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Subgraph-avoiding minimum decycling sets and k-conversion sets in graphs*

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Abstract

A decycling set of a graph G is a set S of vertices such that $G[V - S]$ is acyclic. In this paper we prove that every non-complete graph G of maximum degree $r \geq 3$ has a minimum decycling set S such that $G[S]$ does not contain an $(r-2)$-regular graph as a subgraph. This generalizes a result in [P.A. Catlin and H.-J. Lai, *Discrete Math.* 141(1) (1995), 37–46]. We give several consequences of our main result, including new proofs of known results.

1 Introduction

In a graph G, a set S of vertices is a *decycling set*, or *feedback vertex set*, if and only if $G[V - S]$ is acylic. Early research on these sets was motivated by applications

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in logic networks and circuit theory, first in digraphs [5, 14] and later in undirected graphs [9]. More modern applications are given in [10].

For a given graph G, the *decycling number* of G, denoted by $\phi(G)$, is defined to be the size of a minimum decycling set of G. Clearly, finding a minimum decycling set of G is equivalent to finding a maximum induced forest. The order of such a forest is called the *forest number* of G, and denoted by $a(G)$. Many authors have derived bounds on $\phi(G)$ and $a(G)$, both for general graphs [1] and for special classes of graphs, including planar graphs [7, 8, 12], cubic graphs [2, 11, 15, 19, 21, 22, 25] and other regular graphs [18, 20]. For r-regular graphs, Dreyer and Roberts [6] have shown that the decycling sets coincide with the so-called irreversible $(r-1)$-threshold conversion sets. Given a graph G and a subset S_0 of its vertices, an *irreversible k-threshold conversion process* on G is an iterative process wherein, for each $t = 1, 2, \ldots,$ S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbours in S_{t-1}. We call the set S_0 the *seed set* of the process, and we call it an *irreversible k-threshold conversion set* of G if $S_t = V(G)$ for some $t \geq 0$. Figure 1 illustrates a 2-conversion process in a cubic graph. A detailed survey of results on irreversible k-threshold conversion processes, including results on decycling sets in regular graphs, can be found in [24]. We use the equivalence between decycling sets and irreversible $(r-1)$-threshold conversion sets of r-regular graphs in several proofs.

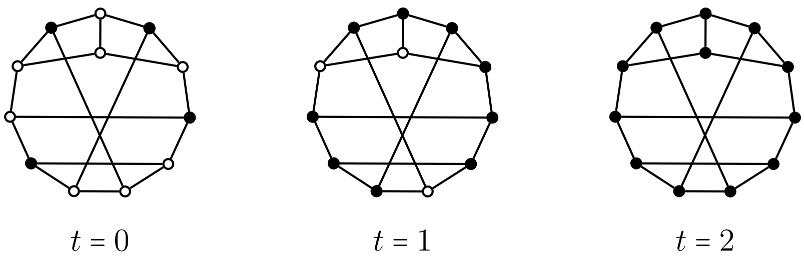

Figure 1: The evolution of a 2-conversion process in a cubic graph, with vertices of S_t shown in black

In this paper we consider structures that can be avoided in minimum decycling sets of graphs, and in minimum k-conversion sets of regular graphs, generalizing Theorem A, a result of Catlin and Lai.

Theorem A. [3, Lemma 2] *Every graph $G \neq K_{r+1}$ with $\Delta(G) = r \geq 3$ has a minimum decycling set S such that $G[S]$ does not contain K_{r-1} as a subgraph.*

Corollary B. [3, Theorem 2] *Every graph $G \neq K_4$ with $\Delta(G) = 3$ has an independent minimum decycling set.*

A *linear forest* is a graph in which every component is a path.

Corollary C. [3, Corollary 1 and Lemma 3] *Every graph $G \neq K_5$ with $\Delta(G) = 4$ has a minimum decycling set S such that $G[S]$ is a linear forest.*

Our main result, which we prove in Section 4, is the following generalization of Theorem A.

Theorem 1.1. *Every graph $G \neq K_{r+1}$ with $\Delta(G) = r \geq 3$ has a minimum decycling set S such that $G[S]$ does not contain an $(r-2)$-regular graph as a subgraph.*

It is easy to see that every vertex v in a minimum decycling set S of G is adjacent to at least two vertices of $G - S$ (and thus has degree at most $\Delta(G) - 2$ in $G[S]$); otherwise $S - \{v\}$ would be a smaller decyling set of G. We note that putting $r = 3$ into Theorem 1.1 gives exactly Corollary B, and using the above observation, the case $r = 4$ gives exactly Corollary C. Furthermore, since every graph of maximum degree r is an induced subgraph of an r-regular graph, we only need to prove Theorem 1.2, below, which we do in Section 3. Theorem 1.1 then follows by embedding a graph G with $\Delta(G) = r$ in a carefully chosen r-regular graph.

Theorem 1.2. *Let $r \geq 3$. Every r-regular graph $G \neq K_{r+1}$ has a minimum decycling set S such that $G[S]$ does not contain an $(r-2)$-regular graph as a subgraph.*

In Section 2 we show that Corollary B does not hold for graphs with maximum degree exceeding 3, and that Corollary C does not hold for graph with maximum degree exceeding 4. We consider several re-interpretations of Corollary B in search of a stronger statement that can be generalized. We show that each re-interpretation that would lead to a stronger generalization than Theorem 1.1 does not hold for larger r, establishing that Theorem 1.1 is the strongest possible natural generalization of Corollaries B and C. The proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 3, respectively. In Section 5 we present some consequences of Theorem 1.1, including new proofs of known results.

2 Possible generalizations of Corollary B

In general, we cannot guarantee the existence of an independent minimum decycling set. In Proposition 2.1 we show that for every $r \geq 4$, there exists an arbitrarily large r-regular graph with no independent decycling set of any size. Proposition 2.2 gives an additional family of counterexamples for the case where r is even.

