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Fully active cops and robbers

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Abstract

We study a variation of the classical pursuit-evasion game of Cops and Robbers in which agents are required to move to an adjacent vertex on every turn. We explore how the minimum number of cops needed to catch the robber can change when this condition is added to the rules of the game. We study this "Fully Active Cops and Robbers" game for a number of classes of graphs and present some open problems for future research.

1 Introduction

The game of Cops and Robbers played on graphs was introduced independently by Quillot [12, 13] and Nowakowski and Winkler [10]. The game is played between a set

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of pursuers (cops) and an evader (robber) who move from vertex to adjacent vertex in a graph. The cops win if at least one cop is able to occupy the same vertex as the robber; the robber wins if he can avoid capture indefinitely. In the original version of the game, the game begins with the cops choosing their starting vertices, followed by the robber choosing his; multiple cops may occupy the same vertex simultaneously. The cops move first, with each cop either moving to a vertex adjacent to her current position or staying on her current vertex. The robber then moves similarly. Players continue to alternate moves in this way.

Many variations on Cops and Robbers have been studied, the most common of which focus on altering the rules by which the players move. For example, Aigner and Fromme [1] and Neufeld and Nowakowski [9] considered the so-called "active" version of the game, in which the robber must move on every robber turn—that is, he cannot remain on his current vertex—and at least one cop must move on every cop turn. Recently, Offner and Ojakian [11] introduced a wide class of Cops and Robbers variants, wherein one specifies how many cops must move on every cop turn, how many must remain in place, and how many may do either. They focused in particular in the case where only one cop may move on each turn; this variant was termed *Lazy Cops and Robbers* by Bal et al. [2] and studied afterwards by several authors (see for example [3, 7, 15]).

Inspired by the "active" game of Aigner and Fromme, we consider the variant of Cops and Robbers in which no player may ever remain on a vertex—that is, every player must move on each turn. We call this variant *Fully Active Cops and Robbers*, and we refer to the original game as *Passive Cops and Robbers*. As in Passive Cops and Robbers, the cops win if at least one cop is able to occupy the same vertex as the robber at any point during the game[1]. The minimum number of cops required for the cops to have a winning strategy in a graph G is called the *cop number* of G. We use $\mathrm{c}(G)$ to denote the cop number of G in the passive setting and $\mathrm{c_a}(G)$ to denote the cop number of G in the fully active setting.

In this paper, we primarily focus on establishing values (or bounds on the value) of $\mathrm{c_a}(G)$. In Section 2, we give bounds on $\mathrm{c_a}$ over several elementary classes of graphs; our main result of this section, Theorem 2.4, states that $\mathrm{c_a}(G) \leq 2$ whenever G is outerplanar. In Section 3, we investigate when and how much $\mathrm{c_a}(G)$ and $\mathrm{c}(G)$ can differ. Theorem 3.1 states that always $\mathrm{c}(G) - 1 \leq \mathrm{c_a}(G) \leq 2\,\mathrm{c}(G)$, while Theorem 3.3 gives a class of graphs on which this upper bound on $\mathrm{c_a}(G)$ is tight. In Section 4, we study the fully active game played on Cartesian products of graphs. For general graphs, Theorem 4.1 states that $\mathrm{c_a}(G \,\Box\, H) \leq \mathrm{c_a}(G)\,\mathrm{c_a}(H)$ provided that G and H are not both bipartite. Theorem 4.4 gives the precise value of the fully active cop number of the Cartesian product of nontrivial trees, while Theorems 4.8 and 4.9 give bounds on the fully active cop number of a product of cycles. We conclude with several tantalizing open questions.

[1]Although this winning condition is generally considered to be the standard definition, other variations have been studied in the literature (see, e.g., [5]), and so we make this explicit.

2 Some simple graph classes

Many of the strategies known for catching a robber in the passive game seem to fail in the fully active game. For example, Aigner and Fromme showed [1] that if P is a shortest uv-path in G (where u and v are any two distinct vertices in G), then one cop can always move along P so that the robber may never occupy a vertex of P; this was the foundation of their proof that $\mathrm{c}(G) \leq 3$ whenever G is planar. However, this strategy relies on the ability of the cop to stay put if needed, and as such, it cannot be applied in the fully active game. Thus, when studying the fully active game, we are forced to return to basics.

We will begin with some elementary results on Fully Active Cops and Robbers. In the case of some simple, classic classes of graphs, it is easy to see that the cop number remains unchanged in the fully active setting.

Proposition 2.1 *1. If T is a tree, then $\mathrm{c_a}(T) = \mathrm{c}(T) = 1$.*

2. If $n \geq 4$ is a positive integer, then $\mathrm{c_a}(C_n) = \mathrm{c}(C_n) = 2$.

3. If n is a positive integer, then $\mathrm{c_a}(K_n) = \mathrm{c}(K_n) = 1$.

4. If $m, n \geq 2$ are positive integers, then $\mathrm{c_a}(K_{m,n}) = \mathrm{c}(K_{m,n}) = 2$.

A graph is called *cop-win* if one cop has a winning strategy (and *robber-win* otherwise). It is not hard to see that cop-win graphs in the passive setting are also cop-win in the active game.

Proposition 2.2 *Let G be a graph. If $\mathrm{c}(G) = 1$, then $\mathrm{c_a}(G) = 1$.*

Proof: Suppose, to the contrary, that G is robber-win in the fully active setting. The robber could then play as follows in the passive setting. When the cop moves, the robber plays his corresponding move from his winning strategy in the fully active setting. When the cop remains in place, so does the robber. This yields a winning strategy for the robber in the passive setting, a contradiction. □

We now provide a simple construction showing that the converse of Proposition 2.2 does not hold and so, in general, $\mathrm{c_a}(G)$ need not equal $\mathrm{c}(G)$.

Proposition 2.3 *Let G be a graph with vertex partition $\{v\} \cup A \cup B$ and the following edges:*

- *v is adjacent to all vertices in A and none in B,*
- *A is an independent set, and each vertex in A is adjacent to all vertices in B, and*
- *B is a clique.*

If $|A| \geq 2$ and $|B| \geq 2$ then $\mathrm{c_a}(G) = 1$ and $\mathrm{c}(G) = 2$.

Proof: In the passive game, the robber can evade one cop as follows. If the cop begins the game on v, then the robber begins on any vertex in B; if the cop begins on some vertex in A, then the robber begins on any other vertex in A; if the cop begins on some vertex in B, then the robber begins on v. Likewise, when the cop moves to v, the robber moves to some vertex in B; when the cop moves to a vertex in A, the robber moves to a different vertex in A; when the cop moves to a vertex in B, the robber moves to v. Hence $\mathrm{c}(G) \geq 2$, and it is easy to check that $\mathrm{c}(G) \leq 2$.

