x_4x_5 is non-contractible. Let C be the component of $G - x_4 - x_5$ containing x_1. Then by the maximality of P and by applying Lemma 5.1 to x_4x_5, $N_G(x_1) \subseteq \{x_2, x_3, x_4, x_5\}$. Suppose $x_5 \in N_G(x_1)$. If x_1x_5 is non-contractible, then by Lemma 3.6, there exists a contractible edge x_1y such that $y \notin \{x_2, x_3, x_4, x_5\}$, a contradiction. Therefore, x_1x_5 is contractible and $x_4x_3x_2x_1x_5Px_n$ has more contractible edges than P, a contradiction. Hence, $x_5 \notin N_G(x_1)$. Since x_2x_3 is contractible, $G - x_2 - x_3$ is connected and $x_1x_4 \in E(G)$. If x_1x_4 is non-contractible, then by Lemma 3.6, there exists a contractible edge x_1y such that $y \notin \{x_2, x_3, x_4, x_5\}$, a contradiction. Hence, x_1x_4 is contractible. Since x_3x_4 is contractible, there exists a x_2-x_5 path Q in $G - x_1 - x_3 - x_4$. Now, $x_3x_4x_1x_2Qx_5Px_n$ is a longer path than P unless Q is the edge x_2x_5. Suppose x_2x_5 is non-contractible. Let D be a component of $G - x_2 - x_5$ not containing x_1, and Q' be an x_2-x_5 path in $G[D \cup \{x_2, x_5\}]$. But $x_3x_4x_1x_2Q'x_5Px_n$ is a longer path than P, a contradiction. Hence, x_2x_5 is contractible but then $x_1x_4x_3x_2x_5Px_n$ has more contractible edges than P which is impossible. Therefore, x_4x_5 is contractible. □

Claim 5.2. *Let x_ix_{i+1} and $x_{i+1}x_{i+2}$ be two consecutive non-contractible edges in P. Then there exists a contractible edge between x_i and x_{i+2}.*

Proof. Let C be the component of $G - x_i - x_{i+1}$ containing x_{i+2}. By Lemma 3.6, there exists a contractible edge x_iy_i such that $y_i \in C$. Let Q be a y_i-P path in C. Apply Lemma 5.1 to $x_{i+1}x_{i+2}$. Since there is no x_1Px_i-$x_{i+3}Px_n$ path in $G - x_{i+1} - x_{i+2}$, $Q \cap P = x_{i+2}$. Define $R := x_1Px_iy_iQx_{i+2}Px_n$. If $|E(Q)| \geq 2$, then R is a longer path than P, a contradiction. If $|E(Q)| = 1$, then R and P have the same length, but R has more contractible edges than P, a contradiction. Therefore, $|E(Q)| = 0$ and $x_i x_{i+2}$ is a contractible edge. □

Claim 5.3. *There are no three consecutive non-contractible edges in P.*

Proof. Suppose there are three consecutive non-contractible edges $x_i x_{i+1}$, $x_{i+1} x_{i+2}$ and $x_{i+2} x_{i+3}$ in P. By Claim 5.2, $x_i x_{i+2}$ and $x_{i+1} x_{i+3}$ are contractible edges. But then $x_1 P x_i x_{i+2} x_{i+1} x_{i+3} P x_n$ has more contractible edges than P, a contradiction. □

Below we will represent contractible and non-contractible edges in P using the following notation. For example, $x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} x_{i+5} := CNCNN = (CN)_2 N$ denotes that $x_i x_{i+1}$ and $x_{i+2} x_{i+3}$ are contractible, and $x_{i+1} x_{i+2}$, $x_{i+3} x_{i+4}$ and $x_{i+4} x_{i+5}$ are non-contractible. Note that NNN is impossible by Claim 5.3.

