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Combinatorics of JENGA

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Abstract

JENGA, a very popular game of physical skill, when played by perfect players, can be seen as a pure combinatorial ruleset. Taking that into account, it is possible to play with more than one tower; a move is made by choosing one of the towers, removing a block from there, that is, a disjunctive sum. JENGA is an impartial combinatorial ruleset, i.e., Left options and Right options are the same for any position and all its followers. In this paper, we illustrate how to determine the Grundy value of a JENGA tower by showing that it may be seen as a bidimensional vector addition game. Also, we propose a class of impartial rulesets, the CLOCK NIM games, JENGA being an example of that class.

1 Introduction

JENGA is a game of physical skill designed by Leslie Scott circa 1973 and marketed in 1982 in London ([14, p. 6]). JENGA became a huge success with more than 80 millions of units sold. JENGA means, in Swahili, “to build”. The game is based on

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Takoradi Bricks that, according to a letter wrote by Scott in 1983, was based on "*a stacking game that children often play with dominoes*" [11].

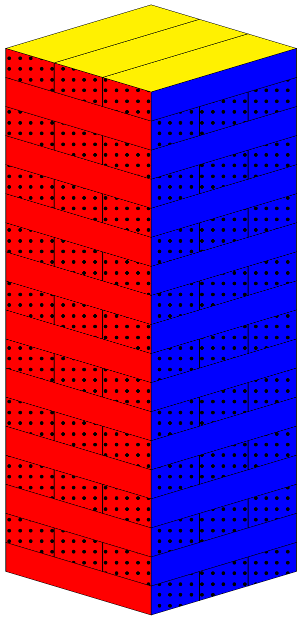


Figure 1: A JENGA tower.

The rules of JENGA are very simple. The game material consists of $3n$ equal blocks (usually of wood) each with a length of thrice its width, and one fifth as thick as its length. The block dimensions are not exact, some random noise is added to provide an extra challenge.

These blocks, initially, form a tower of n levels (n is chosen by the players), where each level consists of three blocks on a right angle with the blocks of the next level. In the commercial version, $n = 18$, that is, 54 blocks, and 18 levels. The player that builds the tower plays first. Each player, on his or her turn (for brevity, the pronoun "he" will be used), must remove a block from any level except from the level below the last incomplete level, and then add that block to the top level until it is completed. Then a new top level begins. This process will progressively increase the height of the tower. Since the building material remains constant, the tower will become more and more unstable until it collapses. The player that places the last block before the tower collapse wins the game.

The known world record for tallest JENGA tower, according to the 2008 Vintage Game Collection edition of JENGA, has $40^2/_3$ levels.

1.1 Perfect players

Each layer may have three blocks: the middle one is the *central block*; the other two are the *lateral blocks*. The idea of "perfect players" is based on two assumptions:

1. The two lateral blocks of any level hold any tower above it;
2. The central block of any level holds any tower above it.

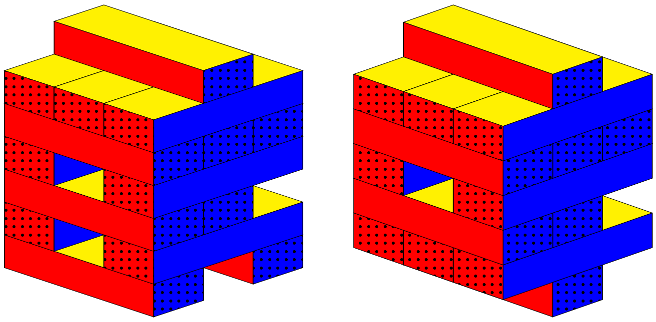


Figure 2: On the left, the two lateral blocks in the first layer hold the tower; on the right, the central block in the first layer holds the tower.

If he wishes, and if it is possible by the rules, a *perfect player* is able to remove a block from a layer, keeping in that layer the two lateral blocks or the central block, in order to prevent the tower from falling.

A *terminal layer* has only the two lateral blocks or the central block. All playable layers of a *terminal tower* are terminal layers (the top one and, eventually, the layer immediately below the top one, by the rules, are not playable).

A perfect player only runs out of moves when, on his turn, all towers are terminal towers.

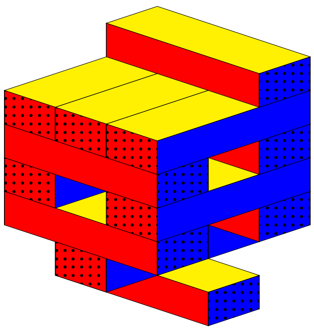


Figure 3: Terminal tower.

1.2 Combinatorial Game Theory, impartial games, and subtraction/addition games

Combinatorial Game Theory studies perfect information games in which there are no chance devices (e.g. dice) and two players take turns moving alternately. Here we are concerned with games under normal play, where the player to move with no disposable moves loses. This paper is self contained; see [1, 2, 4, 15] for background and [6] for a survey. Readers fluent in Combinatorial Game Theory may wish to proceed to Section 2.

1.2.1 Background on relevant Combinatorial Game Theory

The *options* of a game are all those positions which can be reached in one move. Using the notation of Combinatorial Game Theory, where Left and Right are the players, game forms can be expressed recursively as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ where $G^{\mathcal{L}}$ are Left options and $G^{\mathcal{R}}$ are Right options of G. Often, games decompose into components during the play. In those situations, a player has to choose a component in which to play and, so, the concept of *disjunctive* sum is formalized: $G + H = \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G+H^{\mathcal{R}}\}$. A game belongs to one of four outcome classes: $\mathcal{L}$ - $\mathcal{L}$eft wins, regardless of moving first or second; $\mathcal{R}$ - $\mathcal{R}$ight wins, regardless of moving first or second; $\mathcal{N}$ - $\mathcal{N}$ext player wins regardless of whether this is Left or Right; $\mathcal{P}$ - $\mathcal{P}$revious player wins regardless of whether this is Left or Right. When G is *impartial*, Left options and Right options are the same for the game and all its subpositions. In that case, instead of Left and Right, we simply speak about Previous and Next. Also, in that case, we only have two outcome classes, $\mathcal{P}$ and $\mathcal{N}$ and, instead of $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, we use $G = \{G'\}$ where G' is the set of options of G (the options are the same for both players).