In the fully active setting, a single cop can win as follows. The cop begins on some vertex in B. If the robber begins in $A \cup B$, then the cop can capture him immediately. If instead the robber begins on v, then the cop moves to a different vertex of B. The robber must now move to A, after which the cop can capture him. □

As mentioned earlier, a single cop cannot, in general, guard a shortest path in the fully active setting. This strategy was crucial to Aigner and Fromme's proof [1] that $\mathrm{c}(G) \leq 3$ for any planar graph G, and as such, their proof cannot be adapted to the fully active setting. It appears that determining a tight upper bound on $\mathrm{c_a}$ for planar graphs requires some new ideas and may be difficult. However, it is much easier to analyze the fully active game on outerplanar graph. Clarke [6] showed that two cops can capture a robber on any outerplanar graph in the passive game; we show that the same is true in the fully active game. The proof presented below is an adaptation of the proof from [4] of Clarke's result.

Theorem 2.4 *If G is an outerplanar graph, then* $\mathrm{c_a}(G) \leq 2$.

Proof: Let G be an outerplanar graph. Let us first assume that G has no cut vertices. If G is a cycle, then clearly $\mathrm{c_a}(G) \leq 2$. If not, then by definition G may be embedded in the plane so that the outer face is bounded by a Hamiltonian cycle C and all chords lie in the interior of C and are non-crossing. Let $C = v_1 v_2 \cdots v_n v_1$ and let C_1 and C_2 denote the two cops. We say that a cop *controls* a vertex v if that cop's position is in the closed neighbourhood of v. For two vertices v_k and v_ℓ, we define $v_k C v_\ell$ to be the path $v_k v_{k+1} \cdots v_{\ell-1} v_\ell$ (subscripts taken modulo n).

Throughout the game, cop C_1 will control some vertex v_k and C_2 will control some vertex v_ℓ. The robber's position must lie on either $v_k C v_\ell$ or $v_\ell C v_k$; call the interior of this path the *robber territory* and the interior of the other path the *cop territory*, and call v_k and v_ℓ the *endpoints* of the cop territory. We show how the cops can move so that (a) after some finite number of moves the size of the cop territory has increased and (b) at no point does any edge join the robber's position to the cop territory.

The cops begin by placing themselves at the ends of some chord and choosing v_k and v_ℓ to be the vertices occupied by cops C_1 and C_2, respectively. This ensures that no matter where the robber starts, no edge joins the robber's position to the cop territory.

Now consider an arbitrary point during the game just before the cops' turn, and suppose that cop C_1 controls v_k while C_2 controls v_ℓ; note that C_2 must occupy an endpoint of some edge, say e, incident to v_ℓ. Furthermore, suppose without loss of generality that the robber is on v_kCv_ℓ and suppose that no edge from the robber's territory to the interior of the cop territory. If some chord joins v_k to some vertex in the robber territory, then let v_r be the neighbour of v_k in the robber territory which is closest along C to v_ℓ (but not equal to v_ℓ). We may assume that C_1 occupies v_k, because if not she may move to v_k while C_2 moves back and forth along e. If the robber's position is adjacent to v_k then the cops clearly win, so suppose otherwise. If the robber's position is on v_rCv_ℓ, then C_1 moves to v_r and C_2 moves along e. The cops now control v_r and v_ℓ. Note that by choice of v_r, no edge can join the robber's new territory to the interior of $v_\ell Cv_r$: by assumption no edge joined the robber's previous territory to the interior of $v_\ell Cv_k$, the robber's position was not adjacent to v_k, and any edge joining the robber's new territory to the interior of v_kCv_r would have to cross the chord v_kv_r. The cops now set $v_k = v_r$, thereby enlarging the cop territory to $v_\ell Cv_r$. If the robber's position is in v_kCv_r, then C_2 moves to v_r while C_1 moves along v_kv_r as many times as necessary. The cops then set $v_\ell = v_r$, enlarging the cop territory to v_rCv_k. (Note that no edge can join the robber's new territory to the interior of the cop territory, since any such edge would cross the chord v_kv_r.) A symmetric argument shows that the cops can enlarge the cop territory if v_ℓ has a chord to the robber territory but v_k does not. Finally, suppose that neither v_k nor v_ℓ has a chord to the robber territory. In this case, every path from the robber to the cop territory passes through either the edge v_kv_{k+1} or the edge $v_\ell v_{\ell-1}$. In one step, C_1 can move to control v_{k+1} and C_2 can move to control $v_{\ell-1}$. The cops can then set $v_k = v_{k+1}$ and $v_\ell = v_{\ell-1}$, thereby increasing the cop territory to $v_{\ell-1}Cv_{k+1}$.

Now, suppose that G has cut vertices and that the two cops occupy vertices in the same 2-connected block B of G. If the robber's position is also in B, then the cops play as above. If the robber's position is not in B, then there is some cut vertex $v \in B$ which separates the cops' positions from the robber's position. In this case, the cops play as if the robber occupies v. In this way, the cops will either eventually capture the robber in B or will both control v. In the latter case, the cops may now move to a new 2-connected block B' where $V(B') \cap V(B) = \{v\}$ and repeat the strategy. Since v will always belong to the cops' territory in B', the robber can never move back to B. Eventually, the robber will be forced to an end-block of G and will be caught. □

3 Passive versus fully active cop numbers

In this section, we explore the relationship between $\mathrm{c_a}(G)$ and $\mathrm{c}(G)$. We begin with elementary lower and upper bounds on $\mathrm{c_a}(G)$ in terms of $\mathrm{c}(G)$.

Theorem 3.1 *If G is a graph with $\mathrm{c}(G) = k$, then $k - 1 \le \mathrm{c_a}(G) \le 2k$.*

Proof: We first prove the upper bound. Let $\{C_1, \ldots, C_k, D_1, \ldots, D_k\}$ be a set of cops. We place $\{C_1, \ldots, C_k\}$ on $V(G)$ according to a winning strategy in the passive setting and place each D_i on some vertex adjacent to C_i. The cops then play a modification of their winning strategy from the passive game. If the strategy for the passive game requires C_i to move to a new vertex, then she does so and the corresponding cop D_i moves to the vertex formerly occupied by C_i. If C_i stays on her vertex in the passive game strategy, then C_i and D_i switch places and labels. This ensures that at all times, all vertices occupied by cops in the passive game are also occupied by cops in the fully active game; since the cops eventually capture the robber in the passive game, they do so in the fully active game as well.