Claim 5.4. *For any integer $k \geq 0$, there is no $NN(CN)_k N$ in P.*

Proof. The case $k=0$ is Claim 5.3. Suppose $x_i x_{i+1} \ldots x_{i+2k+2} x_{i+2k+3} := NN(CN)_k N$ appears in P where $k \geq 1$. Since $x_{i+2j} x_{i+2j+1}$ is contractible $(1 \leq j \leq k)$, $G - x_{i+2j} - x_{i+2j+1}$ is connected and contains an $x_1 P x_{i+2j-1}$-$x_{i+2j+2} P x_n$ path internally disjoint from P, denoted by Q_j. Apply Lemma 5.1 to $x_{i+2j-1} x_{i+2j}$ and $x_{i+2j+1} x_{i+2j+2}$. Since there is no $x_1 P x_{i+2j-2}$-$x_{i+2j+1} P x_n$ path and no $x_1 P x_{i+2j}$-$x_{i+2j+3} P x_n$ path, $Q_j \cap P = \{x_{i+2j-1}, x_{i+2j+2}\}$ and all Q_j's are pairwise disjoint. By Claim 5.2, $x_i x_{i+2}$ and $x_{i+2k+1} x_{i+2k+3}$ are contractible edges. Consider $P' := x_1 P x_i x_{i+2} x_{i+1} Q_1 x_{i+4} x_{i+3} Q_2$ $x_{i+6} \ldots x_{i+2k-3} Q_{k-1} x_{i+2k} x_{i+2k-1} Q_k x_{i+2k+2} x_{i+2k+1} x_{i+2k+3} P x_n$. Since P is a longest path, all Q_j's are edges. Hence, $Q_j = x_{i+2j-1} x_{i+2j+2}$ and P' is also a longest path. Since $P' - V(Q_j)$ is connected, by Lemma 5.1, Q_j is contractible. But then $x_i P' x_{i+2k+3} = (CN)_{k+1} C$ and P' has more contractible edges than P, a contradiction. □

Claim 5.5. *For any integer $k \geq 0$, every $2k+1$ consecutive edges in P contain at least k contractible edges.*

Proof. For $k = 0$, it is trivial. By Claim 5.3, $k = 1$ is true. Suppose we have proved that for all $0 \leq l \leq k$, every $2l+1$ consecutive edges in P contain at least l contractible edges. Consider any $2k+3$ consecutive edges Q in P. Assume Q contains only k contractible edges. Since the last $2k+1$ edges contain k contractible edges, the first two edges are non-contractible. Similarly, the last two edges are non-contractible. By Claim 5.3, $Q = NNC \ldots CNN$.

Suppose we have proved that $Q = N(NC)_l \ldots (CN)_l N$. Since the last $2(k-l)+1$ edges contain $k-l$ contractible edges and the first $2l+1$ edges has l contractible edges, the $(2l+2)$-th edge is non-contractible. By symmetry, $Q = N(NC)_l N \ldots N(CN)_l N$. From Claim 5.4, $Q = N(NC)_{l+1} \ldots (CN)_{l+1} N$. By induction, $Q = NN(CN)_k N$, which is impossible by Claim 5.4. Hence, Q contains at least $k+1$ contractible edges. □

Claim 5.6. *P has more than $|E(P)|/2$ contractible edges.*

Proof. For $|E(P)| \leq 8$, by Claim 5.1, all edges are contractible. Suppose $|E(P)| > 8$. By Claim 5.1 and Claim 5.5, P contains at least $8 + \frac{|E(P)|-8-1}{2} > \frac{|E(P)|}{2}$ contractible edges if $|E(P)|$ is odd, or at least $8 + \frac{|E(P)|-8-2}{2} > \frac{|E(P)|}{2}$ contractible edges if $|E(P)|$ is even. □

Finally, the bound in Theorem 5.4 is asymptotically best possible as demonstrated by the family of graphs, H_k $(k \geq 0)$, constructed below.
Define $V(H_k) := \{x_1, x_2, \ldots, x_{2k+10}\}$ and

$$E(H_k) := \bigcup_{i=1}^{2k+9} \{x_i x_{i+1}\} \cup \{x_1x_4, x_2x_6, x_{2k+5}x_{2k+9}, x_{2k+7}x_{2k+10}\} \cup \bigcup_{i=1}^{k} \{x_{2i+3}x_{2i+6}\}.$$