Proposition 2.1. *Let $r \geq 4$, and let G be an r-regular graph of order n. Let H be the r-regular graph obtained by replacing each vertex u of G with a copy $K_r(u)$ of K_r, joining $K_r(u)$ to $K_r(v)$ if and only if uv is an edge in G. Then every decycling set of H contains at least $(r-2)n$ edges.*

Proof. For $r \geq 4$ the decycling number of K_r is at least $r - 2 \geq 2$, so any decycling set of H must contain at least $r - 2$ vertices from each copy of K_r. □

Proposition 2.2. *Let L be the line graph of an r-regular graph G, with $r \geq 3$. Then L is a $(2r-2)$-regular graph with no independent decycling set.*

Proof. We show that any independent seed set in L fails to convert any non-seed vertices under a $(2r-3)$-conversion process. Let e be an edge of G, corresponding to vertex v in L. Since e is incident with $r-1$ other edges at each of its endpoints, the closed neighbourhood of v in L consists of two cliques of size r, joined at v (and nowhere else). Therefore, between any three neighbours of v there is at least one edge. That is, no independent set of L contains more than two neighbours of v, so v does not convert at $t=1$ from any independent seed set. Since this holds for all $v \in L$, the result follows. □

Since Corollary B does not hold for larger values of r, we seek in this section an interpretation of that result that generalizes to $r \geq 4$. We can rephrase the result of Corollary B in various ways, two of which are below.

1. Every graph G with $\Delta(G)=3$ has a minimum decycling set S such that $G[S]$ has zero edges.

2. Every graph G with $\Delta(G)=3$ has a minimum decycling set S such that $G[S]$ has maximum degree zero.

One may imagine that a graph G with $\Delta(G) \leq r$ must have a decycling set S such that $G[S]$ has at most $r-3$ edges, or perhaps has maximum degree at most $r-3$. However, both of these proposed generalizations, which would also strengthen Corollary C, are false for $r \geq 4$. We define an infinite class $\mathcal{G}_r$ of r-regular graphs below, and demonstrate in Corollary 2.4 that these graphs are counterexamples to the proposed generalizations.

Let $r \geq 4$. Consider s copies of $K_{r+1}-e$, and for $i=0,\ldots,s-1$, let x_iy_i be the missing edge in the i^{th} copy. Define $\mathcal{G}_r$ to be the family of r-regular graphs constructed from these copies by adding the edges x_iy_{i+1} for all i, where addition is performed modulo s. Figure 2 gives an example of a graph in this class for $r=4$. Proposition 2.3 implies that for every $G \in \mathcal{G}_r$ and for every minimum decycling set S of G, the maximum degree of $G[S]$ is $r-2$. It follows immediately from this result that the generalizations of Corollary B proposed above are false.

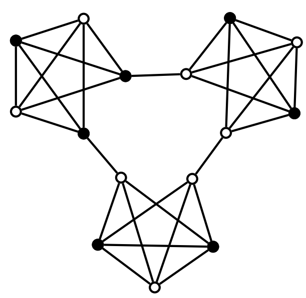

Figure 2: An example of a graph in $\mathcal{G}_4$, and a minimum decycling set, shown in black

Proposition 2.3. *Let $r \geq 4$ and let $G \in \mathcal{G}_r$. Then for every minimum decycling set S of G, one of the copies of $K_{r+1} - e$ in G contains $r - 1$ vertices of S.*

Proof. Let $G \in \mathcal{G}_r$ and let S be a minimum decycling set of G. We argue that some copy of $K_{r+1} - e$ contains $r - 1$ seed vertices. Let v be a vertex that converts at $t = 1$. If v belongs to a copy of $K_{r+1} - e$ with fewer than $r - 1$ seed vertices then $v = x_i$ or y_i for some i. Assume, without loss of generality, that $v = y_1$. Then $x_0 \in S$. Let H be the 0^{th} copy of $K_{r+1} - e$, and assume every vertex outside H has converted by some time t. Let w be a first vertex of $V(H) - S$ to convert. Then w has $r - 1$ converted neighbours (including seed vertices) at time t. If $w = y_0$ then at least $r - 2$ of these converted neighbours are in H, and by definition of w they are all seed vertices. In that case, H contains $r - 2$ seed vertices adjacent to w, as well as x_0, for a total of $r - 1$ seed vertices. On the other hand, if $w \neq y_0$ then all neighbours of w are in H, and therefore $r - 1$ of them are seed vertices. □

Corollary 2.4. *Let $G \in \mathcal{G}_r$ and let S be a minimum decycling set of G. Then $G[S]$ contains $K_{r-1} - e$ as a subgraph, hence it contains at least $r - 3$ vertices of degree $r - 2$.*

For the graphs in the class $\mathcal{G}_r$, every minimum decycling set induces a copy of $K_{r-1} - e$, and these are the only known non-complete graphs with this property. Hence, it is natural to pose the following problem.

Problem 2.5. *Characterize all r-regular graphs G for which every minimum decycling set of G induces a subgraph which contains $K_{r-1} - e$ as a subgraph.*

The proof of Proposition 2.3 shows that if $G \in \mathcal{G}_r$ and S is a minimum decycling set of G then $G[S]$ contains a copy of $K_{r-1} - e$, but not K_{r-1}. This shows that, in terms of subgraphs of K_{r-1}, Theorem A is best possible. Since the minimum decycling sets of $G \in \mathcal{G}_r$ come so close to inducing a copy of K_{r-1}, it is perhaps surprising that we can actually do better than this. In our main result, we prove that every non-complete graph of maximum degree r has a minimum decycling set that does not induce any $(r - 2)$-regular subgraphs. Since an edge is a 1-regular graph while a cycle is a 2-regular sugraph, this is a generalization of Corollaries B and C; it is the generalization we have been looking for.