To prove the lower bound, we suppose that $\mathrm{c_a}(G) = t$ and show how $t + 1$ cops can win the passive game. Let $\{C_1, \dots, C_t, C^*\}$ be a set of cops. We place $\{C_1, \dots, C_t\}$ on $V(G)$ according to a winning strategy in the fully active setting and place C^* arbitrarily. The cops then play a modification of their winning strategy in the fully active game. If the robber moves, then each C_i moves according to the winning strategy and C^* moves to decrease her distance from the robber. If the robber remains in place, then so does each C_i, while C^* again moves to decrease her distance from the robber. Eventually, the game will be in a state where either the robber has been caught by some C_i or C^* occupies a vertex adjacent to the robber's position. In the latter case, the robber can no longer remain in place, so the game proceeds as if it were being played in the fully active setting. Thus, $\mathrm{c}(G) \le \mathrm{c_a}(G) + 1$. □

We have seen that there exist graphs for which $\mathrm{c_a}(G) = \mathrm{c}(G) - 1$ (Proposition 2.3), so the lower bound in Theorem 3.1 is tight. We next show that the upper bound is tight as well by producing, for each positive integer k, a class of graphs G having $\mathrm{c}(G) = k$ and $\mathrm{c_a}(G) = 2k$.

For a graph G and positive integer t, the *t-blowup* of G is a new graph obtained by replacing each vertex v in G with an independent set S_v of size t and replacing each edge uv in G with a complete bipartite graph having partite sets S_u and S_v. We refer to the vertices in S_v as *copies* of v, and we call v the *shadow* of each vertex in S_v. We denote the t-blowup of G by $G^{(t)}$.

We will need the following lemma, which is a special case of a result established by Schröder ([14], Theorem 2.7). Note that the requirement that $\mathrm{c}(G) \ge 2$ cannot be lifted: for $t \ge 2$, a robber may evade a single cop by always occupying a vertex that is distinct from the cop's vertex, yet has the same shadow.

Lemma 3.2 *Let G be a connected graph. If $\mathrm{c}(G) \ge 2$, then $\mathrm{c}(G^{(t)}) = \mathrm{c}(G)$ for all positive integers t.*

We are now ready to give our construction of a class of graphs having passive cop number k and fully active cop number $2k$. In the theorem below, we use $G \,\square\, H$ to denote the Cartesian product of G and H.

Theorem 3.3 *Fix $k \ge 2$, let $T_1, T_2, \dots, T_{2k-1}$ be nontrivial trees, and let $G = T_1 \,\square\, T_2 \,\square\, \cdots \,\square\, T_{2k-1}$. Now $\mathrm{c}(G^{(t)}) = k$ and $\mathrm{c_a}(G^{(t)}) = 2k$ whenever $t \ge 2k$.*

Proof: It follows from a result of Maamoun and Meyniel ([8], Theorem 2) that $\mathrm{c}(G) = k$, hence Lemma 3.2 implies that $\mathrm{c}(G^{(t)}) = k$.

For the active game, Theorem 3.1 shows that $c_a(G^{(t)}) \leq 2\,c(G^{(t)}) = 2k$, so it suffices to show that $c_a(G^{(t)}) > 2k-1$. Suppose the robber plays against $2k-1$ cops on $G^{(t)}$. We view each vertex v in G as a $(2k-1)$-tuple $(v_1, v_2, \dots, v_{2k-1})$, where $v_i \in V(T_i)$ for $1 \leq i \leq 2k-1$; we call v_i the *ith coordinate* of v. Note that every two adjacent vertices in G differ in exactly one coordinate, and any two vertices at distance 2 differ in at most two coordinates. Since each T_i is bipartite, so are G and $G^{(t)}$; let X and Y denote the partite sets of $G^{(t)}$.

Once the cops have chosen their initial positions, some partite set, without loss of generality X, must contain at most $k-1$ cops. For his initial position, the robber chooses any vertex in Y that neither contains a cop nor is adjacent to a cop. To see that this is possible, we first find a vertex in G which is not adjacent to any shadow of a cop in X. Let $v_1, \dots, v_{k-1} \in V(G)$ denote these shadows (note that these vertices may coincide), and let v be a vertex which differs from each v_i in coordinates $2i-1$ and $2i$ and has coordinate $2k-1$ chosen so that v is in the opposite partite set from all v_i. We now have that v is a vertex which is at distance at least two from each shadow in X. By choosing a copy in of v in $G^{(t)}$ which is not occupied by a cop in Y, the robber can find his desired starting position.

The robber's choice of initial position ensures that he cannot lose on the cops' first turn. To show that the robber can avoid losing on subsequent turns, it suffices to show that he can always move to some vertex not containing a cop and not adjacent to a cop. Suppose it is the robber's turn, and assume without loss of generality that the robber occupies some vertex in Y; a similar argument works for the other case. Since $G^{(t)}$ is bipartite and the robber started in Y, he must have taken an even number of turns; consequently, the cops have taken an odd number of turns. Thus every cop who started in X is now in Y, and vice-versa. In particular, there are at most $k-1$ cops in Y, and hence at most $k-1$ cops at distance 2 from the robber. Let v be the shadow in G of the robber's current position, and let $w_1, w_2, \dots, w_{2k-1}$ be the shadows of the cops' positions. Since the robber was not adjacent to a cop after his previous turn, v does not coincide with any of the w_i, and moreover, at most $k-1$ of the w_i are at distance 2 from v. Each w_i at distance 2 from v differs from v in at most two coordinates, so there is some coordinate in which v agrees with all such w_i. Let v' be some neighbor of v that differs only in this coordinate, and note that v' is not adjacent to any w_i (although it might coincide with one or more). Since $t \geq 2k$, there are at least $2k$ copies of v'. As there are only $2k-1$ cops, some copy of v' contains no cop. The robber moves to any such copy of v'; by construction, the robber's new position contains no cop and is not adjacent to any cop, as desired. It follows that the robber may evade capture indefinitely. □

Theorem 3.3 shows that for all $k \geq 2$, there exist graphs with cop number k and fully active cop number $2k$, and thus $c_a(G) - c(G)$ can be arbitrarily large. Note that Proposition 2.2 shows that the requirement of $k \geq 2$ cannot be dropped from the previous statement.

4 Graph products

In this section, we further consider the fully active game played on graph products. For general graphs, we have the following result. Note that the restriction imposed on G_1 and G_2 is not as stringent as it might first appear: for example, any non-bipartite graph meets this requirement, since in a non-bipartite graph the cops can reach any configuration from any other. The restriction also holds for any graph with fully active cop number 1.