It is not difficult to see that the longest path of H_k is either $x_1x_2 \ldots x_{2k+10}$ or $(x_1x_4x_3x_2/x_3x_4x_1x_2)x_6x_5x_8x_7 \ldots x_{6+4i}x_{5+4i}x_{8+4i}x_{7+4i} \ldots x_{2k+4}x_{2k+3}x_{2k+6}x_{2k+5}(x_{2k+9}x_{2k+8}x_{2k+7}x_{2k+10}/x_{2k+9}x_{2k+10}x_{2k+7}x_{2k+8})$, and has the contractible/non-contractible edge pattern: $CCCCN(CN)_kCCCC$.

6 Contractible edges in maximum matchings

This section deals with contractible edges in maximum matchings in 2-connected graphs. First, it will be shown that every maximum matching in a 2-connected graph non-isomorphic to K_3 contains a contractible edge.

Lemma 6.1. *Let G be a 2-connected graph non-isomorphic to K_3 and M be a matching in G. Consider any edge xy in M. Let C be a component of $G-x-y$ such that all edges in $M \cap E(C)$ are non-contractible. Then $G[C \cup xy]$ contains an M-alternating path P such that $xy \in P$, y is an endvertex of P, and the other endvertex of P lies in C and is not incident to any edges in M.*

Proof. Let $x_0 := x$ and x_1 be a neighbor of x in C. Suppose $P := yx_0x_1 \ldots x_{2k}x_{2k+1}$ is a longest M-alternating path in $G[C \cup xy]$ such that $x_1 \ldots x_{2k}x_{2k+1}$ lies in C and $x_{2k}x_{2k+1} \notin M$. If $x_{2k+1} \in V(M)$, then there exists an edge in M incident to x_{2k+1}, say $x_{2k+1}x_{2k+2}$. Obviously, $x_{2k+2} \notin yx_0x_1 \ldots x_{2k}x_{2k+1}$. Since $x_{2k+1}x_{2k+2}$ is non-contractible, x_{2k+2} is adjacent to a vertex, say x_{2k+3}, in a component of $G - x_{2k+1} - x_{2k+2}$ not containing $P - x_{2k+1}$. But then $yxx_1 \ldots x_{2k+2}x_{2k+3}$ is a longer M-alternating path in $G[C \cup xy]$ such that $x_{2k+2}x_{2k+3} \notin M$, a contradiction. Hence, $x_{2k+1} \notin V(M)$ and P is the desired M-alternating path. □

Since an M-augmenting path enables one to construct a larger matching than M, Lemma 6.1 immediately implies the following.

Theorem 6.1. *Every maximum matching in a 2-connected graph non-isomorphic to K_3 contains a contractible edge.*

Proof. Let M be a maximum matching. Suppose all edges in M are non-contractible. Let xy be an edge in M, and C and D be two components of $G-x-y$. By Lemma 6.1,

$G[C \cup xy]$ contains an M-alternating path P such that $xy \in P$, y is an endvertex of P, and the other endvertex of P lies in C and is not incident to any edges in M. By Lemma 6.1, $G[D \cup xy]$ contains an M-alternating path Q such that $xy \in Q$, x is an endvertex of Q, and the other endvertex of Q lies in D and is not incident to any edges in M. Now, $P \cup Q$ is an M-augmenting path which is impossible. □

Next, we characterize all 2-connected graphs with a maximum matching containing precisely one contractible edge. For such purpose, we define the following type of graphs $R_n (n \geq 1)$ with $V(R_n) := \{x_0, y_0, x_1, y_1, \ldots, x_n, y_n, z\}$ and $E(R_n) := \{x_i y_i, x_i x_{i+1}, y_i y_{i+1} : 0 \leq i \leq n-1\} \cup \{x_n y_n, x_n z, y_n z\} \cup F$ where $F \subseteq \{x_i y_{i+1}, y_i x_{i+1} : 0 \leq i \leq n-1\}$.