3 Proof of Theorem 1.2

In the proof of Corollary B, Catlin and Lai show that for any minimum decycling set S of $G \neq K_4$, S can be modified, over a series of steps, to obtain a minimum decycling set S' that is also an independent set. In this section will state and prove a stronger version of Theorem 1.2 that generalizes this stronger version of Corollary B. First, we develop a tool that we will use to modify decycling sets of an r-regular graph G.

Suppose that $x \in S$ has exactly $r - 2$ neighbours in S, and let v be one of the two neighbours of x in $V - S$. We call the operation $(S - \{x\}) \cup \{v\}$, denoted by $x \mapsto v$, a *seed shuffle* from x to v. In the case where x belongs to a $(r-2)$-regular subgraph of S, we call $x \mapsto v$ a *restricted seed shuffle.* Lemma 3.1 guarantees that the seed shuffle operation produces a new decycling set.

Figure 3: Illustration of the shuffle $x \mapsto v$ for $r = 3$ with seed vertices shown in black

Lemma 3.1. *Let G be an r-regular graph, $r \geq 3$, with decycling set S, and suppose that $G[S]$ contains a vertex x of degree $r-2$ in $G[S]$. Let v be a neighbour of x not in S. Then $S' = S - \{x\} \cup \{v\}$ is a decycling set of G of the same size as S. In particular, if S is minimum, then so is S'.*

Proof. Since x has exactly two neighbours outside S, it has exactly one neighbour outside S'. Therefore x does not belong to a cycle in $G[V - S']$. □

We are now ready to state the stronger version of our main result, Theorem 1.2.

Theorem 1.2, Version 2. Let $r \geq 3$. If $G \neq K_{r+1}$ is an r-regular graph then, from any minimum decycling set S of G, it is possible to obtain, via a sequence of restricted seed shuffles, a new minimum decycling set with no $(r-2)$-regular subgraphs.

For $r = 3$, the result of Theorem 1.2 is implied by Corollary B, so we will focus on $r \geq 4$. We begin by noting that, for any minimum decycling set S of G, any $(r-2)$-regular subgraph of $G[S]$ is a component of $G[S]$, since $G[S]$ has maximum degree at most $r-2$. Therefore our goal for the rest of the section is to show that we can obtain a minimum decycling set S' such that $G[S']$ has no $(r-2)$-regular components.

For the rest of Section 3, let $G \neq K_{r+1}$ be an r-regular graph, where $r \geq 4$, and let S be a minimum decycling set of G. We prove the main result (Theorem 1.2) in several steps. We first show in Lemma 3.3 that any minimum decycling set S can be transformed into a minimum decycling set S' with at most one $(r-2)$-regular component, $C(S')$. We then determine a number of conditions that guarantee the unique remaining $(r-2)$-regular component $C(S')$ can be eliminated and we constrain the structure of $G[S']$ and $G[V - S']$ when S' does not satisfy those conditions.

The proof of Lemma 3.3 relies on the observation that restricted shuffling never increases the number of $(r-2)$-regular components.

Lemma 3.2. *If S is a minimum decycling set of G and S' is obtained from S by restricted shuffling, then the number of $(r-2)$-regular components of $G[S']$ is less than or equal to the number of such components of $G[S]$.*

Proof. Suppose S' is obtained from S by the restricted seed shuffle $x \mapsto v$. Since $x \notin S'$, the $(r-2)$-regular component of $G[S]$ containing x is not a component of $G[S']$. On the other hand, any $(r-2)$-regular component of $G[S']$ that is not a component of $G[S]$ contains v, so there is at most one. □

For a graph G with subgraphs H_1 and H_2, we define the *distance (in G)* between H_1 and H_2 to be

$$d_G(H_1, H_2) = \min\{d_G(x,y) \,:\, x \in V(H_1) \text{ and } y \in V(H_2)\}.$$

We say H_1 and H_2 are *adjacent* if they are disjoint and $d_G(H_1, H_2) = 1$.

Lemma 3.3. *Starting from any minimum decycling set S we can, by repeated restricted shuffling, obtain a minimum decycling set S' such that at most one component of $G[S']$ is $(r-2)$-regular.*

Proof. Among all minimum decycling sets of G that are obtainable from S by a sequence of restricted seed shuffles, restrict to those that induce the minimum number c of $(r-2)$-regular components. If $c \le 1$, there is nothing to prove, so assume $c > 1$; we will derive a contradiction. From the restricted set of minimum decycling sets of G, choose S' such that the smallest number arising as the distance (in G) between any two $(r-1)$-regular components of $G[S']$ is as small as possible. Call this distance $d \ge 2$, and let C_1 and C_2 be two $(r-2)$-regular components of $G[S']$ such that $d_G(C_1, C_2) = d$. Let x_1 and x_2 be two vertices of C_1 and C_2, respectively, such that $d_G(x_1, x_2) = d$. If $d = 2$, the restricted seed shuffle sending x_1 to a common neighbour of x_1 and x_2 makes x_2 into a vertex of degree r in the resulting seed set, contradicting the minimality of S'. Thus $d \ge 3$. Let $x_1, a, b, \ldots, x_2$ be a shortest path from x_1 to x_2. By our choice of S', the restricted seed shuffle $x_1 \mapsto a$ decreases the number of $(r-2)$-regular components, or else we end up with two that are closer than before. Therefore $c \le 1$, as desired. □

Given an arbitrary minimum decycling set S, Lemma 3.3 guarantees that it is possible to transform it into a new minimum decycling set with at most one $(r-2)$-regular component. Therefore, for the rest of the section, we may restrict our attention to the minimum decycling sets S that have exactly one $(r-2)$-regular component. We denote that unique $(r-2)$-regular component of S by $C(S)$.