Theorem 4.1 *If G_1 and G_2 are graphs such that $\mathrm{c_a}(G_2)$ cops can win the fully active game on G_2 regardless of their initial positions, then $\mathrm{c_a}(G_1 \square G_2) \leq \mathrm{c_a}(G_1) + \mathrm{c_a}(G_2)$.*

Proof: First suppose that both G_1 and G_2 have the property that $\mathrm{c_a}(G_i)$ cops can win the fully active game on G_i regardless of their initial positions; we explain at the end of the proof how we can eliminate this restriction on G_1. Let $k = \mathrm{c_a}(G_1)$ and $\ell = \mathrm{c_a}(G_2)$; we show that $k + \ell$ cops can win the fully active game on $G_1 \square G_2$. As usual, when a player occupies the vertex (u, v) in $G_1 \square G_2$, we say that u (respectively, v) is that player's *position in* G_1 (respectively, *position in* G_2).

The cops will divide themselves into two teams, "Team G_1" and "Team G_2". Initially, both teams are empty. The cops fix winning strategies for $k + \ell$ cops in the fully active games on both G_1 and G_2. Cop i begins on vertex (u, v), where u denotes her starting position in the game on G_1 and v denotes her starting position on G_2. The cops now play as follows. On their first turn, each cop moves in G_1 according to the cops' winning strategy for the game on G_1. On subsequent turns, the cops respond to the robber's previous move as follows: if the robber moved in G_i, then the cops move in G_i according to a winning strategy for the game on G_i. Eventually, some cop must capture the robber in one of the games. If a cop captures the robber in the game on G_i, then her position in G_i agrees with the robber's, and the cop joins Team G_i. Henceforth, whenever the robber moves in G_i, each cop in Team G_i makes an identical move in G_i, and when the robber moves in G_{3-i} the cops in Team G_i participate in a winning strategy in G_{3-i} without changing their coordinate in G_i. This ensures that each cop in Team G_i always occupies the same position as the robber in G_i.

The cops now repeat this process. Suppose Team G_i has a members and Team G_2 has b members. The remaining cops mimic winning strategies in the fully active games on G_1 and G_2, with the $a + b$ cops playing as described in the previous paragraph. In total, $k + \ell - b$ cops are playing a winning strategy in G_1 and $k + \ell - a$ cops are playing a winning strategy in G_2. This guarantees that some new cop captures the robber either on G_1 (in which case she joins Team G_1) or on G_2 (in which case she joins Team G_2). Eventually either ℓ cops have joined Team G_1 or k cops have joined Team G_2; without loss of generality, assume the former.

The cops now change their strategies slightly. Henceforth, when the robber moves in G_1, the cops in Team G_1 continue to follow him; when the robber moves in G_2, these cops move in G_2 according to a winning strategy for the game on G_2. Note that the robber cannot move in G_2 infinitely often, or some cop in Team G_1 will reach the same position as the robber in G_2, at which point she captures the robber in $G_1 \square G_2$.

The remaining cops play slightly differently. When the robber moves in G_1, the cops in Team G_2 move arbitrarily in G_1, while the unassigned cops move toward the robber in G_2. When the robber moves in G_2, the cops in Team G_2 follow him in G_2, while the unassigned cops again move toward him in G_2. If at any point one of the unassigned cops reaches the same position as the robber in G_2, then that cop joins Team G_2. Since the robber cannot move in G_2 infinitely often, he must move in G_1 infinitely often, so eventually all of the unassigned cops catch up to the robber in G_2 and thus join Team G_2. Thus eventually Team G_2 comes to contain k cops. At this point, whenever the robber moves in G_1, the cops in Team G_2 move through G_1 according to a winning strategy in the game on G_1. (When he moves in G_2, they continue to follow him in G_2.) Since the robber moves in G_1 infinitely often, eventually one of the cops in Team G_2 will capture the robber in G_1 and hence in $G_1 \square G_2$.

This completes the proof when both G_1 and G_2 have the property that $\mathrm{c_a}(G_i)$ cops can capture the robber on G_i regardless of their starting positions. Suppose now that G_2 has this property but G_1 does not. Note that G_1 must necessarily be bipartite; call its partite sets X and Y. Consider a winning strategy for the game on G_1, and suppose that under this strategy k_X cops begin on vertices of X while k_Y begin on vertices of Y. We claim that $k_X + k_Y$ cops can win the game on G_1 for any initial arrangement that places k_X cops in X and k_Y in Y: indeed, from any such arrangement, the cops can reconfigure themselves to their desired starting positions.

The cops amend their strategy on $G_1 \square G_2$ as follows. All cops assigned to Team G_1 or Team G_2 play as normal, but the unassigned cops play slightly differently. After any cop joins a team, before restarting their strategy to capture the robber on G_2, the unassigned cops slightly rearrange their positions. If fewer than k_X cops in Team G_2 currently occupy vertices whose position in G_1 belongs to X, then the unassigned cops move so that, after an even number of steps, every unassigned cop's position in G_1 belongs to X. Should one of these cops join Team G_2 on the next iteration of the strategy, then after an even number of steps, her position in G_1 will belong to X. If instead exactly k_X cops in Team G_2 occupy vertices whose position in G_1 belongs to X, then the unassigned cops move so that after an even number of steps, each occupies a vertices whose position in G_1 belongs to Y; this ensures that if one of these cops joins Team G_2, then her position in G_1 will belong to Y. By playing thus, the cops ensure that once Team G_2 has been completely filled, the cops' positions in G_1 will correspond to an initial configuration from which they can win the game on G_1. From this point, the cops resume following the strategy above, which ensures that they eventually capture the robber. □

Equality in Theorem 4.1 does not hold in general. For example, $\mathrm{c_a}(K_2) = 1$ and $\mathrm{c_a}(C_4) = 2$, but $\mathrm{c_a}(K_2 \square C_4) = 2 < \mathrm{c_a}(K_2) + \mathrm{c_a}(C_4)$. Note also that the restriction on G_2 cannot be lifted. Fix any positive integers k, t with $k \geq 2$ and $t \geq 4k$.