Theorem 6.2. *Let G be a 2-connected graph. Then G has a maximum matching containing precisely one contractible edge if and only if $G \cong R_n$.*

Proof. If $G \cong R_n$, then $\{x_i y_i : 0 \leq i \leq n\}$ is the desired matching. Conversely, let M be a maximum matching containing precisely one contractible edge $x_0 y_0$. Since all edges in K_3 are non-contractible, G has at least four vertices. Note that $G - x_0 - y_0$ is connected and thus contains an edge e. By the 2-connectedness of G, there is a cycle containing $x_0 y_0$ and e. Hence, there exists two distinct vertices x_1 and y_1 such that x_1 is adjacent to x_0 and y_1 is adjacent to y_0. Note that $x_0 x_1 \notin M$ and $y_0 y_1 \notin M$.

We claim that $x_1 y_1 \in M$ and $N_G(\{x_0, y_0\}) = \{x_1, y_1\}$. There are three cases to consider. (1) If $|\{x_1, y_1\} \cap V(M)| = 0$, then $x_1 x_0 y_0 y_1$ is an M-augmenting path, contradicting M being maximum. (2) If $|\{x_1, y_1\} \cap V(M)| = 1$, then we may assume $x_1 \notin V(M)$ and $y_1 \in V(M)$. Let $y_1 y_1' \in M$. Note that $y_1' \neq x_1$ and $y_1 y_1'$ is non-contractible. Let C be a component of $G - y_1 - y_1'$ not containing $x_1 x_0 y_0$. Since each edge in $M \cap E(C)$ is non-contractible, by Lemma 6.1, $G[C \cup y_1 y_1']$ contains an M-alternating path $y_1 y_1' P y$ with $y \notin V(M)$. However, $x_1 x_0 y_0 y_1 y_1' P y$ is an M-augmenting path, a contradiction. (3) Suppose $|\{x_1, y_1\} \cap V(M)| = 2$. Let $x_1 x_1'$ and $y_1 y_1'$ be the edges in M incident to x_1 and y_1 respectively. Assume $x_1' \neq y_1$. Then $x_1 x_1'$ and $y_1 y_1'$ are distinct edges in M, and both are non-contractible. Let C be a component of $G - x_1 - x_1'$ not containing $x_0 y_0 y_1 y_1'$. By Lemma 6.1, $G[C \cup x_1 x_1']$ contains an M-alternating path $x_1 x_1' P x$ with $x \notin V(M)$. Let D be a component of $G - y_1 - y_1'$ not containing $y_0 x_0 x_1 x_1'$. By Lemma 6.1, $G[D \cup y_1 y_1']$ contains an M-alternating path $y_1 y_1' Q y$ with $y \notin V(M)$. But, $x P x_1' x_1 x_0 y_0 y_1 y_1' Q y$ is an M-augmenting path, a contradiction. Therefore, $x_1' = y_1$ and $x_1 y_1 \in M$. Suppose there exists a vertex u in $N_G(\{x_0, y_0\})$ other than x_1 and y_1. Without loss of generality, assume $u \in N_G(x_0)$. Then by repeating the above arguments for $\{u, y_1\}$ in place of $\{x_1, y_1\}$, we obtain $u y_1 \in M$ which is impossible. Therefore, $N_G(\{x_0, y_0\}) = \{x_1, y_1\}$.

Consider $x_1 y_1$. Then $x_0 y_0$ is a component of $G - x_1 - y_1$. Suppose $G - x_1 - y_1$ has another two components C_1 and C_2. By Lemma 6.1, $G[C_1 \cup x_1 y_1]$ contains an M-alternating path $y_1 x_1 P_1 z_1$ with $z_1 \notin V(M)$ and $G[C_2 \cup x_1 y_1]$ contains an M-alternating path $x_1 y_1 P_2 z_2$ with $z_2 \notin V(M)$. But $z_1 P_1 x_1 y_1 P_2 z_2$ is an M-augmenting path, a contradiction. Therefore $G - x_1 - y_1$ has exactly two components. Let C be the component of $G - x_1 - y_1$ other than $x_0 y_0$. If $|V(C)| = 1$, then