Definition 3.4. *For any r-regular graph G, let $\mathcal{S}(G)$ be the set of all minimum decycling sets S of G that induce exactly one $(r-2)$-regular component, and such that no sequence of restricted seed shuffles applied to S yields a minimum decycling set S' with no $(r-2)$-regular component.*

We prove Theorem 1.2 by showing that $\mathcal{S}(G)$ is empty for all $G \ne K_{r+1}$. Through a series of lemmas, we establish a number of properties that must be satisfied by any decycling set in $\mathcal{S}(G)$. Then, in the proof of Theorem 1.2, we derive a contradiction from this collection of properties. The first property we prove about $S \in \mathcal{S}(G)$ is that $\overline{S}$ induces a linear forest; that is, a collection of paths.

Lemma 3.5. *Let S be a minimum decycling set of G with exactly one $(r-2)$-regular component, $C(S)$. If the forest $G[\overline{S}]$* is not *a linear forest then we can, by a sequence of restricted seed shuffles, get a minimum conversion set with no $(r-2)$-regular components. Equivalently, for all $S \in \mathcal{S}(G)$, $G[\overline{S}]$ is a linear forest.*

Proof. If $G[\overline{S}]$ is not a collection of paths then there is a vertex $y \in \overline{S}$ with at least three neighbours in $\overline{S}$. Therefore y has at most $r-3$ neighbours in S. We use induction on the distance from $C(S)$ to y. If $d(C(S), y) = 1$ then some vertex $x_0 \in C(S)$ is adjacent to y. The minimum decycling set S' obtained by the restricted seed shuffle $x_0 \mapsto y$ does not contain $C(S)$, and since y now has at most $r-4$ seed neighbours, the component of $G[S']$ containing y is not $(r-2)$-regular. Furthermore, components of $G[S']$ that do not contain y are components of $G[S]$ distinct from $C(S)$, so they are not $(r-2)$-regular either. Therefore S' does not induce any $(r-2)$-regular components, so $S' \notin \mathcal{S}(G)$.

For the induction hypothesis, assume the statement is true whenever S is a minimum decycling set of G that has exactly one $(r-2)$-regular component $C(S)$ and the distance from $C(S)$ to the nearest vertex of degree at least 3 in $G[\overline{S}]$ is at most $d-1$, with $d \geq 2$. Consider a minimum decycling set S for which this distance equals d. Let $x_0, x_1, \ldots, y$ be a shortest path from $C(S)$ to y, where $x_0 \in C(S)$. If the seed set S' produced by shuffling $x_0 \mapsto x_1$ does not contain an $(r-2)$-regular component, we're done. On the other hand, if it does contain an $(r-2)$-regular component $C(S')$ then $d(C(S'), y) \leq d-1$. Therefore by the induction hypothesis, it is possible to obtain a minimum decycling set S'' from S' by a sequence of restricted seed shuffles. Since the shuffle $x_0 \mapsto x_1$ was itself a restricted seed shuffle, the statement is true for S as well. □

Lemma 3.6. *Let $G \neq K_{r+1}$ be an r-regular graph. If $S \in \mathcal{S}(G)$ then for every vertex $v \in C(S)$, the nonseed neighbours of v are the leaves of a component path of $G[\overline{S}]$.*

Proof. By Lemma 3.5, $G[\overline{S}]$ is a linear forest. If some $x \in C(S)$ is adjacent to a vertex $y \in \overline{S}$ having at least two neighbours in $\overline{S}$, then the restricted shuffle $x \mapsto y$ results in a minimum decycling set S' that does not induce any $(r-2)$-regular components. Indeed, after the shuffle y is adjacent to at least three non-seed vertices, so it is not in an $(r-2)$-regular component of S' and therefore S' does not have any such components. This implies that $S \notin \mathcal{S}(G)$. Therefore, for all $S \in \mathcal{S}(G)$, the neighbours of any vertex $x \in C(S)$ are leaves of $G[\overline{S}]$. To see that the neighbours of x are leaves of the same component path of $G[\overline{S}]$, note that $\overline{S} \cup \{x\}$ contains a cycle, by the minimality of S. □

In light of Lemma 3.6, we may associate with each vertex $x \in C(S)$ a component path of $G[\overline{S}]$.

Definition 3.7. *Let $S \in \mathcal{S}(G)$. For any vertex x of $C(S)$ we denote by $P(x)$ the component path of $G[\overline{S}]$ whose leaves are neighbours of x.*

In the next lemma we constrain the structure of $C(S)$ for $S \in \mathcal{S}(G)$.

Lemma 3.8. *Let $G \neq K_{r+1}$ and let $S \in \mathcal{S}(G)$. If x and y are distinct vertices of $C(S)$ such that $P(x) = P(y)$, then x and y are adjacent.*

Proof. Let u be a leaf of $P(x) = P(y)$. If x and y are nonadjacent then the shuffle $x \mapsto u$ makes y into a seed vertex of degree $r-1$, contradicting the minimality of S. □

Thus, if $S \in \mathcal{S}(G)$, we have the following picture of the unique $(r-2)$-regular component $C(S)$ of $G[S]$. Let $P^1, P^2, \ldots, P^k$ be the distinct component paths of $G[\overline{S}]$ occurring as $P(x)$ for some $x \in C(S)$. Then by Lemma 3.8, for each i, the set of vertices $\{x \in C(S) : P(x) = P^i\}$ induces a complete subgraph of $G[S]$, say of order $n_i < r-1$. Therefore we have a covering of $C(S)$ by disjoint complete graphs K_{n_i} such that all of the vertices x in any one of the K_{n_i} have $P(x) = P^i$.