By Theorem 3.3, we have $c_a(Q^{(t)}_{2k-1}) = 2k$. By Theorem 4.2 (see below), we have $c_a(Q_{2k}) = \lceil 4k/3 \rceil$. But now

$$c_a(Q^{(t)}_{2k-1} \,\Box\, Q_{2k}) = c_a((Q_{2k-1} \,\Box\, Q_{2k})^{(t)}) = c_a(Q^{(t)}_{4k-1}) = 4k$$

by Theorem 3.3, but

$$c_a(Q^{(t)}_{2k-1}) + c_a(Q_{2k}) = 2k + \left\lceil \frac{4k}{3} \right\rceil.$$

We now turn our attention to more restricted classes of Cartesian products. Possibly the most notable of all graph products is the n-dimensional hypercube, denoted Q_n. Offner and Ojakian [11] studied a wide class of variants of Cops and Robbers played on the hypercube, in which some cops must move on every cop turn, while others have the option of remaining in place. The theorem below is a special case of their results.

Theorem 4.2 *For any positive integer n, we have* $c_a(Q_n) = \left\lceil \frac{2n}{3} \right\rceil$.

We extend Theorem 4.2 to the more general setting where the game is played on the Cartesian product of arbitrary nontrivial trees. We begin with a lemma.

Lemma 4.3 *Let T_1 and T_2 be nontrivial trees, and consider the fully active game played with a single cop on $T_1 \,\Box\, T_2$. If the cop and robber begin the game at odd distance from each other, then the cop can capture the robber.*

Proof: We view vertices of $T_1 \,\Box\, T_2$ as pairs (v_1, v_2) with $v_i \in V(T_i)$; when a player is located at this vertex, we call v_i that player's *position in* T_i. We give a winning strategy for the cop. The cop's strategy is simple. At each point in the game, let d_i denote the distance (in T_i) from the cop's position in T_i to the robber's position in T_i. On each cop turn, if $d_1 > d_2$, then the cop takes one step closer to the robber in T_1; otherwise, she takes one step closer to the robber in T_2. Note that since $T_1 \,\Box\, T_2$ is bipartite and the cop is at an odd distance from the robber before her first turn, she must be at odd distance from the robber before each of her turns for the duration of the game. Hence $d_1 \neq d_2$ and, consequently, the cop's move always decreases $\max\{d_1, d_2\}$ by 1.

To see that the cop eventually captures the robber, it suffices to show that $\max\{d_1, d_2\}$ gradually decreases throughout the game. It is clear that $\max\{d_1, d_2\}$ never increases over the course of a full round (that is, a cop turn together with the subsequent robber turn): the cop's move decreases $\max\{d_1, d_2\}$ by 1, while the robber's move increases it by at most 1. Moreover, the robber can increase $\max\{d_1, d_2\}$ only by moving away from the cop in the appropriate tree. However, he cannot do this forever: the robber can take at most $\mathrm{diam}(T_1)$ steps away from the cop in T_1 and at most $\mathrm{diam}(T_2)$ steps in T_2, so after at most $\mathrm{diam}(T_1) + \mathrm{diam}(T_2)$ rounds the robber must take at least one step toward the cop. Thus, in this round $\max\{d_1, d_2\}$ decreases. Consequently, $\max\{d_1, d_2\}$ eventually reaches 0, at which point the cop has captured the robber. □

Theorem 4.4 *Let* $\{T_1, T_2, \ldots, T_k\}$ *be nontrivial trees. If* $G = T_1 \,\square\, T_2 \,\square\, \cdots \,\square\, T_k$*, then* $\mathrm{c_a}(G) = \left\lceil \frac{2k}{3} \right\rceil$.

Proof: The claim is clearly true when $k = 1$, so suppose $k \geq 2$. We first show that $\lceil 2k/3 \rceil$ cops can capture the robber. Initially, the cops split themselves into groups whose sizes differ by at most 1. The cops in one group choose some vertex v and all begin the game on v; cops in the other group begin on any neighbor of v. Once the robber has chosen his initial position, let c denote the number of cops at even distance from the robber (call them *even cops*) and d the number of cops at odd distance (*odd cops*). Note that since G is bipartite, there will be c even cops and d odd cops prior to every cop turn throughout the duration of the game.

We assign to each cop either one or two *inactive coordinates*; coordinates which are not inactive are *active coordinates*. To the c even cops we assign coordinates $1, 2, \ldots, c$, respectively; to the d odd cops we assign coordinates $c+1$ and $c+2, c+3$ and $c+4, \ldots, c+2d-1$ and $c+2d$, respectively. If necessary, we "round down" active coordinates to k, so it may be that multiple cops have k as an inactive coordinate. Through case analysis depending on the congruence class of k modulo 3, it is easily verified that $c + 2d \geq k$, so every coordinate is an inactive coordinate for at least one cop.

Each cop moves as follows. If the cop's and robber's positions disagree in any of the cop's active coordinates, then the cop takes one step closer to the robber in any such coordinate. Otherwise, the cop restricts her attention to her inactive coordinates and pretends she is playing a game on the Cartesian product of the corresponding trees. If the cop has only one inactive coordinate, then she simply moves one step closer to the robber in that coordinate. If instead she has two inactive coordinates, then she follows the strategy outlined in Lemma 4.3. Note that every cop with two inactive coordinates is an odd cop, and hence must be at an odd distance from the robber; since the cop under consideration agrees with the robber in all but her two inactive coordinates, she must be at odd distance even when considering only those two coordinates. Thus, she can indeed follow the winning strategy in Lemma 4.3. Once the cop captures the robber in this new game, she has in fact captured the robber in the "real" game.

It remains to show that this is a winning strategy for the cops. Consider a single round of the game, consisting of a robber turn followed by a cop turn. It is clear from the cops' strategy that each cop's total distance to the robber across all active coordinates cannot increase throughout the course of a full round. Suppose now that the robber moved in coordinate i. Some cop has i as an inactive coordinate. If, on the cops' turn, that cop had not yet caught up to the robber in all of her active coordinates, then by the end of the round her total distance to the robber across all active coordinates has decreased by one. If instead that cop had already caught up to the robber in her active coordinates, then on her turn, she was able to focus on her inactive coordinates and take one step closer to winning in that game. Thus, on each turn, at least one cop makes progress, either toward catching up to the robber in her active coordinates or toward capturing the robber in her inactive coordinates; moreover, no other cop loses progress toward either of these goals. Hence, eventually, some cop captures the robber.

We have thus shown that $\mathrm{c_a}(G) \le \lceil 2k/3 \rceil$. To show that $\mathrm{c_a}(G) \ge \lceil 2k/3 \rceil$, we give a strategy for the robber to evade $\lceil 2k/3 \rceil - 1$ cops. Denote the partite sets of G by X and Y, and suppose that initially c cops begin in X while d cops begin in Y. Without loss of generality suppose $c \le d$.