An example of a combinatorial impartial ruleset is the classic game of NIM, first studied by C. Bouton [3]. NIM is played with piles of stones. On his turn, each player can remove any number of stones from any pile. The winner is the player who takes the last stone. The values involved in NIM are called *nimbers*:

$$*k = \{0, *, \dots, *(k-1)\}.$$

It is a surprising fact that all impartial rulesets take only nimbers as values (Sprague-Grundy Theory, see [7, 16]).

The *minimum excluded value* of a set S is the least nonnegative integer which is not included in S and is denoted $\operatorname{mex}(S)$. The *Grundy value* of an impartial game G, denoted by $\mathcal{G}(G)$, is given by

$$\mathcal{G}(G) = \operatorname{mex}\{\mathcal{G}(H) : H \text{ is an option of } G\}.$$

The value of an impartial game G is the nimber $*\mathcal{G}(G)$. The game G is a previous player win, i.e. the next player has no good move ($G \in \mathcal{P}$), if and only if $\mathcal{G}(G) = 0$.

The *nim-sum* of two nonnegative integers is the exclusive or (XOR), written $\oplus$, of their binary representations. It can also be described as adding the numbers in binary without carrying. One fundamental result about impartial games is the following: if $G = H + K$ (disjunctive sum), then $\mathcal{G}(G) = \mathcal{G}(H) \oplus \mathcal{G}(K)$ (see [1, 2, 4, 15]).

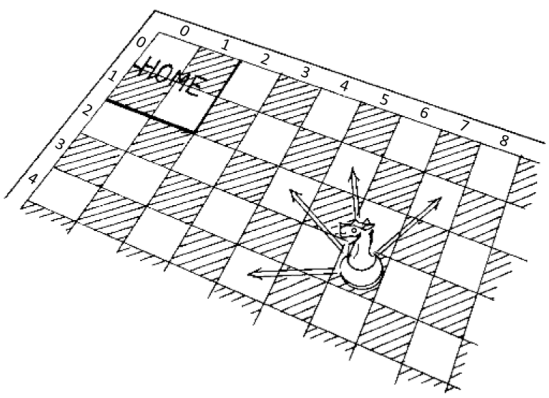
JENGA is an impartial combinatorial ruleset. Because of Sprague-Grundy Theory and its relation to disjunctive sum, it is important to know how to find the Grundy values of JENGA positions. That will be explained in this paper.

1.2.2 Subtraction/addition games

By slightly changing the rules of NIM, the class of subtraction games is defined. A subtraction game, denoted SUBTRACTION$(s_1, ..., s_k)$, is played like NIM but a player can only remove a number of stones if that number is an element of $\{s_1, ..., s_k\}$. Given a SUBTRACTION$(s_1, ..., s_k)$, if n is the size of a pile, then

$$\mathcal{G}(n) = \text{mex}\{\mathcal{G}(n - s_1), \mathcal{G}(n - s_2), ..., \mathcal{G}(n - s_k)\}.$$

It is known that Grundy sequences of subtraction games are periodic [1, 2, 4, 15]. However, the usual proof provides an huge bound for the period. An open problem, proposed by Richard Guy, is to find if the period of SUBTRACTION$(s_1, ..., s_k)$ is bounded by some polynomial of degree C_2^k [8].



0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1
0	0	*2	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1
*1	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	
*1	*1	*2	*1	*4	*3	*2	*3	*3	*3	*2	*3	*3	*3	*2	*3	*3	*3	*2	
0	0	*3	*4	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0		
0	0	*2	*3	0	0	*2	*1	0	0	*1	*1	0	0	*1	*1	0	0		
*1	*1	*2	*2	*1	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3			
*1	*1	*2	*3	*1	*1	*2	*1	*4	*3	*2	*3	*3	*3	*2	*3	*3			
0	0	*3	*3	0	0	*3	*4	0	0	*1	*1	0	0	*1	*1				
0	0	*2	*3	0	0	*2	*3	0	0	*2	*1	0	0	*1	*1				
*1	*1	*2	*2	*1	*1	*2	*2	*1	*2	*2	*2	*3	*2						
*1	*1	*2	*3	*1	*1	*2	*3	*1	*1	*2	*1								
0	0	*3	*3	0	0	*3	*3	0	0										
0	0	*2	*3	0	0	*2	*3												
*1	*1	*2	*2	*1	*1														
*1	*1	*2	*3																
0	0																		

Figure 4: Grundy values of WHITE KNIGHT (images adapted from [2]).

Regarding the n-dimensional case, we will use the standard notation for vector addition games [10], where we add vectors instead of subtracting. For instance, ADDITION$((-2, 1), (-2, -1), (-1, -2), (1, -2))$ is the well-known WHITE KNIGHT; the four vectors correspond to the legal moves of the knight (a position on the board is indicated by a pair $(column, row)$). The table of the Grundy values presents a periodic behaviour. However, contrary to what happens with the unidimensional case, it is not known if the multidimensional version presents that kind of behaviour in all cases.

In order to determine the Grundy values of JENGA positions, we will show that a JENGA tower is a well-behaved bidimensional vector addition game.