Lemma 3.9. *Let $G \neq K_{r+1}$ be an r-regular graph and let $S \in \mathcal{S}(G)$. If P is a path in $G[\overline{S}]$ such that $V_P = \{x \in C(S) : P(x) = P\}$ has at least two elements, then*

(a) V_P is a clique in S,

(b) P is an edge, and

(c) every $x \in V_P$ is a cut vertex of $C(S)$.

Proof. The first statement follows immediately from Lemma 3.8.

For the second statement, we show that if P has length at least 2 then there is a restricted seed shuffle $S \mapsto S' \notin \mathcal{S}(G)$, so $S \notin \mathcal{S}(G)$. Write $P = u_0 u_1 \ldots u_\ell$, and assume that $\ell \geq 2$. By assumption there exist two (adjacent) vertices $x, y \in C(S)$ with $P(x) = P(y) = P$. Let S' be the minimum decycling set obtained from S by performing the restricted seed shuffle $x \mapsto u_0$. By assumption, $S' \in \mathcal{S}(G)$, so $G[S']$ has an $(r-2)$-regular component $C(S')$, specifically, the component of $G[S']$ containing u_0. Denote by P' the path $u_1 \ldots u_\ell x$, which is the path associated with u_0 in $G[\overline{S'}]$. Since y and u_0 are adjacent, $y \in C(S')$, and since y is adjacent to x, the path associated with y in $G[\overline{S'}]$ is also P'. Therefore y is adjacent to the other endpoint u_1 of P'. Since $\ell \geq 2$, u_0, u_1 and u_ℓ are all distinct vertices of $\overline{S}$, and we now see that y is adjacent to all of them. This contradicts the assumption that y is in an $(r-2)$-regular component of $G[S]$.

For the third statement, again start with two vertices $x, y \in C(S)$ such that $P(x) = P(y) = P = uv$ and suppose x is not a cut vertex. We derive a contradiction by showing that, in this case, $G = K_{r+1}$. Let S' be the minimum decycling set of G obtained by the restricted seed shuffle $x \mapsto u$. By assumption $S' \in \mathcal{S}(G)$. Since x is not a cut vertex of $C(S)$, every vertex of $C(S) - \{x\}$ belongs to $C(S')$, because u is connected to y and $y \in C(S') - \{x\}$. In particular, the neighbourhood N of x in $C(S)$ belongs to $C(S')$ (including y). Therefore every vertex of N still has $r-2$ seed neighbours after the shuffle $x \mapsto u$, so they are all adjacent to u. Thus, for every $z \in N \cup \{x\}$, $P(z) = P$. Hence, by Lemma 3.8, all of the $r-2$ vertices in N are also adjacent to each other. We have shown that the $r+1$ vertices in $N \cup \{x\} \cup \{P\}$ are pairwise adjacent, so $G = K_{r+1}$. □

For $x \in C(S)$, we say x is a *sharing vertex* if $P(x) = P(y)$ for some $y \in C(S), y \neq x$, and in this case we call $P(x)$ a *shared path*.

Lemma 3.9 (2) and (3) imply that for all $S \in \mathcal{S}(G)$, all shared paths (if there are any) are merely edges and all sharing vertices (if there are any) are cut vertices of $C(S)$. For the proof of Theorem 1.2 we require $C(S)$ to have at least two non-cut vertices. This is guaranteed by Lemma 3.10.

Lemma 3.10. *Let $G \neq K_{r+1}$ and suppose $S \in \mathcal{S}(G)$. Then $C(S)$ contains at least two non-cut, non-sharing vertices.*

Proof. The leaves of any spanning tree of $C(S)$ are not cut vertices of $C(S)$. By Lemma 3.9 (3), they are not sharing vertices. □

Lemma 3.11. *Let $S \in \mathcal{S}(G)$, let x be a non-cut (hence non-sharing) vertex of $C(S)$ and let v be a leaf of $P(x)$. Let $S' \in \mathcal{S}(G)$ be the minimum decycling set of G obtained from the restricted seed shuffle $x \mapsto v$. Then $C(S') \cap C(S) = \varnothing$.*

Proof. Since x is a non-cut vertex of $C(S)$, it is also a non-sharing vertex, by Lemma 3.9 (3). Suppose for a contradiction that some vertex $y \in C(S) - \{x\}$ is in $C(S')$. Then, since x is not a cut vertex, every vertex of $C(S) - \{x\}$ is in the component $C(S')$ (that is, $C(S') = (C(S) - \{x\}) \cup \{v\}$). This implies that every neighbour u of x in $C(S)$ is adjacent to v, since they must have degree $r-2$ in $C(S')$, and therefore $P(u) = P(x)$. This makes x a sharing vertex, which is a contradiction. □

We have established a large set of properties that must be satisfied by any minimum k-conversion set $S \in \mathcal{S}(G)$. In the proof of Theorem 1.2 we will show that in fact $\mathcal{S}(G)$ is empty by showing that these properties are contradictory.

Theorem 1.2 (again). If $G \neq K_{r+1}$ is an r-regular graph, then $\mathcal{S}(G) = \varnothing$. That is, if S_0 is any minimum decycling set of G then, by a sequence of restricted seed shuffles, S_0 can be transformed into a minimum decycling set S_ℓ of G with no $(r-2)$-regular component.

Proof. Assume for a contradiction that $S_0 \in \mathcal{S}(G)$. We define a sequence $S_0 \mapsto S_1 \mapsto \cdots \mapsto S_\ell$ of minimum decycling sets of G, each of which obtained from the last by a restricted seed shuffle. By assumption, $S_i \in \mathcal{S}(G)$ for all i.