$G \cong R_1$. If $|V(C)| > 1$, then there exists two distinct vertices x_2 and y_2 such that x_2 is adjacent to x_1 and y_2 is adjacent to y_1. By arguing as above, $x_2y_2 \in M$ and $N_G(x_1, y_1) = \{x_0, y_0, x_2, y_2\}$. We can continue this process with $x_2y_2, x_3y_3, \ldots$ and prove that $G \cong R_n$. □

Note that R_n contains not only a maximum matching with exactly one contractible edge but also a maximum matching all of whose edges are contractible. It is natural to ask whether every 2-connected graph non-isomorphic to K_3 contains a maximum matching with many contractible edges. The answer is given by Theorem 6.4 below, and we need a result by Grossman and Häggkvist [14] concerning properly colored cycles in edge-colored graphs. A cycle is *properly colored* if adjacent edges have different colors.

Theorem 6.3 (Grossman and Häggkvist [14]). *Let G be a 2-connected graph with its edges colored by two colors. If every vertex is incident to at least one edge of each color, then G has a properly colored cycle.*

Theorem 6.4. *Let G be a 2-connected graph non-isomorphic to K_3 and M be a maximum matching that contains as many contractible edges as possible. Then M contains at least $2(|M| + 1)/3$ contractible edges.*

Proof. We make use of a tree construction similar to the block-cut tree with the cut being non-contractible edges in M. First, define $M_{NC} := M \cap E_{NC}(G)$ and $M_C := M \cap E_C(G)$. We say that a subgraph H in G has property $(*)$ if for each edge $e \in M_{NC} \cap E(H)$, $H - V(e)$ is connected. Define $\mathcal{H}$ to be the set of all maximal 2-connected induced subgraphs in G having property $(*)$.

Claim 6.1. *Every edge in G belongs to an element of $\mathcal{H}$.*

Proof. Consider a shortest cycle C containing the edge. Obviously, C is 2-connected, induced, and has property $(*)$. The maximal 2-connected induced subgraph containing C with property $(*)$ is the desired subgraph. □

Claim 6.2. *Every edge $e \in M_{NC}$ belongs to at least two elements of $\mathcal{H}$.*

Proof. Let D_1 and D_2 be two components of $G - V(e)$. Consider a shortest cycle C_i in $G[D_i \cup e]$ containing e. Then C_i has property $(*)$. But no element of $\mathcal{H}$ contains both C_1 and C_2 since e is non-contractible. □

Claim 6.3. *Let $H \in \mathcal{H}$. If there is an x-y H-path in G, then $xy \in M_{NC}$.*

Proof. Let P be a shortest x-y H-path. Since $G[H \cup P]$ does not have property $(*)$, there exists $e \in M_{NC} \cap E(G[H \cup P])$ such that $G[H \cup P] - V(e)$ is not connected. As H is 2-connected and has property $(*)$, this implies $|V(e) \cap P| = 2$ and $e \notin E(P)$. Hence, e is a chord of P contradicting P being shortest. □

Claim 6.4. *Let H_1 and H_2 be two distinct elements of $\mathcal{H}$ such that $H_1 \cap H_2 \neq \emptyset$. Then $H_1 \cap H_2$ is an edge in M_{NC} and $G[H_1 \cup H_2 - V(e)]$ is not connected.*

Proof. Suppose $|V(H_1 \cap H_2)| = 1$ and let $x = H_1 \cap H_2$. Since G is 2-connected, there exists an H_1-H_2 path in $G - x$, say P. Let $y_1 = P \cap H_1$ and $y_2 = P \cap H_2$. By Claim 6.3, $xy_1, xy_2 \in M_{NC}$ which is impossible.