To finish this introductory section, we propose an exercise. The solution will be presented at the end of Section 3.

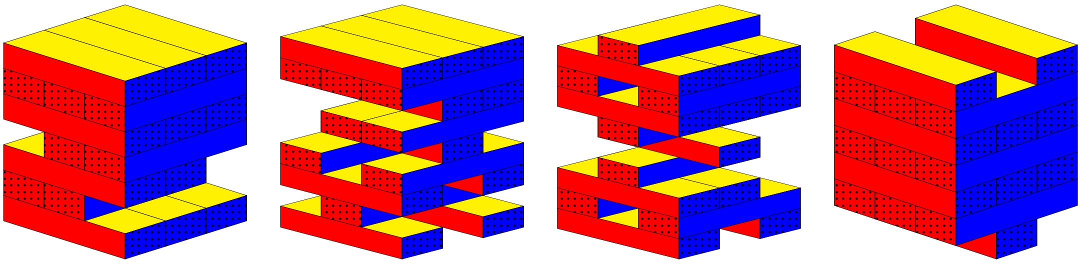


Figure 5: Find a good move in this disjunctive sum.

2 CLOCK NIM

Suppose that you play NIM with the following extra rule: *Every* 5 *moves, a new pile of* 4 *stones appears on the table.* Of course, in 5 moves, at least 5 stones are removed. Thus, in 5 moves, the total number of stones decreases at least by 1 unit. This explains why the game is finite (it always has an end). A CLOCK NIM position is defined in the following way.

Definition 2.1. Let k be the largest pile size in a disjunctive sum. Then $(a_1, a_2, \ldots, a_k|s)_{p,n}$, where $p > n > 0$, is a CLOCK NIM position such that a_i denotes the number of piles of size i, p denotes the periodicity of the arising of a new pile of size n, and $s \in \{0, \ldots, p-1\}$ denotes the state of the clock. After a move, the clock increases by one unit, except when $s = p - 1$, for which it changes from $p - 1$ to 0 and a new pile of size n arises.

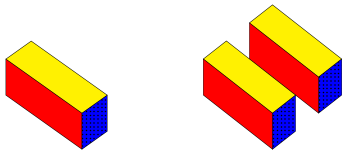
Example 2.2. In the position $(2, 4|1)_{3,2}$, there are 2 piles of size 1, and 4 piles of size 2. The possible options are $(1, 4|2)_{3,2}$ (removal of a pile of size 1), $(3, 3|2)_{3,2}$ (removal of 1 stone from a pile of size 2), and $(2, 3|2)_{3,2}$ (removal of a pile of size 2). In all cases, the state of the clock changes from 1 to 2 (a move is made).

Example 2.3. In the position $(2, 4|2)_{3,2}$, the possible options are $(1, 5|0)_{3,2}$ (removal of a pile of size 1), $(3, 4|0)_{3,2}$ (removal of 1 stone from a pile of size 2), and $(2, 4|0)_{3,2}$ (removal of a pile of size 2). In all cases, the state of the clock changes from 2 to 0, and a new pile of size 2 arises.

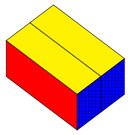
2.1 JENGA towers as CLOCK NIM positions

There are five possible configurations for a JENGA layer.

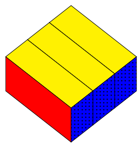
– Two configurations behave like piles of size 0 (terminal layers).



– Two symmetrical configurations behave like piles of size 1 because the only possible option leads to a terminal layer.



– One configuration behaves like a pile of size 2 because one of the options leads to a terminal layer (removal of the central block), and the other options lead to a configuration that behaves like a pile of size 1 (removals of a lateral block).



Naturally, a JENGA tower may be seen as a CLOCK NIM position $(a_1, a_2|s)_{3,2}$; a_1 is the number of layers that behave like piles of size 1, and a_2 is the number of layers that behave like piles of size 2 (for ease, if there are no layers like that, we will write $a_2 = 0$ anyway). Every three moves, the topmost layer is completed and, because of that, the layer immediately below, behaving like a pile of size 2, becomes available. Therefore, the state of the clock is given by the number of blocks above the topmost complete layer (a layer with three blocks). If there is no complete layer, the state of the clock is given by the number of blocks on the ground.

In the exercise of Figure 5, from the left to the right, we have the CLOCK NIM positions $(3, 3|0)_{3,2}$, $(2, 1|0)_{3,2}$, $(1, 1|1)_{3,2}$, and $(0, 4|2)_{3,2}$.

2.2 CLOCK NIM positions of the type $(a_1, a_2|s)_{3,2}$

In the previous subsection, we have seen that JENGA towers behave like CLOCK NIM positions of the type $(a_1, a_2|s)_{3,2}$; the layers behave like NIM piles of sizes 0,1 or 2, and the top exhibits the state of the clock.

In general, the determination of the Grundy values of CLOCK NIM positions seems difficult. However, $(a_1, a_2|s)_{3,2}$ is a particular case with only two possible sizes larger than zero and, because of that, it is worth to look at the Grundy values table.