By Lemma 3.10, there is a non-cut vertex $x_0 \in C(S_0)$ (in fact there are two). Let v_1 be a leaf of $P(x_0)$ (a non-shared path) and denote by S_1 the minimum decycling set obtained by shuffling $x_0 \mapsto v_1$.

For $1 \leq i \leq \ell - 1$, the set S_{i+1} is obtained from S_i as follows.

Since $S_i \in \mathcal{S}(G)$, $G[S_i]$ contains a unique $(r-2)$-regular component, $C(S_i)$, namely the component containing the vertex v_i. Furthermore, by Lemma 3.10, $C(S_i)$

has at least two non-cut, non-sharing vertices. If v_i is not adjacent to any vertex of $C(S_j) - \{x_j\}$ for any $j < i$ (that is, if the component $C(S_i)$ does not contain any remaining vertices of a previous $(r-2)$-regular component), then let $x_i \neq v_i$ be a non-cut (and hence non-sharing) vertex of $C(S_i)$. Let v_{i+1} be an endpoint of (the nonshared path) $P(x_i)$, and shuffle $x_i \mapsto v_{i+1}$ to obtain the minimum decycling set S_{i+1}.

The process terminates when some v_ℓ is adjacent to a vertex of $C(S_j) - \{x_j\}$ for some $j < \ell$. This is illustrated in Figure 4.

By Lemma 3.11, $j \neq \ell - 1$. In fact, by ignoring $S_0, \dots, S_{j-1}$ and reindexing, we may assume that $j = 0$. That is, we may assume that v_ℓ is adjacent to some vertex of $C(S_0) - \{x_0\}$. Since x_0 is not a cut vertex of $C(S_0)$, every vertex of $C(S_0) - \{x_0\}$ is therefore in the same component of $G[S_\ell]$ as v_ℓ. By assumption (since $S_\ell \in \mathcal{S}(G)$), this component is $(r-2)$-regular, so we call it $C(S_\ell)$.

Let N denote the $(r-2)$-element set of neighbours of x_0 in $C(S_0)$. We claim that v_ℓ is adjacent to every vertex $w \in N$. First, note that the only difference between S_0 and S_ℓ is that all of the x_i's have been removed from the seed set and all of the v_i's have been added, and these are all distinct vertices. When x_0 was removed from the seed set, the number of seed-neighbours of every $w \in N$ was reduced to $r-3$. In $G[S_\ell]$, these vertices all have degree $r-2$, since they are in $(C(S_0) - \{x_0\}) \subseteq C(S_\ell)$, so each one must be adjacent to exactly one vertex from $\{v_1, \dots, v_\ell\}$. However, for $1 \leq i < \ell$, v_i is not adjacent to any vertex of $S_0 - \{x_0\}$, or the algorithm would have terminated sooner. Thus, v_ℓ is adjacent to all $r-2$ vertices of N, and therefore, by regularity, $C(S_\ell) = (C(S_0) - \{x_0\}) \cup \{v_\ell\}$.

Our next claim is that $v_\ell \neq x_0$. Indeed, x_0 is adjacent to $r-1$ vertices in S_ℓ, namely v_1 and the $r-2$ vertices of N, so by the minimality of S_ℓ, $x_0 \notin S_\ell$.

Since $x_0 \neq v_\ell$, $x_0 \notin S_\ell$. Let $y \in N$. Then performing the shuffle $y \mapsto x_0$ followed by the shuffle $v_\ell \mapsto y$ results in a decycling set S'_ℓ containing x_0, N and v_1. This contradicts the minimality of S_ℓ, since $|S'_\ell| = |S_\ell|$ but $S'_\ell - \{x_0\}$ is still a decycling set. □

4 Proof of Theorem 1.1

We now deduce Theorem 1.1 from Theorem 1.2 by carefully embedding a graph of maximum degree r in an r-regular graph.

For a vertex v of a graph G of maximum degree Δ, we define the *deficiency* of v to be $\text{def}_\Delta(v) = \Delta - \deg(v)$. We define the *deficiency* of G to be

$$\text{def}_\Delta(G) = \sum_{v \in V(G)} \text{def}_\Delta(v).$$

Clearly, if G has n vertices and m edges, then $\text{def}(G) = n\Delta - 2m$.

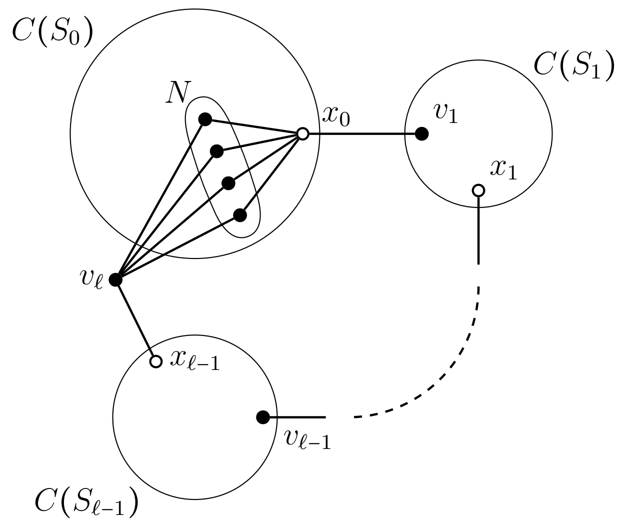

Figure 4: The outcome of the algorithm from the proof of Theorem 1.2, with vertices of S_ℓ shown in black

Theorem 1.1 (again). Every graph $G \neq K_{r+1}$ with $\Delta(G) = r \geq 3$ has a minimum decycling set S such that $G[S]$ does not contain an $(r-2)$-regular graph as a subgraph.

