On the dimension of downsets of integer partitions and compositions

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Abstract

We characterize the downsets of integer partitions (ordered by containment of Ferrers diagrams) and compositions (ordered by the generalized subword order) which have finite dimension in the sense of Dushnik and Miller. In the case of partitions, while the set of all partitions has infinite dimension, we show that every proper downset of partitions has finite dimension. For compositions we identify four minimal downsets of infinite dimension and establish that every downset which does not contain one of these four has finite dimension.

1   Introduction

The notion of the *dimension* of a poset $P = (X, \leqslant)$ was introduced by Dushnik and Miller [3], who defined it as the least d so that P embeds into a product of d linear orders. In particular, the dimension of a countable poset P is the least d so that P embeds into $\mathbb{R}^d$, the definition given by Ore [11]. Here we consider the dimension of downsets of integer partitions and compositions.

The partial order on partitions we consider is simply the one of Young's lattice, namely containment of Ferrers diagrams, and we establish the result below.

Theorem 1.1. *A downset of integer partitions is finite dimensional if and only if it does not contain every partition.*

We go on to study the dimension of downsets of compositions under the *generalized subword order*. In this order we view compositions as words over the positive integers

$\mathbb{P}$; we denote the set of these words by $\mathbb{P}^*$ and the empty word by $\varepsilon \in \mathbb{P}^*$. Given two compositions $u = u(1) \cdots u(k)$ and $w = w(1) \cdots w(n)$, we say that u is *contained* in w and write $u \leqslant w$ if there are indices $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that $u(j) \leqslant w(i_j)$ for all j.

This order can be illustrated graphically by way of skyline diagrams (which are merely a symmetry of Ferrers diagrams). The *skyline diagram* of the composition $w = w(1) \cdots w(n)$ consists of n columns of cells, with the ith column having $w(i)$ cells. For compositions u and w, we have $u \leqslant w$ if the skyline diagram of u can be embedded into that of w. For example, the diagrams below show that $3413 \leqslant 141421143$.
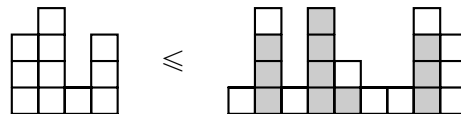
The generalized subword order on compositions has received some attention since it was first considered by Bergeron, Bousquet-Mélou, and Dulucq [1], who studied saturated chains in this poset. Snellman [14] extended their work. Later, Sagan and Vatter [13] determined the Möbius function of this poset, and Björner and Sagan [2] showed that this Möbius function has a rational generating function. Finally, Vatter [15] considered the analogue of the Reconstruction Conjecture in this poset. We also remark that Young's lattice (the poset of partitions) is the subposet of this poset of compositions consisting precisely of the compositions whose parts are weakly decreasing.

To state the analogue of Theorem 1.1 for compositions, we need to introduce a bit more notation and extend our viewpoint to include infinite compositions. A possibly-infinite composition is represented by a word over the alphabet $\mathbb{P} \cup \{n^\omega : n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\}$. In such a word, n^ω stands for an infinite number of parts all equal to n, ω stands for an infinite part, and ω^ω stands for an infinite number of infinite parts. Given a word u over the alphabet $\mathbb{P} \cup \{n^\omega : n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\}$, the *age* of u, denoted $\mathrm{Age}(u)$ is the set of all compositions which embed into it (this term dates to Fraïssé [4]). For example, $\mathrm{Age}(\omega\omega\omega)$ is the set of compositions with at most three parts, $\mathrm{Age}(2^\omega)$ is the set of all compositions with all parts at most two, and $\mathrm{Age}(1^\omega\omega2131^\omega)$ consists of all compositions which embed into the skyline diagram below.
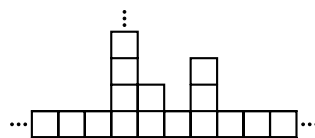

We can now state our result for compositions.

Theorem 1.2. *A downset of compositions in the generalized subword order is finite dimensional if and only if it does not contain* $\mathrm{Age}(\omega\omega\omega)$, $\mathrm{Age}(1^\omega21^\omega21^\omega)$, $\mathrm{Age}(\omega1^\omega\omega1^\omega)$, *or* $\mathrm{Age}(1^\omega\omega1^\omega\omega)$.

We also use the concept of ages in the partition setting, where the age of a word u over the alphabet $\mathbb{P} \cup \{n^\omega : n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\}$ is the set of all (finite) integer partitions which embed into u. While notationally identical, it will always be clear from the context whether an age consists of partitions or compositions.

Dimension is a monotone property in that the dimension of a poset is at least that of any of its subposets. Thus to show that a poset is infinite dimensional we show that it contains subposets of arbitrarily large dimension. In particular, we recall that the *crown* on the $2n$ elements $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is the poset in which the only comparisons are of the form $a_i < b_j$ for $i \neq j$, as depicted in the Hasse diagram below.
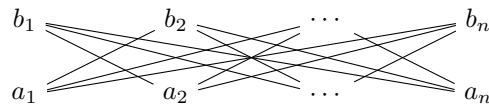

It is easily seen that the crown on $2n$ elements has dimension n, so we refer to it as the crown of dimension n. It is often referred to as the *standard example* of dimension n.