The robber begins the game on any vertex in Y that is neither occupied by nor adjacent to any cop. An argument similar to that used in the proof of Theorem 3.3 shows that this is always possible. It suffices to show that, on every robber turn, the robber can move to a vertex that is neither occupied by nor adjacent to any cop. Since cops at distance three or greater from the robber pose no immediate threat to him, we focus only on cops that are either at distance 1 or distance 2. When it is the robber's turn, the cops have taken one more turn than the robber, so c cops occupy the same partite set as the robber while d cops occupy the other partite set; consequently, at most c cops are at distance 2, while at most d are at distance 1. Cops at distance 2 agree with the robber in all but at most two coordinates, while those at distance 1 agree in all but at most one coordinate. Now note that

$$2c + d = c + (c + d) \le \frac{3}{2}\left(\left\lceil \frac{2k}{3} \right\rceil - 1\right) < k,$$

so there is at least one coordinate, say coordinate i, in which all cops at distance 1 or 2 agree with the robber. The robber now simply takes one step in any direction in T_i. This increases his distance to those cops at distance 1 or 2, while it decreases his distance to all other cops by at most 1. Thus the robber ends his turn on a vertex that is neither occupied by nor adjacent to any cop, as desired. □

We next seek to determine the fully active cop numbers of products of cycles. We will need the following lemma.

Lemma 4.5 *Let G be a bipartite graph. Let $k = \mathrm{c}(G)$, and consider the fully active game on G with k cops. If, after the initial placement by both players, all cops and the robber occupy the same partite set of G, then the cops can ensure capture of the robber.*

Proof: Label the cops $C_1, C_2, \ldots, C_k$. Suppose that the cops and robber have chosen their initial positions for the fully active game on G, and suppose further that all cops and the robber occupy the same partite set. The cops "imagine" an instance of the passive game on G and use a winning strategy in that game to guide their play in the fully active game. Initially, all cops and the robber occupy the same vertices in the imagined passive game as in the fully active game. (Note that when $\mathrm{c}(G)$ cops play the passive game on G, they can win regardless of which positions they initially occupy, since they can begin the game by gradually moving to whichever positions they might have preferred to start at.) For positive integers i, t with $1 \leq i \leq k$, let $v_i^{(t)}$ denote the position of cop C_i in the passive game after t cop turns.

In the passive game, the cops follow a winning strategy. They would like to employ the same strategy in the fully active game; however, cops in the passive game may choose to remain in place, while those in the fully active game cannot. Hence the cops cannot ensure that each cop always occupies the same vertex in both games. However, the cops can ensure that for all positive integers t, after t cop turns, each cop C_i occupies either vertex $v_i^{(t)}$ or one of its neighbors. This is clearly true after the cops' initial placement (adopting the convention that $v_i^{(0)}$ denotes C_i's initial position), so we need only show how the cops can maintain this invariant from round to round. The robber will always occupy the same vertex in both games.

The cops play as follows. Whenever the robber makes a move in the fully active game, the cops imagine that he makes the same move in the passive game, and they respond (in the passive game) as dictated by their winning strategy. In the fully active game, the cops mimic this response in the following manner. Suppose the cops have played t turns in the passive game. By the invariant, each cop C_i occupies $v_i^{(t)}$ in the passive game and either $v_i^{(t-1)}$ or one of its neighbors in the fully active game. If C_i occupies some neighbor of $v_i^{(t-1)}$ in the fully active game, then he moves to $v_i^{(t-1)}$ itself; since $v_i^{(t-1)}$ and $v_i^{(t)}$ must be adjacent, this maintains the invariant. If instead C_i occupies $v_i^{(t-1)}$ itself and $v_i^{(t-1)} \neq v_i^{(t)}$, then he moves to $v_i^{(t)}$. Finally, if C_i occupies $v_i^{(t-1)}$ and $v_i^{(t-1)} = v_i^{(t)}$, then he moves to any neighbor of $v_i^{(t)}$. In any case, the invariant is maintained.

Eventually, some cop C_i captures the robber in the passive game. When this happens, suppose C_i and the robber occupy vertex v in the passive game, while C_i occupies vertex u in the fully active game. By the cops' invariant, either $u = v$ or $u \in N(v)$. If the passive game capture happens on a robber turn, then regardless of whether $u = v$ or $u \in N(v)$, cop C_i either has already captured the robber in the fully active game or can capture him on the ensuing cop turn. If instead the passive game capture happens on a cop turn, then the number of cop moves and number of robber moves have the same parity; since the cops all began on the same partite set as the robber, in the fully active game, the cops and robber must all occupy the same partite set. In particular, u and v belong to the same partite set, so we cannot have $u \in N(v)$, and must therefore have $u = v$: that is, cop C_i has in fact captured the robber in the fully active game. In either case, the cops win the fully active game. □

Later in the section, we will apply Lemma 4.5 to a special type of graph—a *covering graph*.

Definition 4.6 *Given graphs G and H, a* covering map *from H onto G is a mapping from $V(H)$ to $V(G)$ that is surjective and locally isomorphic—that is, for each vertex v in H, the neighborhood of v maps bijectively onto the neighborhood of its image in G. When a covering map from H to G exists, we say that H is a* covering graph *of G.*

Lemma 4.7 *Let G and H be graphs, and let $\phi : H \to G$ be a covering map. Consider a multiset $\mathcal{C}$ of cop positions and a robber position R in H. If cops who begin the game on $\mathcal{C}$ can capture a robber who begins on R in the game on H, then cops who begin on $\phi(\mathcal{C})$ can capture a robber who begins on $\phi(R)$ in the game on G.*

Proof: When playing the game on G, the cops "imagine" a game on H and use a winning strategy in that game to guide their play on G. In the game on H, the cops initially occupy the multiset $\mathcal{C}$ of positions, while the robber occupies position R; in the game on G, the cops occupy $\phi(\mathcal{C})$, while the robber occupies $\phi(R)$. The cops maintain the invariant that the position of each entity (that is, every cop and the robber) in the game on H maps, under ϕ, to the position of that entity in the game on G. This is clearly true at the beginning of the game. When the robber on G moves from some vertex u to some adjacent vertex v, the cops imagine that the robber on H moves to some vertex v' with $\phi(v') = v$; this is possible since, by the invariant, the robber currently occupies some vertex u' with $u = \phi(u')$, and since ϕ is isomorphic on $N(u')$. On the cops' turn in the game on G, each cop first moves on H according to some winning strategy for that game, and then moves, in G, to the image of his new position; as before, this is possible because ϕ is locally isomorphic. Since the cops play a winning strategy on H, eventually some cop on H occupies the same vertex (say x) as the robber. At this point, by the invariant, that cop and the robber both occupy $\phi(x)$ in the game on G, so the cops must eventually capture the robber. □

We now have the tools we need to analyze the fully active game played on the Cartesian product of cycles. In the theorem below, we make use of the fact that $\mathrm{c}(C_{n_1} \,\square\, C_{n_2} \,\square\, \cdots \,\square\, C_{n_k}) = k + 1$ for any positive integers $k, n_1, n_2, \ldots, n_k$, a result due to Neufeld and Nowakowski [9].