	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8	9	9	9	10	10	10	11	11	11	12	12	12	13	13
0	0	0	2	1	1	0	0	2	1	0	0	2	1	1	0	0	2	1	0	0	2	1	1	0	0	2	1	0	0	2	1	1	0	0	2	1	0	0	2	1
1	1	0	0	2	1	1	3	2	1	1	0	0	2	1	1	3	2	1	1	0	0	2	1	1	3	2	1	1	0	0	2	1	1	3	2	1	1	0	0	2
2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2
3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1
4	1	0	0	2	1	1	2	2	1	1	0	0	2	1	1	2	2	1	1	0	0	2	1	1	2	2	1	1	0	0	2	1	1	2	2	1	1	0	0	2
5	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2
6	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1
7	1	0	0	2	1	1	2	2	1	1	0	0	2	1	1	2	2	1	1	0	0	2	1	1	2	2	1	1	0	0	2	1	1	2	2	1	1	0	0	2
8	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2	0	0	3	2	2	1	1	3	2
9	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1	1	0	0	3	3	0	0	3	1

Table 1: The rows are numbered with the number of piles of size 1 (labels are blue); for instance, the cells of the fifth row (labeled with the number 4) correspond to CLOCK NIM positions with four piles of size 1. The columns are numbered with the number of piles of size 2; for instance, the cells of the fourth column (labeled with the number 1) correspond to CLOCK NIM positions with one pile of size 2. The repetitions of the labels of the columns happen due to the possible states of the clock – green corresponds to $s = 0$, yellow corresponds to $s = 1$, and red corresponds to $s = 2$; for instance, the cells of the fourth column correspond to CLOCK NIM positions with one pile of size 2, and $s = 1$. The cell in the fifth row and fourth column corresponds to the CLOCK NIM position $(4, 1|1)_{3,2}$.

Observation 2.4. The CLOCK NIM position $(a_1, a_2|s)_{3,2}$ corresponds to the cell $(3a_2 + s - 1, a_1)$ of Table 1.

Observation 2.5. The green column labeled with 0 was removed from Table 1. The reason for that is the fact that a position $(a_1, 0|0)_{3,2}$ cannot be reached from any CLOCK NIM position, that is, a position $(a_1, 0|0)_{3,2}$ is a Garden of Eden configuration.

It is also possible to understand how to make the three types of move on Table 1.

1. The removal of a pile of size 2 corresponds to the move $(a, b) \to (a - 2, b)$, described by the vector $(-2, 0)$. That happens because, when the state of the clock is different than 2, the number of piles of size 2 decreases by one, and the number of piles of size 1 does not change. When the state of the clock is 2, the numbers of piles of size 2 and size 1 do not change.

2. The removal of a stone from a pile of size 2 corresponds to the move $(a, b) \to (a - 2, b + 1)$, described by the vector $(-2, 1)$. That happens because, when the state of the clock is different than 2, the number of piles of size 2 decreases by 1, and the number of piles of size 1 increases by one. When the state of the clock is 2, the number of piles of size 2 does not change, and the number of piles of size 1 increases by one.

3. The removal of a pile of size 1 corresponds to the move $(a, b) \to (a + 1, b - 1)$, described by the vector $(1, -1)$. That happens because, when the state of the clock is different than 2, the number of piles of size 2 does not change, and the number of piles of size 1 decreases by one. When the state of the clock is 2, the number of piles of size 2 increases by one, and the number of piles of size 1 decreases by one.

Table 1 was organized in order to respect all cases.

As an example, the CLOCK NIM position $(6,3|0)_{3,2}$ corresponds to the cell $(8,6)$ of Table 1. From $(6,3|0)_{3,2}$, a player can move to $(6,2|1)_{3,2}$, $(7,2|1)_{3,2}$ or $(5,3|1)_{3,2}$, corresponding to the cells $(6,6)$, $(6,7)$ or $(9,5)$ (see the three vectors on the left). Due to the fact that $\mathcal{G}(6,6) = 0$, $\mathcal{G}(6,7) = 2$, and $\mathcal{G}(9,5) = 1$, it is possible to determine $\mathcal{G}(8,6) = \text{mex}\{0,1,2\} = 3$. Another example, the CLOCK NIM position $(3,4|2)_{3,2}$ corresponds to the cell $(13,3)$ of Table 1. From $(3,4|2)_{3,2}$, a player can move to $(3,4|0)_{3,2}$, $(4,4|2)_{3,2}$ or $(2,5|2)_{3,2}$, corresponding to the cells $(11,3)$, $(11,4)$ or $(14,2)$ (see the three vectors on the right). Due to the fact that $\mathcal{G}(11,3) = 3$, $\mathcal{G}(11,4) = 0$, and $\mathcal{G}(14,2) = 0$, it is possible to determine $\mathcal{G}(13,3) = \text{mex}\{0,3\} = 1$.

With this interpretation, a JENGA tower reveals itself a position of the ADDITION $((-2,0),(-2,1),(1,-1))$, the subject of the next section.

3 ADDITION$((-2,0),(-2,1),(1,-1))$

The table of the Grundy values of the ADDITION$((-2,0),(-2,1),(1,-1))$ shows a periodic configuration. In the following theorem, all reddish parallelograms have the same pattern, all green triangles have the same pattern, and all blue triangles have the same pattern. Therefore, in each row or column, the period is 9. The first two lines are a kind of preperiod (very usual in unidimensional subtraction games).

It is easier to present the table than a closed formula. Suppose that we want to know the Grundy value $\mathcal{G}(47,234)$. In order to avoid the problem of the preperiod (first two lines), we consider $234 - 2 = 232$. Next, $232 = 7 \mod 9$ and, due to that, the first nine cells of the 234th line are 1 0 0 2 1 1 2 2 1 – corresponding to the 8th line of the motif (a square with a blue triangle and a green triangle). Thinking now about columns, $47 = 2 \mod 9$, and the wanted Grundy value corresponds to the third position of the referred line. Therefore, $\mathcal{G}(47,234) = 0$.

Theorem 3.1. *The pattern of the Grundy values of the bidimensional vector addition game stated by $S = \{(-2,0),(-2,1),(1,-1)\}$ is given by the table presented in Figure 6.*

Proof. We will use a pictorial argument to build an inductive proof.
First, filling the table diagonally, we obtain an initial set of Grundy values, as shown in Figure 7. That set of values is the base case for the induction.