Suppose $|V(H_1 \cap H_2)| \geq 3$. Since H_1 is 2-connected, we can find an H_2-path in H_1, say xPy. By Claim 6.3, $xy \in M_{NC}$. By property $(*)$, $H_1 - x - y$ is connected. Let Q be a shortest $(V(H_1 \cap H_2) \setminus \{x, y\})$-$V(P - x - y)$ path in $H_1 - x - y$ such that $z = V(Q) \cap (V(H_1 \cap H_2) \setminus \{x, y\})$ and $a = V(Q) \cap V(P - x - y)$. Then $xPaQz$ is an H_2-path. By Claim 6.3, $xz \in M_{NC}$ which is impossible.

Therefore, $|V(H_1 \cap H_2)| = 2$ and let $\{x, y\} = H_1 \cap H_2$. Consider an x-y H_2-path in H_1. By Claim 6.3, $xy \in M_{NC}$. Since H_1 and H_2 have property $(*)$, $G[H_1 \cup H_2 - V(e)]$ is not connected. □

Now, define an auxillary bipartite graph A with the bipartite vertex sets $\mathcal{H}$ and M_{NC} respectively such that there exists an edge between $H \in \mathcal{H}$ and $e \in M_{NC}$ in A if and only if $e \in E(H)$.

Claim 6.5. *A is a tree.*

Proof. First, we show that A is connected. By Claim 6.2, it suffices to prove that for any $H_1, H_2 \in \mathcal{H}$, there is a path between H_1 and H_2 in A. For $H_1 = H_2$, it is trivial. For $H_1 \cap H_2 \neq \emptyset$, it is true by Claim 6.4. For $H_1 \cap H_2 = \emptyset$, let P be an H_1-H_2 path in G. By applying Claim 6.1 to each edge in P and using Claim 6.4, it is easy to see that there is a path between H_1 and H_2 in A.

Next, we show that A is acyclic. Suppose there is a cycle in A, say $H_1e_1H_2e_2 \ldots H_ke_kH_1$ where $H_i \in \mathcal{H}$ and $e_i \in M_{NC}$. But then $G[H_1 \cup H_2 \cup \ldots \cup H_k]$ has property $(*)$, a contradiction. □

Claim 6.6. *For any $H \in \mathcal{H}$, H is not K_3.*

Proof. Suppose H is K_3 with vertices x, y, z. Without loss of generality, assume there is an x-y H-path. By Claim 6.3, $xy \in M_{NC}$. By Claim 6.4, z cannot belong to other elements of $\mathcal{H}$. Hence, z has degree two in G. By Lemma 3.2, zx is contractible. But then $M - xy + zx$ is a maximum matching containing more contractible edges than M, a contradiction. □

Claim 6.7. *Let $H \in \mathcal{H}$ and e be an edge in H. If e is non-contractible in H, then e is non-contractible in G. If e is contractible in H, then either e is contractible in G or $e \in M_{NC}$.*

Proof. Suppose e is non-contractible in H and contractible in G. Let C_1 and C_2 be two components of $H - V(e)$. Since $G - V(e)$ is connected, there exists a shortest C_1-C_2 path in $G - V(e)$, say P. But then $G[H \cup P]$ is 2-connected and has property $(*)$, a contradiction.

Suppose e is contractible in H and non-contractible in G. Let C be a component of $G - V(e)$ not containing $H - V(e)$. Let D be a shortest cycle in $G[C \cup e]$ containing e, and H' be an element of $\mathcal{H}$ containing D. Obviously, $H \neq H'$. By Claim 6.4, $e \in M_{NC}$. □

Claim 6.8. $|\mathcal{H}| \geq |M_{NC}| + 1$.

Proof. By Claims 6.5 and 6.2, we have

$$\begin{aligned} 2(|\mathcal{H}| + |M_{NC}| - 1) &= 2(|V(A)| - 1) \\ &= \sum_{H \in \mathcal{H}} \deg_A(H) + \sum_{e \in M_{NC}} \deg_A(e) \\ &= 2 \sum_{e \in M_{NC}} \deg_A(e) \;\geq\; 4|M_{NC}|. \end{aligned}$$