Theorem 4.8 *Let $G = C_{n_1} \,\square\, C_{n_2} \,\square\, \cdots \,\square\, C_{n_k}$. If any of the n_i is odd, then $\mathrm{c_a}(G) \leq k + 1$.*

Proof: Suppose without loss of generality that n_1 is odd. Let $H = C_{2n_1} \,\square\, C_{2n_2} \,\square\, \cdots \,\square\, C_{2n_k}$. We represent the vertices of G by ordered k-tuples $(w_1, w_2, \ldots, w_k)$ with $0 \leq w_i < n_i$ in the usual way: two vertices are adjacent if and only if they agree in all but one coordinate, where they differ by 1 (modulo the length of the corresponding factor cycle). Likewise, we represent the vertices of H by ordered k-tuples $(x_1, x_2, \ldots, x_k)$ with $0 \leq x_i < n_{2i}$. For $1 \leq i \leq k$, let ϕ_i be the covering map from C_{2n_i} onto C_{n_i} defined by

$$\phi_i(x) = \begin{cases} x & \text{if } x < n_i; \\ x - n_i & \text{if } x \geq n_i \end{cases}$$

It is easily verified that the map $\psi : V(H) \to V(G)$ defined by $\psi((v_1, \ldots, v_k)) = (\phi_1(v_1), \ldots, \phi_k(v_k))$ is a covering map from H onto G.

Consider the fully active game on G, played with $k + 1$ cops. We show how the cops can use an "imagined" game on H to guide them in playing on G. In the game on G, the cops all begin at vertex $(0, 0, \dots, 0)$. Suppose the robber begins at $(r_1, r_2, \dots, r_k)$. Let $(r'_1, r'_2, \dots, r'_k)$ be any vertex of H whose image (under ψ) is $(r_1, r_2, \dots, r_k)$; the cops imagine that the robber begins the game on H at vertex $(r'_1, r'_2, \dots, r'_k)$. If $(0, 0, \dots, 0)$ and $(r'_1, r'_2, \dots, r'_k)$ belong to the same partite set of H, then the cops imagine that they all begin the game on H at $(0, 0, \dots, 0)$; otherwise, the cops imagine that they all begin at $(n_1, 0, 0, \dots, 0)$. In either case, in the game on H, the cops and robber all occupy the same partite set. Since $\mathrm{c}(H) = k + 1$, Lemma 4.5 implies that the cops have a winning strategy on H. Consequently, by Lemma 4.7, they have a winning strategy on G as well, so $\mathrm{c_a}(G) \le k+1$, as claimed. □

Theorem 4.8 and Theorem 3.1 together show that when $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$ is non-bipartite, then its fully active cop number is either k or $k + 1$. As it turns out, when the graph is bipartite, the fully active cop number behaves much differently.

Theorem 4.9 *Let $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_k}$. If the n_i are all even, then* $\lceil \frac{4k}{3} \rceil \le \mathrm{c_a}(G) \le \lceil \frac{4k+4}{3} \rceil$.

Proof: The lower bound follows from an argument very similar to that used in the proof of Theorem 4.4; as before, the roles of even distance cops and odd distance cops cannot be changed since the graph is bipartite, and double the cops are required compared to the product of trees since the robber can change each coordinate in two ways.

For the upper bound, suppose each n_i is even and let $m = \lceil \frac{4k+4}{3} \rceil$. We give a strategy for m cops to capture the robber on G. We represent the vertices of G as ordered k-tuples in the usual way. Each player's move consists of either incrementing or decrementing one coordinate of his current position; when a player increments (respectively, decrements) his ith coordinate, we call this *moving forward* (respectively, *moving backward*) in dimension i. We consider several cases, depending on the congruence class of k modulo 3.

Case 1: $k = 3\ell$ for some integer ℓ. In this case, $m = 4\ell + 2$. Out of the $4\ell + 2$ cops, $2\ell + 1$ begin the game on vertex $(0, 0, \dots, 0)$, while the other $2\ell + 1$ begin on $(1, 0, \dots, 0)$. No matter where the robber starts, exactly $2\ell + 1$ cops begin in the same partite set as the robber, while $2\ell + 1$ begin in the other partite set. Let S be the set of cops that begin in the same partite set as the robber, and partition 2ℓ of the remaining cops into sets $T_1, T_2, \dots, T_\ell$, each of size 2. (The final remaining cop is not needed and may move arbitrarily.)

The cops now employ the following strategy. The cops in S initially aim to make their positions agree with the robber's in dimensions $2\ell + 1, 2\ell + 2, \dots, 3\ell$: they do this greedily, always moving closer to the robber in one of these dimensions. After the cops have achieved this goal, they employ a different strategy. Whenever the robber moves in dimensions $2\ell + 1, 2\ell + 2, \dots, 3\ell$, the cops in S mirror this action, moving in the same dimension and in the same direction. When the robber instead moves in one of the first 2ℓ dimensions, the cops in S take one more step in a winning strategy in the projection of the game onto the first 2ℓ dimensions. (The existence of such a strategy is guaranteed by Lemma 4.5.) The cops in each T_i employ a similar strategy. First, they greedily attempt to catch up to the robber in all dimensions other than $2\ell + i$; once they have done so, whenever the robber moves in dimension $2\ell+i$ the cops in T_i take one more step in a winning strategy in the projection of the game onto dimension $2\ell + i$, and whenever the robber moves in any other dimension the cops move in the same dimension and the same direction.

Within the first $n_{2\ell+1} + n_{2\ell+2} + \ldots + n_{3\ell}$ times the robber moves in one of the first 2ℓ dimensions, the cops in S catch up to him in dimensions $2\ell + 1, 2\ell + 2, \ldots, 3\ell$; after finitely many more robber moves in one of the first 2ℓ dimensions, some cop in S captures the robber. Thus, the robber can move in one of the first 2ℓ dimensions only finitely many times without being captured. Likewise, for $1 \leq i \leq \ell$, the robber can move only finitely many times in dimension $2\ell + i$ before he is captured by some cop in T_i. In other words, whenever the robber moves in dimensions $1, 2, \ldots, 2\ell$, the cops in S get closer to capturing him, and whenever the robber moves in dimension $2\ell+i$, the cops in T_i get closer to capturing him. Eventually, some cop must capture the robber.