Second, for the next diagonals (inductive step), we make three observations:

a) A first group of Grundy values, until the last entry side by side with the first triangle, is obtained exactly in the same way as the Grundy values obtained nine diagonals before (Figure 8).

b) A second group of Grundy values, until the entry below the penultimate triangle, is obtained exactly in the same way as the Grundy values of the same diagonal obtained nine rows above (Figure 9).

c) A third group of Grundy values, until the end of the diagonal, is obtained exactly in the same way as the Grundy values obtained nine diagonals before (Figure 10).

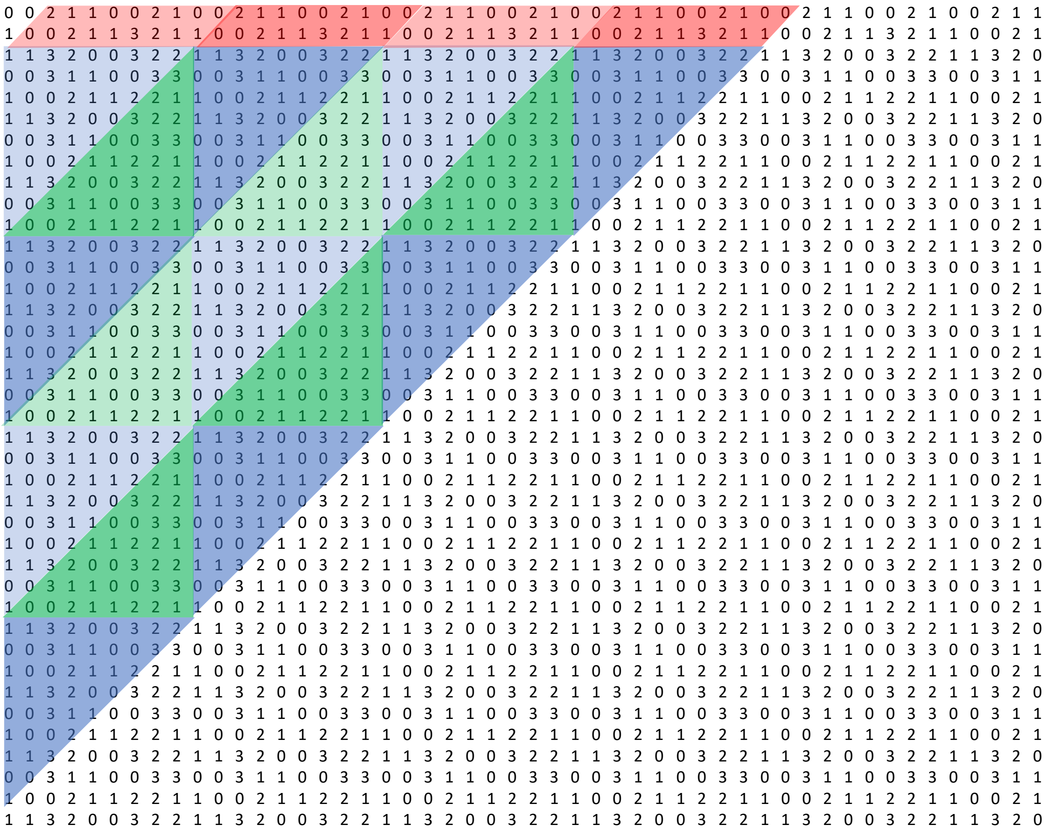


Figure 6: Pattern of ADDITION$((-2,0),(-2,1),(1,-1))$.

The next diagonals are obtained by applying the mex rule in circumstances analogous to previous situations, as explained in these three observations. That fact justifies the spread of the pattern observed in the base case. Hence, the proof is finished. □

Solution of the exercise of Figure 5: The CLOCK NIM positions $(3,3|0)_{3,2}$, $(2,1|0)_{3,2}$, $(1,1|1)_{3,2}$, and $(0,4|2)_{3,2}$, are the cells $(8,3)$, $(2,2)$, $(3,1)$, and $(13,0)$ of the ADDITION$((-2,0),(-2,1),(1,-1))$. By Theorem 3.1, $\mathcal{G}(8,3)=3$, $\mathcal{G}(2,2)=3$, $\mathcal{G}(3,1)=2$, and $\mathcal{G}(13,0)=1$.

We have the disjunctive sum $*3+*3+*2+*$, and three possible winning moves: removal of the central block from a complete layer in the first tower ($*3+*3+*2+* \to *3+*2+*$); removal of the lateral block from a layer with two adjacent blocks in the third tower ($*3+*3+*2+* \to *3+*3+*+*$);removal of a lateral block from a complete layer in the third tower($*3+*3+*2+* \to *3+*3+*+*$).

Besides the solution of the proposed exercise, we observe that the "commercial tower" with 18 levels corresponds to seventeen copies of $*2$ and zero as the state of the clock. Therefore, its Grundy value is zero, that is, the commercial tower is a $\mathcal{P}$-position.