To establish that the poset of all integer partitions is infinite dimensional, it suffices to find arbitrarily large crowns of partitions. One such family of crowns is defined by taking

$$a_i = (n-i)^i \quad \text{and} \quad b_i = \bigvee_{j \neq i} a_j,$$

i.e., taking a_i to be the partition consisting of i parts equal to $n-i$ and b_i to be the join (in Young's lattice) of all a_j for $j \neq i$.

Similarly, one direction of Theorem 1.2 can be established by finding arbitrarily large crowns in the four stated ages. For example, we see that $\text{Age}(\omega\omega\omega)$ contains the crown of dimension $n-3$ shown below for all $n \geqslant 5$.
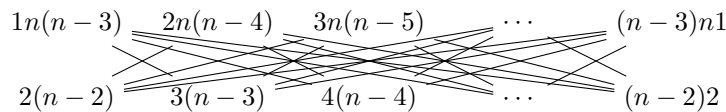
A slight modification of this crown shows that $\text{Age}(1^\omega 21^\omega 21^\omega)$ is infinite dimension, as it contains the crown of dimension $n-3$ shown below for all $n \geqslant 5$.
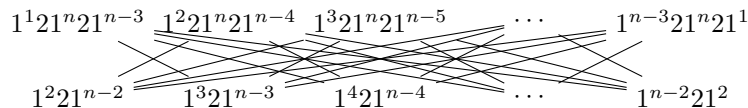
The last two ages stated in Theorem 1.2 are isomorphic, so it suffices to show that $\text{Age}(\omega 1^\omega \omega 1^\omega)$ is infinite dimensional. This age contains the crown of dimension $n-1$ shown below for all $n \geqslant 3$.

$$11^0 n 1^{n-1} \quad 21^1 n 1^{n-2} \quad 31^2 n 1^{n-3} \quad \cdots \quad (n-1)1^{n-2}n1^1$$
$$21^n \qquad 31^{n-1} \qquad 41^{n-2} \qquad \cdots \qquad n1^2$$
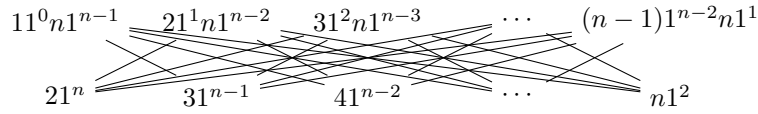
Thus it suffices to prove that downsets of compositions not containing any of these four ages are finite dimensional. Note that $\mathrm{Age}(2^\omega)$ is infinite dimensional—this follows from the fact that it contains $\mathrm{Age}(1^\omega 21^\omega 21^\omega)$, or more easily by observing that it contains the crown of dimension n defined by $a_i = 1^{i-1}21^{n-i}$ and $b_i = 2^{i-1}12^{n-i}$. Consequently, the age of any (infinite) composition which includes any symbol of the form ω^ω or n^ω for $n \geqslant 2$ is necessarily infinite dimensional. Therefore when characterizing the finite dimensional ages of infinite compositions we may restrict our attention to ages of words over the alphabet $\mathbb{P} \cup \{1^\omega, \omega\}$.

2   Tools

In this section we introduce the tools we use to establish the other directions of Theorems 1.1 and 1.2. A poset P is *well quasi-ordered* if it contains neither infinite antichains nor infinite strictly decreasing chains, i.e., $x_0 > x_1 > \cdots$. We begin by recalling the following well-known result.

Higman's Lemma [5]. *If $(P, \leqslant)$ is well quasi-ordered then P^*, the poset of words over P ordered by the generalized subword order, is also well quasi-ordered.*

As the poset of partitions is a subposet of $\mathbb{P}^*$ and the poset of compositions is precisely the poset $\mathbb{P}^*$, Higman's Lemma implies that both posets are well quasi-ordered. This allows us to appeal to the following result.

Proposition 2.1. *Downsets of well quasi-orders satisfy the* descending chain condition*, i.e., there does not exist a sequence of downsets satisfying $\mathcal{C}^0 \supsetneq \mathcal{C}^1 \supsetneq \mathcal{C}^2 \supsetneq \cdots$.*

Proof. Suppose to the contrary that the well quasi-ordered downset $\mathcal{C}$ were to contain an infinite strictly decreasing sequence of subdownsets $\mathcal{C} = \mathcal{C}^0 \supsetneq \mathcal{C}^1 \supsetneq \mathcal{C}^2 \supsetneq \cdots$. For each $i \geqslant 1$, choose $x_i \in \mathcal{C}^{i-1} \backslash \mathcal{C}^i$. The set of minimal elements of $\{x_1, x_2, \ldots\}$ is an antichain and therefore finite, so there is an integer m such that $\{x_1, x_2, \ldots, x_m\}$ contains these minimal elements. In particular, $x_{m+1} \geqslant x_i$ for some $1 \leqslant i \leqslant m$. However, we chose $x_{m+1} \in \mathcal{C}^m \backslash \mathcal{C}^{m+1}$, and because $x_{m+1} \geqslant x_i$, x_{m+1} does not lie in $\mathcal{C}^i$ and thus cannot lie in $\mathcal{C}^m$, a contradiction. □

Because of Proposition 2.1, we can consider minimal (with respect to set containment) counterexamples to prove Theorems 1.1 and 1.2. Our next result shows that such minimal counterexamples cannot be unions of two proper subdownsets, but before proving it we need to make some more general remarks about dimension, and in particular, our approach to establishing that downsets are finite dimensional.