Proof. Let $d = \mathrm{def}_r(G)$. We may assume G is connected, hence $0 \leq \mathrm{def}_r(v) \leq r-1$ for each vertex v. We embed G in an r-regular graph H for which some minimum decycling set S_H satisfies

- $H[S_H]$ does not contain an $(r-2)$-regular subgraph, and
- $S = S_H \cap V(G)$ is a minimum decycling set of G.

We consider two cases, depending on the parity of r.

Case 1: r is odd. Let F be the graph obtained from K_{r+2} by deleting a maximum matching plus one additional edge incident with the vertex unsaturated by the matching. Then F has exactly one vertex of degree $r-1$, all other vertices being of degree r. Construct the (unique) r-regular graph H by adding d copies of F to G, joining each vertex v of G to $\mathrm{def}_r(v)$ copies of F in the obvious way (thus forming d bridges). Let S_H be any minimum decycling set of H such that $H[S_H]$ does not contain an $(r-2)$-regular subgraph. Since each cycle of H is contained either entirely in G or entirely in a copy of F, $S = S_H \cap V(G)$ is a minimum decycling set of G and has the desired property.

Case 2: r is even. Then G has an even number of vertices with odd deficiencies and d is even. For $\ell \geq 0$, let $u_1, v_1, \ldots, u_\ell, v_\ell$ be the vertices of G with odd deficiency and let $v_{\ell+1}, \ldots, v_m$ be the vertices with even positive deficiency. Then $2 \leq \mathrm{def}_r(u_i) + \mathrm{def}_r(v_i) \leq 2(r-1)$ for $i \leq \ell$, and $2 \leq \mathrm{def}_r(v_i) \leq r-2$ for $i > \ell$. Let $B_1, \ldots, B_m$ be disjoint copies of $K_{r-1,r-1}$; say B_i has partite sets A_i^1 and A_i^2. For $i = 1, \dots, \ell$, let H_i be the graph obtained from B_i by joining vertices in A_i^1 (A_i^2, respectively) by independent edges until $\sum_{x \in A_i^1} \mathrm{def}_r(x) = \mathrm{def}_r(u_i)$ ($\sum_{x \in A_i^2} \mathrm{def}_r(x) = \mathrm{def}_r(v_i)$, respectively). For $i = \ell+1, \dots, m$, let H_i be a graph obtained by joining (nonadjacent) vertices of B_i by independent edges until $\sum_{x \in V(H_i)} \mathrm{def}_r(x) = \mathrm{def}_r(v_i)$. Construct an r-regular graph H by joining u_i to vertices in A_i^1, and v_i to vertices in A_i^2, for $i = 1, \dots, \ell$, and v_i to H_i, $i = \ell+1, \dots, m$, in the obvious way.

Among all minimum decycling sets of H that do not induce an $(r-2)$-regular component, let S_H be one that contains as many vertices of $H_1, \dots, H_m$ as possible. We claim that

$$S_H \text{ contains exactly } r-1 \text{ vertices of each } H_i. \tag{1}$$

Consider A_i^1 (say) and note that $H_i - A_i^1 = H_i[A_i^2]$ consists of isolated vertices and disjoint copies of K_2. The isolated vertices, having r-deficiency 1 in H_i, are adjacent to v_i, while the copies of K_2, having r-deficiency 0 in H_i, are not. Hence no cycle of $H - A_i^1$ contains a vertex of H_i. By minimality, therefore, S_H contains at most $r-1$ vertices of each H_i.

Suppose S_H contains fewer than $r-2$ vertices of H_i. Then each of A_i^1 and A_i^2 contains at most $r-3$ vertices of S_H. But then $H_i - S_H$ contains a copy of C_4 (with two vertices in each of A_i^1 and A_i^2), contrary to S_H being a decycling set.

Now assume that S_H contains exactly $r-2$ vertices of H_i. If S_H has nonempty intersection with both of A_i^1 and A_i^2, then each of A_i^1 and A_i^2 contains at most $r-3$ vertices of S_H and we obtain a contradiction as above. Hence assume S_H contains $r-2$ vertices in A_i^1 and let $A_i^1 - S_H = \{x_i\}$. Then A_i^2 is independent, otherwise x_i lies on a triangle of $A_i - S_H$. Since H is r-regular, v_i is adjacent to all $r-1$ vertices in A_i^2, thus forming 4-cycles containing x_i, and v_i has degree 1 in G. To destroy these 4-cycles, $v_i \in S_H$. Since v_i has degree 1 in G, any cycle of H containing v_i intersects H_i. Hence $(S_H - \{v_i\}) \cup \{x_i\}$ is a minimum decycling set of H that does not induce an $(r-2)$-regular component but contains more vertices of $H_1, \dots, H_m$ than S_H does, contrary to the choice of S_H. Therefore (1) holds.

Let $S = S_H \cap V(G)$. Then $|S| = |S_H| - m(r-1)$. Since no vertex of any H_i belongs to a cycle of G, S is a decycling set of G. Moreover, S is a minimum decycling set of G, for if not, let B be a decycling set of G such that $|B| < |S|$. Then $S' = B \cup A_1^1 \cup \dots \cup A_m^1$ is a decycling set of H such that $|S'| = |B| + m(r-1) < |S_H|$, which is impossible. Since $H[S_H]$ does not contain an $(r-2)$-regular component, neither does $G[S]$ and the proof is complete. □

5 Consequences of Theorem 1.1

In this section we present some corollaries of Theorem 1.1, with emphasis on the corollaries that yield structural results. Additional corollaries that give bounds on the decycling number are stated in [24].

Catlin and Lai [3] prove that the vertex set of every graph $G \neq K_4$ of maximum degree 3 can be partitioned into two sets that induce a forest and an independent set, respectively. As a consequence of Theorem 1.1, we obtain a generalization to graphs of maximum degree r.