Therefore, $|\mathcal{H}| \geq |M_{NC}| + 1$. □

Claim 6.9. *For each $H \in \mathcal{H}$, H contains at least two edges in M_C.*

Proof. Suppose H contains at most one edge in M_C. Since H is not K_3 by Claim 6.6, by applying Lemmas 3.2, 3.3 and 3.4 to H, we can delete all non-contractible edges in H so that the resulting graph H' is 2-connected and all edges in H' are contractible in H. By Claim 6.7, every contractible edge in G that lies in H is contractible in H. By the definition of $(*)$, every edge in $M_{NC} \cap E(H)$ is contractible in H. Therefore, none of the edges in $M \cap E(H)$ are deleted in forming H'. Consider any vertex x in H'. Suppose x is incident to an edge in M, say xy. If $y \notin H'$, then by Claim 6.1, xy belongs to an element of $\mathcal{H}$ other than H, say K. By Claim 6.4, $K \cap H$ is an edge in M_{NC} incident to x, which is impossible. Therefore, any edge in M incident to a vertex in H' lies in H'.

We claim that every vertex in H' is incident to an edge in $M \cap E(H')$. Suppose x is a vertex in H' not incident to any edges in $M \cap E(H')$. Let y be any neighbor of x in H'. By Claim 6.7, xy is contractible in G. By the maximality of M, y is incident to an edge in $M \cap E(H')$, say yz. If $yz \in M_{NC}$, then $M - yz + xy$ contains more contractible edges than M, a contradiction. Hence, $yz \in M_C$. Since y is an arbitrary neighbor of x in H' and H' contains at most one edge in M_C, this implies that y and z are the only neighbors of x in H'. But then yz is non-contractible in H', a contradiction.

Summing up, every edge in H' belongs to either M or $E_C(G) \setminus M$, and every vertex in H' is incident to an edge in $M \cap E(H')$ and an edge in $(E_C(G) \setminus M) \cap E(H')$. Color all edges in $M \cap E(H')$ with one color and the rest with another color. By Theorem 6.3, there exists a cycle $x_1 x_2 \ldots x_{2k} x_1$ in H' such that $F := \{x_1x_2, x_3x_4, \ldots, x_{2k-1}x_{2k}\} \subseteq M$ and $F' := \{x_2x_3, x_4x_5, \ldots, x_{2k}x_1\} \subseteq E_C(G) \setminus M$. Since H contains at most one edge in M_C, $M - F + F'$ has more contractible edges than M, a contradiction. □

Claim 6.10. *M contains at least $2(|M| + 1)/3$ contractible edges.*

Proof. By Claim 6.4, no two distinct elements of $\mathcal{H}$ share an edge in M_C. By Claims 6.9 and 6.8, $|M_C| \geq 2|\mathcal{H}| \geq 2(|M_{NC}|+1)$. Therefore, $3|M_C| \geq 2(|M_C|+|M_{NC}|+1) = 2(|M| + 1)$. □

The proof of Theorem 6.4 is complete. □

Lastly, the bound in Theorem 6.4 is best possible as demonstrated by the family of graphs below. The building blocks are cycles of length four, C_4, and K_2's. Define $V(C_4^i) := \{x_1^i, y_1^i, x_2^i, y_2^i\}$ and $E(C_4^i) := \{x_1^i y_1^i, y_1^i x_2^i, x_2^i y_2^i, y_2^i x_1^i\}$, and $V(K_2^i) := \{z_1^i, z_2^i\}$ and $E(K_2^i) := \{z_1^i z_2^i\}$. Now, we construct the family of graphs G_n inductively. Define

$$V(G_1) := V(C_4^1) \cup V(K_2^1) \cup V(C_4^2)$$

and

$$E(G_1) := E(C_4^1) \cup E(K_2^1) \cup E(C_4^2) \cup \{y_1^1 z_1^1, y_2^1 z_2^1, z_1^1 y_1^2, z_2^1 y_2^2\}.$$

Suppose we have constructed G_n. Define

$$V(G_{n+1}) := V(G_n) \cup V(K_2^{n+1}) \cup V(C_4^{n+2})$$

and

$$E(G_{n+1}) := E(G_n) \cup E(K_2^{n+1}) \cup E(C_4^{n+2}) \cup \{y_1^{n+1} z_1^{n+1}, y_2^{n+1} z_2^{n+1}, z_1^{n+1} y_1^{n+2}, z_2^{n+1} y_2^{n+2}\}.$$

Notice that any maximum matching of G_n is in fact a perfect matching, and must contain two independent edges of every C_4 which are contractible and all the K_2's which are non-contractible.