A *realizer* of the poset P is a collection $\mathcal{R}$ of linear extensions of the poset such that $x \leqslant_P y$ if and only if $x \leqslant_L y$ for each $L \in \mathcal{R}$. If $\mathcal{R}$ is a realizer of the poset P, we say that $\mathcal{R}$ *realizes* P. Given that the elements of a realizer are extensions of the original poset, this is equivalent to saying that for each pair $x, y \in P$ of incomparable elements, there is some $L \in \mathcal{R}$ such that $y \leqslant_L x$.

A *refinement* of the poset P is another partial order, say $\leqslant_R$, such that $x \leqslant_R y$ for all pairs $x, y \in P$ with $x \leqslant_P y$. Because every refinement can be extended to a linear extension, to establish that the dimension of the poset P is at most n, it suffices to find a collection $\mathcal{R}$ of n refinements of P such that $x \leqslant_P y$ if and only if $x \leqslant_R y$ for each $R \in \mathcal{R}$. Frequently we go a step further than this. As every refinement of a subposet of P can be extended to a linear extension of P, to show that P has dimension at most n it suffices to find a collection $\mathcal{R}$ of n *partial refinements* (meaning refinements of subposets of P) with this property.

In constructing and analyzing these refinements or partial refinements, we use two additional terms. If the refinements R_1 and R_2 satisfy $x <_{R_1} y$ and $y <_{R_2} x$ (or vice versa) then we say that the pair R_1, R_2 *breaks* the incomparability between x and y. Finally, every homomorphism between a poset (or subposet of it) to a totally ordered set (typically $\mathbb{N}$ here) induces a refinement or partial refinement on the poset. In this situation we often say that the induced refinement *sorts* the objects of P according to the homomorphism. For example, a natural refinement of the either the poset of partitions or of compositions is the one that sorts them according to length (number of parts).

The following result, which is fundamental to our approach, is folklore, but we include a proof for completeness.

Proposition 2.2. *Let $(P, \leqslant)$ be a poset, and let $\mathcal{C}, \mathcal{D} \subseteq P$ be downsets of dimension m and n respectively. Then $\mathcal{C} \cup \mathcal{D}$ is a downset of dimension at most $m + n$.*

Proof. Certainly $\mathcal{C} \cup \mathcal{D}$ is a downset, so it suffices to show it has dimension at most $m+n$. Let $\{R_1, R_2, \ldots, R_m\}$ and $\{S_1, S_2, \ldots, S_n\}$ be realizers of $\mathcal{C}$ and $\mathcal{D}$ respectively. First, note that every member of $\mathcal{C} \backslash \mathcal{D}$ is incomparable with every member of $\mathcal{D} \backslash \mathcal{C}$. Define the refinements

$$R_1' = R_1 \oplus (\mathcal{D} \backslash \mathcal{C}), \ldots, R_m' = R_m \oplus (\mathcal{D} \backslash \mathcal{C})$$

and

$$S_1' = S_1 \oplus (\mathcal{C} \backslash \mathcal{D}), \ldots, S_n' = S_n \oplus (\mathcal{C} \backslash \mathcal{D}),$$

where $A \oplus B$ is the *ordinal sum* of A and B, including all relations within both A and B, as well as all relations of the form $a < b$ where $a \in A$ and $b \in B$.

The collection $\{R_1', \ldots, R_m', S_1', \ldots, S_n'\}$ realizes $\mathcal{C} \cup \mathcal{D}$, as it breaks all incomparabilities between elements of $\mathcal{C} \backslash \mathcal{D}$ and $\mathcal{D} \backslash \mathcal{C}$ and realizes each of $\mathcal{C}$ and $\mathcal{D}$. This shows that $\mathcal{C} \cup \mathcal{D}$ has dimension at most $m + n$. □

We note that the hypothesis that $\mathcal{C}$ and $\mathcal{D}$ are both downsets in Proposition 2.2 is essential, as shown by the fact that the crown of dimension n can be expressed as the union of two antichains (which are thus each 2-dimensional).

The downsets of compositions which are not unions of proper subdownsets are precisely the ages, as shown by the following theorem of Fraïssé (which we have specialized to our contexts here). This result implies it suffices to prove Theorems 1.1 and 1.2 for ages.