Corollary 5.1. *Let $G \neq K_{r+1}$ be a graph of maximum degree $r \geq 3$. Then $V(G)$ can be partitioned into sets X and $\overline{X}$ such that $G[X]$ has maximum degree at most $r-2$ but does not contain an $(r-2)$-regular subgraph, and $G[\overline{X}]$ is a maximum forest.*

In Corollary 5.3 we provide an alternative proof of Brooks' Theorem, using Corollary 5.1 and the following lemma, from [4]. Recall that a *proper vertex colouring* of G is a partition of $V(G)$ into independent sets, and the *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of sets in a proper vertex colouring of G. We write $H \trianglelefteq G$ if H is an induced subgraph of G.

Lemma 5.2. [4] *For every graph G, $\chi(G) \leq 1 + \max\{\delta(H) : H \trianglelefteq G\}$.*

Corollary 5.3 (Brooks' Theorem). *If $r \geq 3$ and $G \neq K_{r+1}$ is a graph of maximum degree r, then $\chi(G) \leq r$.*

Proof. By Corollary 5.1, $V(G)$ can be partitioned into sets X and $\overline{X}$ such that $G[X]$ has maximum degree at most $r-2$ but does not have an $(r-2)$-regular subgraph, and $G[\overline{X}]$ is acyclic. Therefore $G[\overline{X}]$ can be 2-coloured.

Let H be any induced subgraph of $G[X]$. Then $\Delta(H) \leq r-2$ and H is not $(r-2)$-regular, hence $\delta(H) < r-2$. Therefore $\max\{\delta(H) : H \trianglelefteq G\} \leq r-3$. By Lemma 5.2, it follows that $\chi(G[X]) \leq r-2$.

Any $(r-2)$-colouring of $G[X]$ and any 2-colouring of $G[\overline{X}]$ now give an r-colouring of G. ☐

One generalization of proper vertex colourings involves partitioning $V(G)$ into sets such that each set induces a forest. We call such a partition a *forest partition* of G. The minimum number of sets in a forest partition is called the *vertex arboricity* of G and denoted by $a(G)$. Vertex arboricity, which is studied in [3] and [13], is a variation on the *arboricity* of a graph G, defined as the minimum number of sets needed to partition the edge set of G such that each set induces a forest. Arboricity was first studied in the early 1960s by Nash-Williams and Tutte [16, 17, 23].

Corollary 5.4 guarantees the existence of forest partitions with certain properties in a graph G. We then obtain an existing bound on the vertex arboricity of G, originally proved by Kronk and Mitchem [13], as a consequence of Corollaries 5.4. This is stated in Corollary 5.5. Corollary 5.4 is a minor strengthening of [3, Theorems 1 and 2(a)].

Corollary 5.4. *Let $G \neq K_{r+1}$ be a graph of maximum degree $\Delta = r \geq 3$. If r is odd then G has a forest partition into at most $\frac{r+1}{2}$ sets of which one set induces a maximum forest and another is independent, and if r is even then G has a forest partition into at most $\frac{r}{2}$ sets of which one set induces a maximum forest and another induces a linear forest.*

Proof. If $\Delta = 3$ or $\Delta = 4$ the statement follows immediately from Corollary 5.1. Let $r \geq 5$ and assume the statement holds all non-complete graphs of maximum degree $3 \leq \Delta < r$. Let $G \neq K_{r+1}$ be a graph of maximum degree r. Let $V(G) = X \cup \overline{X}$ be a partition of $V(G)$ as described in Corollary 5.1. Then $G[\overline{X}]$ is a forest and $G[X]$ has maximum degree $D \leq r-2$ which does not contain K_{r-1}, since it does not contain any $(r-2)$-regular subgraph. By the induction hypothesis, if D is odd then $G[X]$ has a forest partition into at most $\frac{D+1}{2} \leq \frac{r+1}{2} - 1$ sets, one of which is independent, and if D is even then $G[X]$ has a forest partition into at most $\frac{D}{2} \leq \frac{r}{2} - 1$ sets, one of which induces a linear forest. If r is even or D is odd then we are done (noting that an independent set is a linear forest, for the case where r is even and D is odd). If r is odd and D is even then $D \leq r-3$, so G has a forest partition into at most $\frac{r-3}{2} + 1 = \frac{r-1}{2}$ sets, one of which induces a linear forest. Then, by further partitioning one of the forests, we may obtain a forest partition into at most $\frac{r-3}{2} + 2 = \frac{r+1}{2}$ sets, one of which is independent. □

We now state the Kronk-Mitchem bound on the vertex arboricity, $a(G)$, which follows from Corollary 5.4.

Corollary 5.5 (The Kronk-Mitchem Bound)**.** [13] *If G is neither a cycle nor an odd clique then G has vertex arboricity at most $\left\lceil \frac{\Delta(G)}{2} \right\rceil$.*

Proof. The bound is trivial for $\Delta(G) < 3$, and for non-complete graphs of maximum degree $\Delta(G) \geq 3$ it follows immediately from Corollary 5.4. If $G = K_{2n}$ then any three vertices induce a cycle, so $a(G) = n$ and the result follows since $\Delta(G) = 2n - 1$. □

We conclude with an open problem. Recall that in the process of determining an appropriate generalization of Theorem B for $r \geq 4$, we found that a class $\mathcal{G}_r$ of graphs (defined on page 291) provided counterexamples to two restatements of Theorem B for general r. This led us to Theorem 1.1, but we note that stronger results may hold for all but a relatively small number of graphs. This prompts the following question.

Problem 5.6. *Which stronger results of the form, "Every graph $G \notin \mathcal{K}$ of maximum degree r has a minimum decycling set S such that $G[S]$ does not contain the subgraph H," can be obtained by allowing $\mathcal{K}$ to contain more than just K_{r+1}?*

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