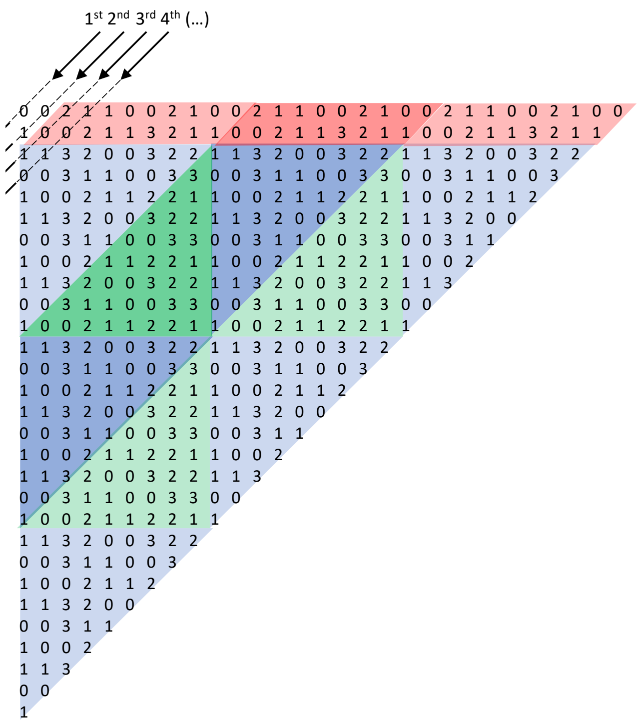


Figure 7: Base case.

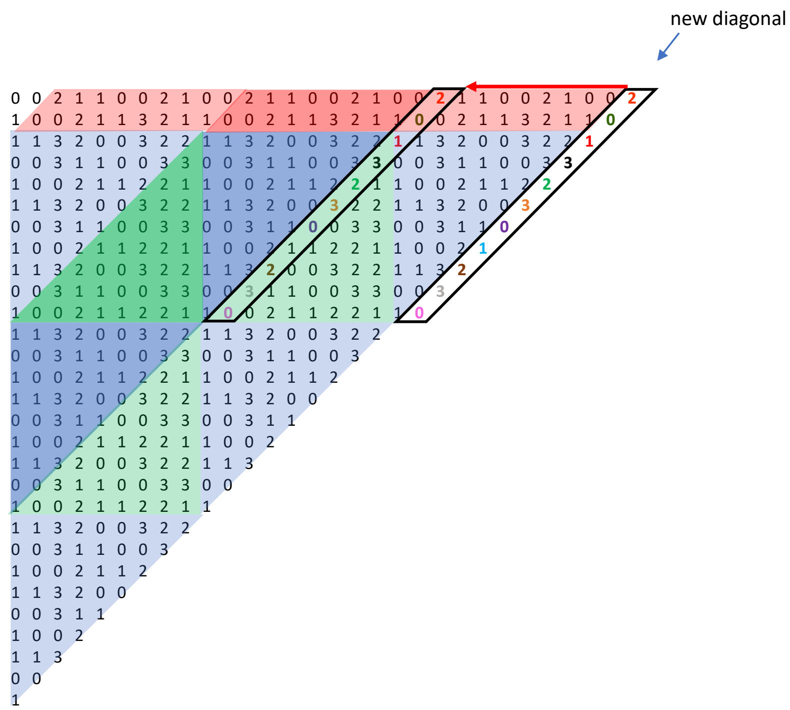


Figure 8: First group.

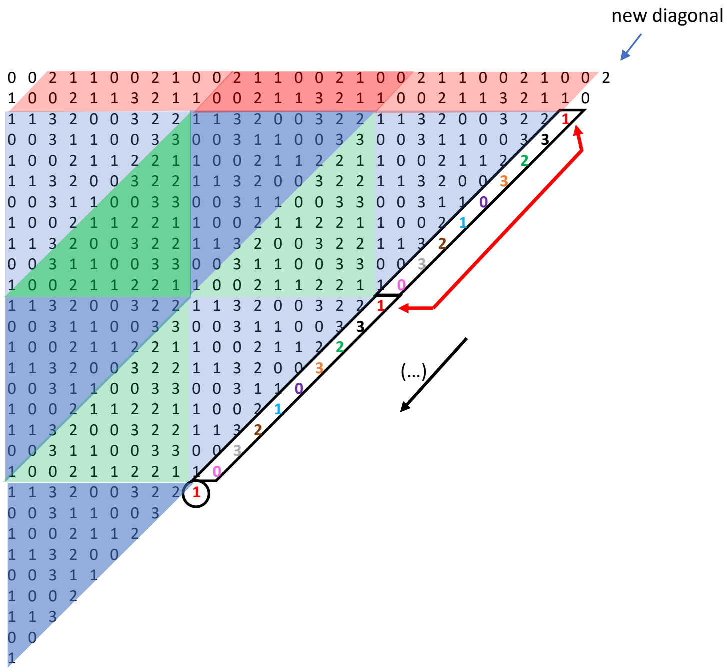


Figure 9: Second group.

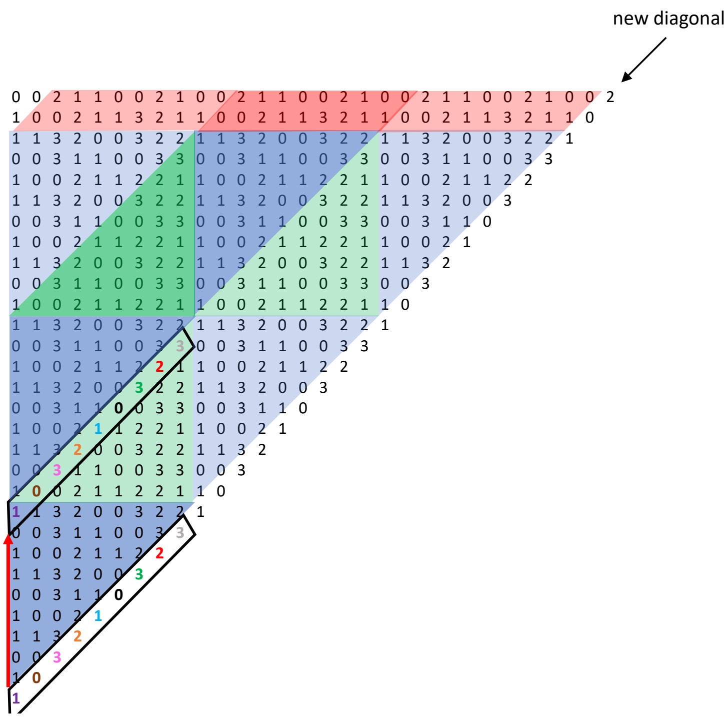
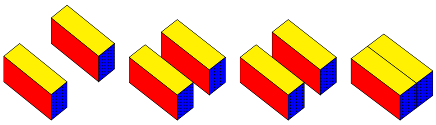


Figure 10: Third group.