Theorem 2.3 (Fraïssé [4]; see also Hodges [6, Section 7.1]). *The following are equivalent for a downset $\mathcal{C}$ of integer partitions or compositions:*

(1) $\mathcal{C}$ *cannot be expressed as the union of two proper subdownsets,*

(2) $\mathcal{C}$ *satisfies the* joint embedding property *meaning that for every $a, b \in \mathcal{C}$ there is some $c \in \mathcal{C}$ such that $a, b \leqslant c$, and*

(3) $\mathcal{C} = \mathrm{Age}(u)$ *for some word $u \in (\mathbb{P} \cup \{n^\omega \ : \ n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\})^*$.*

We conclude this section by providing the only specific dimension results of the paper. To realize the downset $\mathrm{Age}(\omega\omega)$ of compositions, we use a pair of linear extensions L_1 and L_2 and a refinement R_3. The first, L_1, orders compositions according to the *shortlex order*, which sorts compositions first by their length, and within each length sorts compositions according to the lexicographical ordering. The second, L_2, orders compositions according to the *shortcolex order*, which sorts compositions first by their length, and within each length sorts compositions according to the colexicographical ordering (lexicographical order, but sorting from right to left). Lastly, the refinement R_3 sorts compositions first by their largest part and then by their second largest part. Note that this sometimes leaves a composition and its reverse incomparable, and thus is not a linear extension.

These three refinements constitute a realizer of $\mathrm{Age}(\omega\omega)$, implying that the dimension of $\mathrm{Age}(\omega\omega)$ is at most 3. Observing that this age contains the crown of dimension 3 below allows us to conclude that the dimension of $\mathrm{Age}(\omega\omega)$ equals 3.
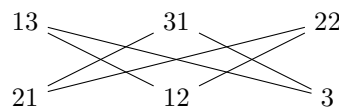
Proposition 2.4. *The dimension of $\mathrm{Age}(\omega\omega)$ is 3.*

Similar methods can be applied to show that the dimension of $\mathrm{Age}(\omega 1^\omega)$ is 2, and that the dimensions of $\mathrm{Age}(1^\omega \omega 1^\omega)$, $\mathrm{Age}(\omega \omega 1^\omega)$, and $\mathrm{Age}(\omega 1^\omega \omega)$ are each 4.
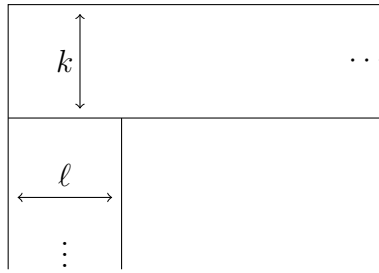
Figure 1: The Ferrers diagram of the infinite partition $\omega^k \ell^\omega$.

3   Partitions

Having observed in the introduction that the poset of all integer partitions is infinite dimensional, Theorem 1.1 will follow once we show that all proper downsets of partitions are finite dimensional, which we do here. By Theorem 2.3, every *proper* downset of partitions can be written as a finite union of ages of the form $\mathrm{Age}(u)$ for some word $u \in (\mathbb{P} \cup \{n^\omega \ : \ n \in \mathbb{P}\} \cup \{\omega\})^*$. Because the parts of partitions are ordered, each such age is contained in an age of the form $\mathrm{Age}(\omega^k \ell^\omega)$ for nonnegative integers k and ℓ. The Ferrers diagram of such a partition is shown in Figure 1.

By Proposition 2.2, it suffices to show that each such age is finite dimensional. We see that $\mathrm{Age}(\omega^k \ell^\omega)$ is isomorphic (as a poset) to the product $\mathrm{Age}(\omega^k) \times \mathrm{Age}(\ell^\omega)$. The first of these ages is finite dimensional because it is isomorphic to a subposet of $\mathbb{N}^k$. The second of these ages is finite dimensional because it is isomorphic to $\mathrm{Age}(\omega^\ell)$, via conjugation, and that age is in turn isomorphic to a subposet of $\mathbb{N}^\ell$. Thus the dimension of $\mathrm{Age}(\omega^k \ell^\omega)$ is at most $k + \ell$. This completes the proof of Theorem 1.1.

4   Compositions

We have shown in Section 1 that $\mathrm{Age}(\omega\omega\omega)$, $\mathrm{Age}(1^\omega 2 1^\omega 2 1^\omega)$, $\mathrm{Age}(\omega 1^\omega \omega 1^\omega)$, and $\mathrm{Age}(1^\omega \omega 1^\omega \omega)$ are infinite dimensional, and in Section 2 we showed that it suffices to show that the maximal ages not containing the four distinguished infinite dimensional ages are finite dimensional. The two types of these maximal ages are those of the forms $\mathrm{Age}(a\omega b 1^\omega c 1^\omega d\omega e)$ and $\mathrm{Age}(a 1^\omega b\omega c\omega d 1^\omega e)$ for finite compositions a, b, c, d, and e.

We establish the finite dimensionality of these two types of ages with a series of results. Our first such result implies that we may assume a and e are empty.

Proposition 4.1. *If* $\mathrm{Age}(u)$ *is finite dimensional for* $u \in (\mathbb{P} \cup \{1^\omega, \omega\})^*$, *then* $\mathrm{Age}(ku)$ *is finite dimensional for all* $k \in \mathbb{N}$.