Acknowledgments

The author would like to thank the referees for pointing out errors and giving suggestions that have greatly improved the accuracy and presentation of the paper.

References

[1] R. E. L. Aldred, J. Chen and R. L. Hemminger, The 3-connected graphs with a Hamilton path containing precisely two contractible edges, *Congr. Numer.* 97 (1993), 209–222.

[2] R. E. L. Aldred, J. Chen and R. L. Hemminger, The 3-connected graphs with a longest path containing precisely two contractible edges, *Australas. J. Combin.* 13 (1996), 3–13.

[3] R. E. L. Aldred, R. L. Hemminger and K. Ota, The 3-connected graphs having a longest cycle containing only three contractible edges, *J. Graph Theory* 17 (1993), 361–371.

[4] R. E. L. Aldred, R. L. Hemminger and X. Yu, Maximum matchings in 3-connected graphs contain contractible edges, *Australas. J. Combin.* 6 (1992), 23–34.

[5] R. E. L. Aldred, R. L. Hemminger and X. Yu, The 3-connected graphs with a maximum matching containing precisely one contractible edge, *J. Graph Theory* 18 (1994), 37–50.

[6] K. Ando, H. Enomoto and A. Saito, Contractible edges in 3-connected graphs, *J. Combin. Theory Ser. B* 42 (1987), 87–93.

[7] N. Dean, R. L. Hemminger and K. Ota, Longest cycles in 3-connected graphs contain three contractible edges, *J. Graph Theory* 13 (1989), 17–21.

[8] R.Diestel, *Graph theory* (4th edition), Springer-Verlag, 2010.

[9] Y. Egawa, K. Ota, A. Saito and X. Yu, Non-contractible edges in a 3-connected graph, *Combinatorica* 15 (1995), 357–364.

[10] M. N. Ellingham, R. L. Hemminger and K. E. Johnson, Contractible edges in longest cycles in non-Hamiltonian graphs, *Discrete Math.* 133 (1994), 89–98.

[11] A. Elmasry, K. Mehlhorn and J. M. Schmidt, Every DFS tree of a 3-connected graph contains a contractible edge, *J. Graph Theory* 72 (2013), 112–121.

[12] K. Fujita, Maximum number of contractible edges on longest cycles of a 3-connected graph, *Ars Combin.* 74 (2005), 129–149.

[13] K. Fujita, Lower bound on the maximum number of contractible edges on longest cycles of a 3-connected graph, *Far East J. Applied Math.* 22 (2006), 55–86.

[14] J. W. Grossman and R. Häggkvist, Alternating cycles in edge-partitioned graphs, *J. Combin. Theory Ser. B* 34 (1983), 77–81.

[15] J. Kahn and P. Seymour, On forbidden minors for GF(3), *Proc. Amer. Math. Soc.* 265 (1981), 47–58.

[16] M. Kriesell, Average degree and contractibility, *J. Graph Theory* 51 (2006), 205–224.

[17] W. McCuaig, Edge contractions in 3-connected graphs, *Ars Combin.* 29 (1990), 299–308.

[18] K. Ota, The number of contractible edges in 3-connected graphs, *Graphs Combin.* 4 (1988), 333–354.

[19] J. G. Oxley, *Matroid Theory*, Oxford University Press, 1992.

[20] W. T. Tutte, A theory of 3-connected graphs, *Nederl. Akad. Wet. Proc. Ser A* 64 (1961), 441–455.

[21] H. Wu, Distribution of contractible elements in 2-connected matroids, *Discrete Math.* 305 (2005), 386–393.

(Received 21 Feb 2018; revised 30 Nov 2019, 8 July 2020)