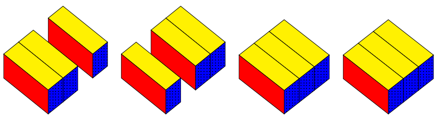
4 Final remarks

Consider the JENGA variant with four blocks in each layer, and nine possible configurations for the layers:

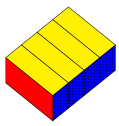
- Four configurations ("b..b", "b.b.", ".b.b", ".bb.") behave like piles of size 0 (terminal layers).



- Four configurations ("bb.b", "b.bb", ".bbb", "bbb.") behave like piles of size 1 because the only possible option leads to a terminal layer.



- One configuration behaves like the literal form $\{* \mid *\}$ because moving to a pile of size 1 is the only option that a player has.



Using the same approach made for classic JENGA, the positions can be seen as $(a_1, a_2|s)_{4,\{*\,|\,*\}}$, where a_1 denotes the number of piles of size 1, a_2 denotes the number of forms $\{* \mid *\}$, and $s \in \{0, 1, 2, 3\}$ denotes the state of the clock. When s changes from 3 to 0, a $\{* \mid *\}$ form arises. It is not a "pure" CLOCK NIM because the literal form $\{* \mid *\}$ is not a NIM pile. However, everything works fine, and the game is the ADDITION$((-3, 1), (1, -1))$. In this case, the periodicity is trivial: it is a "loves me, loves me not" game, that is, parity is all that matters. In "loves me, loves me not" situations, a player never decides wrongly, making a "bad" move; it is impossible to make mistakes. That happens because the only Grundy values are 0 and 1, all the options of 0 (if there are options) are equal to $*$, and all the options of $*$ are equal to 0.

It may be thought that bidimensional vector addition games always present periodic patterns. But it is not known if that is true. Research done in [5, 10, 12, 13] shows how difficult the problem is. For instance, the ADDITION$((2, -3), (-1, -1), (-3, 2))$ is very difficult to understand [9] (Figures 12 and 13).

0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1

Figure 11: Pattern of ADDITION$((-3,1),(1,-1))$.

0	0	0	1	0	1	0	0	0	1	0	1
0	1	1	2	0	1	0	1	1	2	0	2
0	1	0	2	1	2	0	2	0	2	0	1
1	2	2	2	3	3	1	3	1	2	3	3
0	0	1	3	1	2	0	3	2	0	1	2
1	1	2	3	2	0	1	1	2	3	2	0
0	0	0	1	0	1	2	0	0	1	0	1
0	1	2	3	3	1	0	1	2	3	0	2
0	1	0	1	2	2	0	2	0	1	2	1
1	2	2	2	0	3	1	3	1	2	0	3
0	0	0	3	1	2	0	0	2	0	0	2
1	2	1	3	2	0	1	2	1	3	2	1

Figure 12: ADDITION$((2,-3),(-1,-1),(-3,2))$.