Proof. We proceed by induction on k. The base case of $k = 0$ is tautological, so let $k \in \mathbb{P}$ be given, and assume $\mathrm{Age}((k-1)u)$ is finite dimensional. Let $A = \mathrm{Age}(u)$, let

$B = \mathrm{Age}(ku) \backslash A$, and for each $1 \leqslant j \leqslant k$, define

$$A_j = \{ja \in A\} \quad \text{and} \quad B_j = \{ja \in B\}.$$

as well as $A_{>k} = \{\ell a \in A : \ell > k\}$.

By induction, $A \cup B_j$ is finite dimensional for each $1 \leqslant j \leqslant k-1$. Furthermore, B is finite dimensional as it is isomorphic to a subposet of $\mathbb{N} \times A$ by the mapping that sends the word ℓa to (ℓ, a); note that if $\ell a \leqslant \ell' a'$ for $\ell a, \ell' a' \in B$ then we must have $\ell \leqslant \ell'$ and $a \leqslant a'$ because otherwise we would have $\ell a \leqslant a'$, but that would imply that $\ell a \in A$, a contradiction to its choice.

We claim that it suffices to establish that $A_j \cup B_k$ and $A_{>k} \cup B_k$ are finite dimensional for each $1 \leqslant j \leqslant k$. To see this, suppose that each of these unions is finite dimensional. This means that there are finite realizers of $A \cup B_j$ for all $1 \leqslant j \leqslant k-1$, of B, and of $A_j \cup B_k$ and $A_{>k} \cup B_k$ for all $1 \leqslant j \leqslant k$. We can consider each of these finite realizers as a finite set of partial refinements of the entire poset $A \cup B$, and it follows that the union of all of those partial refinements does indeed break all of the incomparabilities that need to be broken.

Now fix $1 \leqslant j \leqslant k$. Given $ja_1 \in A_j$ and $ka_2 \in B_k$, we have $ja_1 \leqslant ka_2$ if and only if $a_1 \leqslant a_2$. For this reason, we define

$$A'_j = \{a : ja \in A_j\} \quad \text{and} \quad B'_k = \{a : ka \in B_k\},$$

and consider a realizer $\{L_1, \dots, L_n\}$ of $A'_j \cup B'_k$, which is finite dimensional as it is contained in A. For each $1 \leqslant i \leqslant n$, we expand L_i into a linear extension $\hat{L}_i$ of a set containing $A_j \cup B_k$. To do so, we replace the instance of each composition v in L_i with the two element chain $\{jv, kv\}$. If $ja_1 \in A_j$ and $ka_2 \in B_k$ with $ja_1 \not\leqslant ka_2$, then $a_1 \not\leqslant a_2$. Thus a_2 precedes a_1 in some L_i, meaning ka_2 precedes ja_1 in $\hat{L}_i$.

Lastly, given $\ell a_1 \in A_{>k}$ and $ka_2 \in B_k$, we have $\ell a_1 \leqslant ka_2$ if and only if $\ell a_1 \leqslant a_2$. Let $\{R_1, \dots, R_m\}$ be a realizer of $A_{>k} \cup B'_k$, which is finite dimensional as it is contained in A. For each $1 \leqslant i \leqslant m$, we expand R_i into a linear extension $\hat{R}_i$ of $A_{>k} \cup B_k$. To do so, we replace the instance of $a \in B'_k$ in R_i with ka.

Then, if $\ell a_1 \in A_{>k}$ and $ka_2 \in B_k$ with $\ell a_1 \not\leqslant ka_2$, then $\ell a_1 \not\leqslant a_2$. Thus a_2 precedes ℓa_1 in some R_i, meaning ka_2 precedes ℓa_1 in $\hat{R}_i$. $\qquad\square$

By applying Proposition 4.1 twice, we obtain the following.

Corollary 4.2. *For all compositions a and b, both $\mathrm{Age}(a1^\omega b)$ and $\mathrm{Age}(a\omega b)$ are finite dimensional.*

The proof of our next result is more complicated.

Proposition 4.3. *For all compositions c, $\mathrm{Age}(1^\omega c 1^\omega)$ is finite dimensional.*

Proof. We partition the age of interest into a finite collection of intervals and then construct a family of linear extensions which break the incomparabilities between these intervals. These intervals are $[a, 1^\omega a 1^\omega) = \{d \in \mathrm{Age}(1^\omega a 1^\omega) \,:\, d \geqslant a\}$ for each $a = a(1) \cdots a(m) \in \mathrm{Age}(c)$, where the first and last parts of a are at least 2. Because of this condition on a, we see that

$$[a, 1^\omega a 1^\omega) = \{1^i a 1^j \,:\, i, j \in \mathbb{N}\},$$

so each such interval is isomorphic to $\mathbb{N}^2$ and thus finite dimensional. Let $\mathcal{R}$ denote the (finite) collection of linear extensions realizing each $[a, 1^\omega a 1^\omega)$.