Case 2: $k = 3\ell + 1$ for some integer ℓ. We now have $m = 4\ell + 3$. We proceed similarly to the previous case, starting $2\ell + 1$ cops on $(0, 0, \ldots, 0)$ and the other $2\ell + 2$ on $(1, 0, \ldots, 0)$. At least $2\ell + 1$ cops must begin in the same partite set as the robber; let these cops comprise the set S, and partition the remaining $2\ell + 2$ cops into pairs $T_1, T_2, \ldots, T_{\ell+1}$. As in Case 1, the cops in S first catch up to the robber in dimensions $2\ell + 1, 2\ell + 2, \ldots, 3\ell + 1$, then attempt to employ a winning strategy in the projection of the game onto the first 2ℓ dimensions; the cops in T_i first catch up to the robber in all dimensions other than $T_{2\ell+i}$, then attempt to employ a winning strategy in the projection of the game onto dimension $2\ell + i$. As before, some cop eventually captures the robber.

Case 3: $k = 3\ell+2$ for some integer ℓ. This time, $m = 4\ell+4$. Now $2\ell+2$ cops begin on $(0, 0, \ldots, 0)$, while the other $2\ell+2$ begin on $(1, 0, \ldots, 0)$. Exactly $2\ell+2$ cops must begin in the same partite set as the robber; let these cops comprise the set S, and partition the remaining $2\ell + 2$ cops into pairs $T_1, T_2, \ldots, T_{\ell+1}$. As in the previous cases, the cops in S first catch up to the robber in dimensions $2\ell+2, 2\ell+3, \ldots, 3\ell+2$, then attempt to employ a winning strategy in the projection of the game onto the first $2\ell + 1$ dimensions; the cops in T_i first catch up to the robber in all dimensions other than $T_{2\ell+1+i}$, then attempt to employ a winning strategy in the projection of the game onto dimension $2\ell + 1 + i$. Once again, some cop eventually captures the robber.

In any case, m cops suffice to capture a robber on G. $\square$

5 Open problems

Several natural questions on the fully active game remain open. In Proposition 2.3, is was shown that it is possible for a graph G to satisfy $\mathrm{c_a}(G) < \mathrm{c}(G)$. However, the construction given is only for the case when $\mathrm{c_a}(G) = 1$.

Question 1 *Is it true that for any positive integer k, there exists a graph G such that $k < \mathrm{c_a}(G)$ and $\mathrm{c_a}(G) < \mathrm{c}(G)$?*

Trivially, $\mathrm{c_a}(G) \leq 6$ for every planar graph G, following from Aigner and Fromme's proof [1] that planar graphs have cop number at most 3 and from Theorem 3.1.

Question 2 *What is the smallest constant c such that $\mathrm{c_a}(G) \leq c$ for every planar graph G?*

In light of Theorems 4.1, 4.8, and 4.9, it seems that the game works very differently on bipartite and non-bipartite graphs. We can trivially construct examples of non-bipartite graphs with $\mathrm{c_a}(G) > \mathrm{c}(G)$ by taking a bipartite graph as constructed in Theorem 3.3 and pasting a triangle onto a single vertex. However, we do not know of any "non-trivial" constructions of non-bipartite graphs which satisfy this inequality, nor do we know of bipartite graphs for which $\mathrm{c_a}(G) < \mathrm{c}(G)$.

Question 3 *Is $\mathrm{c_a}(G) \geq \mathrm{c}(G)$ for every bipartite graph G?*

Question 4 *Is $\mathrm{c_a}(G) \leq \mathrm{c}(G)$ for every sufficiently connected non-bipartite graph G?*

As demonstrated by Cartesian products of even cycles (Theorem 4.9) and blowups of Cartesian products of trees (Theorem 3.3), we know that the active cop number of a graph can be roughly 4/3 or 2 times the usual cop number, but we have no construction that forces the parameters to differ by a prescribed additive constant.

Question 5 *Let c and k be positive integers, with $k < c$. Does there necessarily exist a graph G with $\mathrm{c}(G) = c$ and $\mathrm{c_a}(G) = c + k$?*

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References

[1] M. Aigner and M. Fromme, A game of cops and robbers, *Discrete Appl. Math.* 8 (1) (1984), 1–12.

[2] D. Bal, A. Bonato, W. B. Kinnersley and P. Prałat, Lazy cops and robbers on hypercubes, *Combin. Probab. Comput.* 24 (6) (2015), 829–837.

[3] D. Bal, A. Bonato, W. B. Kinnersley and P. Prałat, Lazy cops and robbers played on random graphs and graphs on surfaces, *J. Comb.* 7 (2016), 627–642.

[4] A. Bonato and R. Nowakowski, The game of cops and robbers on graphs, *Student Math. Library* 61, American Mathematical Society, Providence, RI, 2011.

[5] G. R. Brightwell and P. Winkler, Gibbs Measures and Dismantlable Graphs, *J. Combin. Theory Ser. B* 78 (1) (2000), 141–166.

[6] N. E. Clarke, Constrained Cops and Robber, *PhD Thesis*, Dalhousie University, 2002.

[7] Z. Gao and B. Yang, The cop number of the one-cop-moves game on planar graphs, *Combinatorial optimization and applications (Part II)*, 199–213, Lec. Notes in Comp. Sci. 10628, Springer, Cham, 2017.

[8] M. Maamoun and H. Meyniel, On a game of policemen and robber, *Discrete Appl. Math.* 17 (3) (1987), 307–309.

[9] S. Neufeld and R. Nowakowski, A game of cops and robbers played on products of graphs, *Discrete Math.* 186 (1-3) (1998), 253–268.

[10] R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math.* 43 (2) (1983), 235–239.

[11] D. Offner and K. Ojakian, Variations of Cops and Robber on the hypercube, *Australas. J. Combin.* 59 (2) (2014), 229–250.

[12] A. Quilliot, Jeux et Points Fixes sur les graphes, *Thèse de 3ème cycle*, Université de Paris VI, 1978.

[13] A. Quilliot, Problèmes de jeux, de point fixe, de connectivité et d representation sur des graphes, des ensembles ordonnés et des hypergraphe. *Thèse d'Etat*, Université de Paris VI, 1983.

[14] B. S. Schröder, The copnumber for lexicographic products and sums of graphs, *Contrib. Discrete Math.* 9 (2) (2014), 45–49.

[15] B. W. Sullivan, N. Townsend and M. L. Werzanski, An introduction to lazy cops and robbers on graphs, *College Math. J.* 48 (2017), 322–333.

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