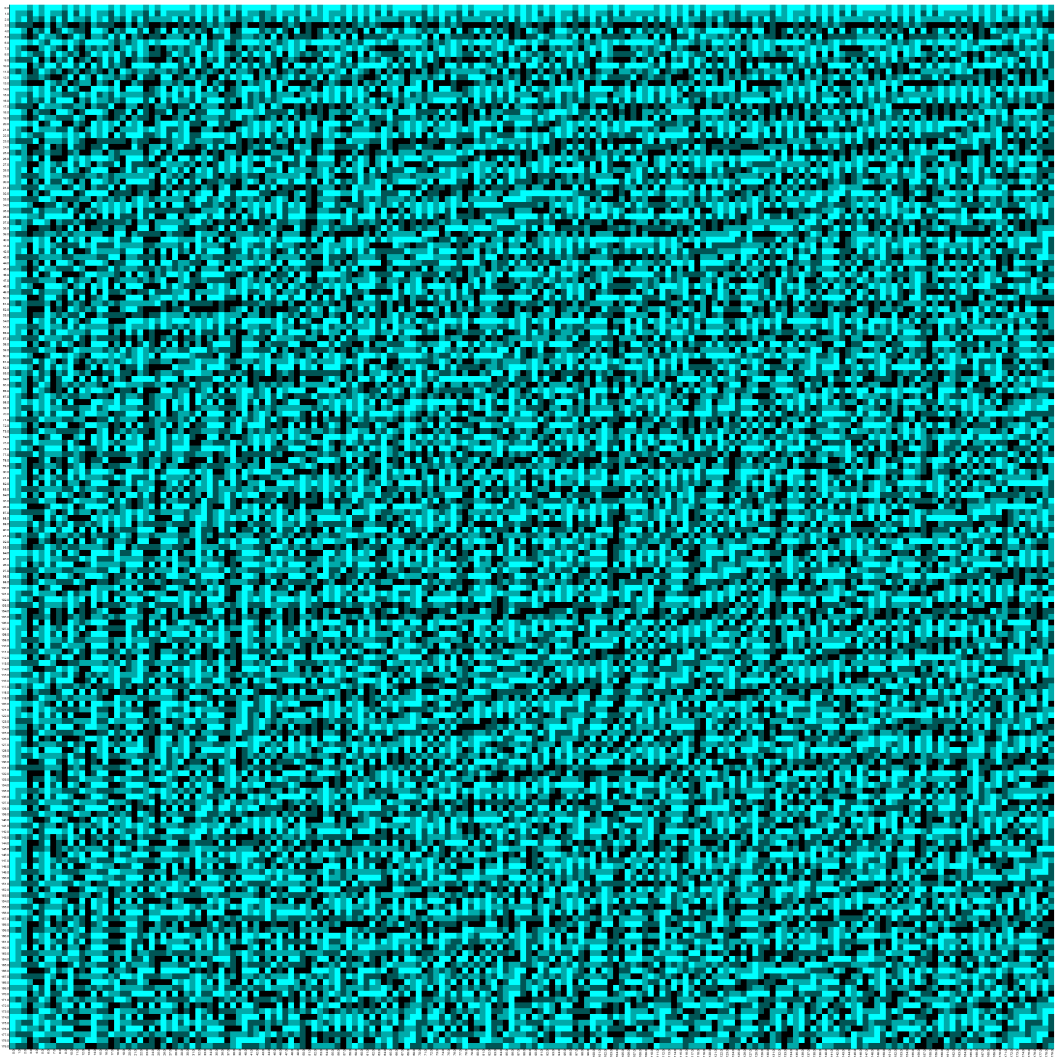


Figure 13: Is there any pattern for ADDITION$((2, -3), (-1, -1), (-3, 2))$?

What is an indisputable fact is that knowledge about bidimensional (and multi-dimensional) vector addition games is important, since there are rulesets (JENGA is an example) that behave, in a camouflaged way, like those games.

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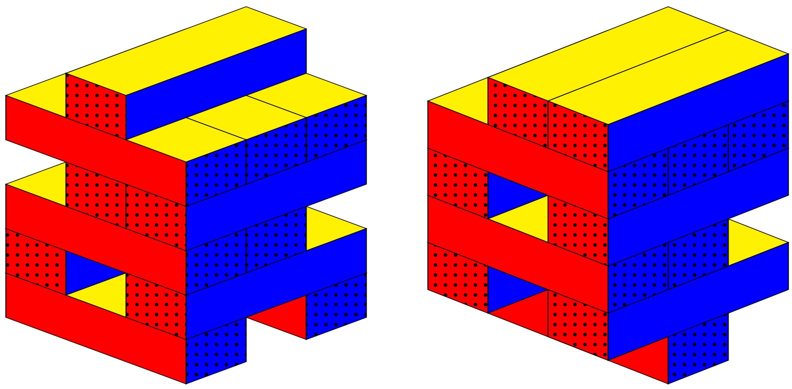
Appendices

Drawing Jenga diagrams

The paper's Jenga diagrams were created using a LaTeX macro available at `https://github.com/jpneto/jengaCGT`. Typing

```
\jengaTower{{1,0,1},{1,0,1},{0,1,1},{0,1,1},{1,1,1},{0,1,0}}
\jengaTower{{0,1,0},{1,0,1},{0,1,1},{1,0,1},{1,1,1},{0,1,1}}
```

will produce:



CGSUITE program for CLOCK NIM (v0.7)

CGSUITE is a computer program, coded by A. Siegel, for evaluating combinatorial games (`http://cgsuite.sourceforge.net`). Next, the code of a procedure for CLOCK NIM.

```
Clock := proc (pos)
local i,l,r,t,aux,len;
option remember;
l := [];
r := [];
len := Length(pos);
for i from 4 to len do
      if (pos[i]>0) then
         aux:=Clone(pos);
         aux[i]:= aux[i]-1;
         if (pos[3]==pos[2]-1) then
            aux[3]:=0;
            aux[pos[1]+3]:=aux[pos[1]+3]+1;
         fi;
         if (pos[3]<pos[2]-1) then
            aux[3]:=aux[3]+1;
```

```
        fi;
        Add(l,Clock(aux));
        Add(r,Clock(aux));
        for t from 4 to i-1 do
              aux:=Clone(pos);
              aux[i]:= aux[i]-1;
              aux[t]:=aux[t]+1;
              if (pos[3]==pos[2]-1) then
                  aux[3]:=0;
                  aux[pos[1]+3]:=aux[pos[1]+3]+1;
              fi;
              if (pos[3]<pos[2]-1) then
                 aux[3]:=aux[3]+1;
              fi;
              Add(l,Clock(aux));
              Add(r,Clock(aux));
        od;
     fi;
od;
return {l | r};
end;
```

Syntax: CLOCK$[\mathrm{n}, \mathrm{p}, \mathrm{s}, \mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_\mathrm{k}]$

- n is the size of the pile that arises every p moves (0<n<p)
- s is the state of the clock (0<s<p-1)
- a_i is the number of piles of size i
- k is the largest pile size
  it should be larger or equal than n (a_k=0 if necessary)

Examples:

CLOCK$[2, 3, 0, 3, 3]$ gives the game value of the position $(3, 3|0)_{3,2}$ (it is $*3$);

CLOCK$[2, 3, 0, 3, 3]$+CLOCK$[2, 3, 0, 2, 1]$+CLOCK$[2, 3, 1, 1, 1]$+CLOCK$[2, 3, 2, 0, 4]$ gives the result of the disjunctive sum exhibited in the exercise of Figure 5 (it is $*3$);

CLOCK$[4, 5, 3, 6, 3, 0, 0]$ gives the game value of the position $(6, 3, 0, 0|3)_{5,4}$ (zeros until the seventh position are needed (it is $*2$).

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