It suffices to consider the union of a pair of such intervals. Let $a, b \leqslant c$ where $a = a(1) \cdots a(m)$ and $b = b(1) \cdots b(n)$ have the property that the first and last parts of each a and b are at least 2. Note that there are only finitely many such pairs a, b because c is a finite composition. First, if a and b are such that $b \not\leqslant a$, then none of the elements of $[b, 1^\omega b 1^\omega)$ embed into any of the elements of $[a, 1^\omega a 1^\omega)$, and these incomparabilities can be broken with the refinement $[a, 1^\omega a 1^\omega) \oplus [b, 1^\omega b 1^\omega)$. Let $\mathcal{S}$ be the (finite) collection of these refinements for each a, b with $b \not\leqslant a$.

This leaves us to consider the case where a and b are comparable with $a < b$, and the only incomparabilities left to break are those of the form $1^i a 1^j \not\leqslant 1^k b 1^\ell$.

The bulk of the proof consists of contending with the fact that a may have several embeddings into b. Of these, it suffices to consider the *compact* embeddings, meaning those which cannot be shrunk. More precisely, let $\alpha_1 < \cdots < \alpha_q$ denote the beginnings of these compact embeddings and $\beta_1 < \cdots < \beta_q$ denote the ends. Because these are embeddings, for all p we have

$$a \leqslant b(\alpha_p) b(\alpha_p + 1) \cdots b(\beta_p),$$

and because they are compact, we have both

$$a \not\leqslant b(\alpha_p + 1) b(\alpha_p + 2) \cdots b(\beta_p),$$
$$a \not\leqslant b(\alpha_p) b(\alpha_p + 1) \cdots b(\beta_p - 1).$$

Consider an incomparability between elements of these two intervals, $1^i a 1^j \not\leqslant 1^k b 1^\ell$. This means that, in $\mathbb{N}^2$, we have incomparabilities of the form

$$(i, j) \not\leqslant (k + \alpha_p - 1, \ell + n - \beta_p)$$

for each $1 \leqslant p \leqslant q$. The set of points $\{(k + \alpha_p - 1, \ell + n - \beta_p) : 1 \leqslant p \leqslant q\}$ is an antichain in $\mathbb{N}^2$ that lies weakly above and to the right of (k, ℓ) in the plane, as shown on the left of Figure 2.

We now introduce two refinements of $[a, 1^\omega a 1^\omega) \cup [b, 1^\omega b 1^\omega)$. The first sorts compositions by the largest r such that $1^r a$ is contained in them, while the second sorts compositions by the largest s such that $a 1^s$ is contained in them. For a given k and ℓ, these two refinements break all incomparabilities of the form $1^i a 1^j \not\leqslant 1^k b 1^\ell$ where
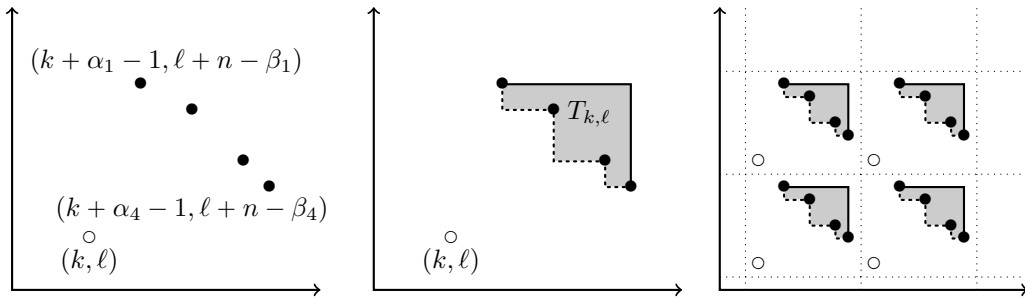
Figure 2: (Left) A point (k, ℓ) representing $1^k b 1^\ell$ together with associated points representing the minimal compositions of the form $1^i a 1^j$ which do not embed into $1^k b 1^\ell$. (Center) A point (k, ℓ) representing $1^k b 1^\ell$ and its associated set $T_{k,\ell}$ representing compositions of the form $1^i a 1^j$. (Right) The shaded regions indicate part of a family of compositions included in one refinement constructed at the end of the proof of Proposition 4.3.

$i > k + \alpha_q - 1$ or $j > \ell + n - \beta_1$. Still thinking of k and ℓ as fixed, this leaves us with a finite set of incomparabilities of the form $1^i a 1^j \not\leqslant 1^k b 1^\ell$ to break, as illustrated in the center of Figure 2. Let $T_{k,\ell}$ denote the finite set of compositions of the form $1^i a 1^j$ whose incomparabilities with $1^k b 1^\ell$ have not been dealt with. Thus $T_{k,\ell}$ is the set

$$\{1^i a 1^j \quad : \quad (i, j) \leqslant (k + \alpha_q - 1, \ell + n - \beta_1) \text{ and}$$
$$(i, j) \not\leqslant (k + \alpha_p - 1, \ell + n - \beta_p) \text{ for all } 1 \leqslant p \leqslant q\}.$$

We identify each composition $1^i a 1^j \in T_{k,\ell}$ with the point (i, j) in the plane. Thus the points corresponding to the compositions in $T_{k,\ell}$ are contained in the rectangle

$$[k, k + \alpha_q - 1] \times [\ell, \ell + n - \beta_1].$$

Given k, ℓ, we define a refinement $R_{k,\ell}$ of $\{1^k b 1^\ell\} \cup T_{k,\ell}$ in which $1^k b 1^\ell$ is less than each element of $T_{k,\ell}$. All that remains is to combine the collection of refinements $R_{k,\ell}$ into finitely many refinements of $[a, 1^\omega a 1^\omega) \cup [b, 1^\omega b 1^\omega)$. We achieve this by partitioning $\mathbb{N}^2$ into equivalence classes with respect to the equivalence relation $(k, \ell) \sim (k', \ell')$ if $k \equiv k' \bmod \alpha_q$ and $\ell \equiv \ell' \bmod n - \beta_1 + 1$. We further write $[(k, \ell)]$ to denote the equivalence class containing (k, ℓ). Note that there are only finitely many such equivalences classes.

The motivation for this equivalence relation is that if $(k, \ell) \sim (k', \ell')$ then the relations defined by $R_{k,\ell}$ and $R_{k',\ell'}$ do not conflict. Thus for any $(k, \ell) \in \mathbb{N}^2$, all of the relations

$$\bigcup_{(k',\ell') \in [(k,\ell)]} R_{k',\ell'}$$

can be combined into a single refinement. The compositions involved in one such refinement are drawn on the right of Figure 2.

As there are only finitely many such equivalence classes in $\mathbb{N}^2$, and only finitely many pairs a, b with $a \leqslant b \leqslant c$, this (finite) set of refinements, together with the refinements of $\mathcal{R}$ and $\mathcal{S}$, realizes $\mathrm{Age}(1^\omega c 1^\omega)$, completing the proof. $\qquad \square$

With Proposition 4.3 established, showing that ages of the forms $\text{Age}(\omega a 1^\omega b 1^\omega c \omega)$ and $\text{Age}(1^\omega a \omega b \omega c 1^\omega)$ are finite dimensional is accomplished by first proving that ages of the forms $\text{Age}(\omega a 1^\omega b 1^\omega)$ and $\text{Age}(1^\omega a \omega b 1^\omega)$ are finite dimensional. Each of these steps relies on Proposition 4.3.

Proposition 4.4. *For all compositions a and b, $\text{Age}(\omega a 1^\omega b 1^\omega)$ is finite dimensional.*

Proof. Let m denote the maximum entry in a or b and let $\bar{m} = m + 1$. By Propositions 4.1 and 4.3 we have that $\text{Age}(\bar{m} a 1^\omega b 1^\omega)$ is finite dimensional, so let $\{L_1, \ldots L_n\}$ be a realizer of it. For each $1 \leq i \leq n$, we expand L_i into a linear extension $\hat{L}_i$ of $\text{Age}(\omega a 1^\omega b 1^\omega)$. To do so, we replace the instance of $\bar{m} x$ in L_i with the linearly ordered interval $[\bar{m} x, \omega x)$.

The only incomparabilities yet to be handled are those of the form $k_1 x_1 \nleq k_2 x_2$ where $\bar{m} x_1 \leq \bar{m} x_2$ and $k_1 > k_2 \geq \bar{m}$. These are fixed by including a single refinement which sorts elements of $\text{Age}(\omega a 1^\omega b 1^\omega)$ by their largest entry. $\square$

Proposition 4.5. *For all compositions a, b, and c, $\text{Age}(\omega a 1^\omega b 1^\omega c \omega)$ is finite dimensional.*

Proof. We proceed by defining six sets, each of which is finite dimensional and whose union is the age of interest, and then construct a family of refinements which break the incomparabilities between the sets. Let m denote the maximum entry in a, b, or c, let $\bar{m} = m + 1$, let $\bar{\bar{m}} = \bar{m} + 1$, and define

$$
\begin{aligned}
A &= [\varepsilon, m a 1^\omega b 1^\omega c m), \\
B_1 &= [\bar{m}, m a 1^\omega b 1^\omega c \bar{m}), \\
B_2 &= [m, m a 1^\omega b 1^\omega c m), \\
C_1 &= [\bar{m}\bar{m}, \omega a 1^\omega b 1^\omega c \bar{m}), \\
C_2 &= [\bar{m}\bar{m}, \bar{m} a 1^\omega b 1^\omega c \omega), \\
D &= [\bar{\bar{m}}\bar{\bar{m}}, \omega a 1^\omega b 1^\omega c \omega),
\end{aligned}
$$

where we recall that ε denotes the empty word. Now, the complement of D,

$$
\begin{aligned}
\text{Age}(\omega a 1^\omega b 1^\omega c \omega) \setminus D &= A \cup B_1 \cup B_2 \cup C_1 \cup C_2 \\
&= \text{Age}(\omega a 1^\omega b 1^\omega c \bar{m}) \cup \text{Age}(\bar{m} a 1^\omega b 1^\omega c \omega)
\end{aligned}
$$

is finite dimensional by Propositions 2.2, 4.1, and 4.4. Also, $C_1 \cup C_2 \cup D$ is finite dimensional as it is isomorphic to a subposet of $\mathbb{N} \times \text{Age}(a 1^\omega b 1^\omega c) \times \mathbb{N}$. Thus it suffices to show that the incomparabilities between $A \cup B_1 \cup B_2$ and D can be broken with finitely many refinements.

Let $\{L_1, \ldots, L_n\}$ be a realizer for $\text{Age}(\bar{m} a 1^\omega b 1^\omega c \bar{m})$. For each $1 \leq i \leq n$, we expand L_i into a refinement $\hat{L}_i$ of $\text{Age}(\omega a 1^\omega b 1^\omega c \omega)$. To do so, for each v, we replace the instance of $\bar{m} v \bar{m}$ in L_i with the interval $[\bar{m} v \bar{m}, \omega v \omega)$. If $u \in A \cup B_1 \cup B_2$ and $k_1 v k_2 \in D$ for integers $k_1, k_2 \geq \bar{\bar{m}}$ and $u \nleq k_1 v k_2$, then $u \nleq \bar{m} v \bar{m}$, so $\bar{m} v \bar{m}$ is less than u in some L_i, and thus $k_1 v k_2$ is less than u in $\hat{L}_i$. This completes the proof. $\square$

The proofs of our next two results are very similar to those of Propositions 4.4 and 4.5.

Proposition 4.6. *For all compositions a and b, $\mathrm{Age}(1^\omega a \omega b 1^\omega)$ is finite dimensional.*

Proof. Let m denote the maximum entry in a or b and let $\bar{m} = m + 1$. By Proposition 4.3 we have that $\mathrm{Age}(1^\omega a \bar{m} b 1^\omega)$ is finite dimensional, so let $\{L_1, \ldots L_n\}$ be a realizer of it. For each $1 \leqslant i \leqslant n$, we expand L_i into a linear extension $\hat{L}_i$ of $\mathrm{Age}(1^\omega a \omega b 1^\omega)$. To do so, we replace the instance of $x\bar{m}y$ in L_i with the linearly ordered interval $[x\bar{m}y, x\omega y)$.

The only incomparabilities yet to be handled are those of the form $x_1 k_1 y_1 \not\leqslant x_2 k_2 y_2$ where $x_1 \bar{m} y_1 \leqslant x_2 \bar{m} y_2$ and $k_1 > k_2 \geqslant \bar{m}$. These are fixed by including a single refinement which sorts elements of $\mathrm{Age}(1^\omega a \omega b 1^\omega)$ by their largest entry. $\square$

Proposition 4.7. *For all compositions a, b, and c, $\mathrm{Age}(1^\omega a \omega b \omega c 1^\omega)$ is finite dimensional.*

Proof. Let m denote the maximum entry in a, b, or c, let $\bar{m} = m + 1$, let $\bar{\bar{m}} = \bar{m} + 1$, and define
$$
\begin{aligned}
A &= [\varepsilon, 1^\omega a m b m c 1^\omega), \\
B_1 &= [\bar{m}, 1^\omega a m b \bar{m} c 1^\omega), \\
B_2 &= [\bar{m}, 1^\omega a \bar{m} b m c 1^\omega), \\
C_1 &= [\bar{m}\bar{m}, 1^\omega a \omega b \bar{m} c 1^\omega), \\
C_2 &= [\bar{m}\bar{m}, 1^\omega a \bar{m} b \omega c 1^\omega), \\
D &= [\bar{\bar{m}}\bar{\bar{m}}, 1^\omega a \omega b \omega c 1^\omega).
\end{aligned}
$$
The complement of D,
$$
\begin{aligned}
\mathrm{Age}(1^\omega a \omega b \omega c 1^\omega) &= A \cup B_1 \cup B_2 \cup C_1 \cup C_2 \\
&= \mathrm{Age}(1^\omega a \bar{m} b \omega c 1^\omega) \cup \mathrm{Age}(1^\omega a \omega b \bar{m} c 1^\omega)
\end{aligned}
$$
is finite dimensional by Propositions 2.2, 4.1, and 4.6. Also, $C_1 \cup C_2 \cup D$ is finite dimensional as it is isomorphic to a subposet of $\mathrm{Age}(1^\omega a) \times \mathbb{N} \times \mathrm{Age}(b) \times \mathbb{N} \times \mathrm{Age}(c 1^\omega)$. Thus it suffices to show that the incomparabilities between $A \cup B_1 \cup B_2$ and D can be broken with finitely many refinements.

Let $\{L_1, \ldots, L_n\}$ be a realizer for $\mathrm{Age}(1^\omega a \bar{m} b \bar{m} c 1^\omega)$. For each $1 \leqslant i \leqslant n$, we expand L_i into a refinement $\hat{L}_i$ of $A \cup B_1 \cup B_2 \cup D$. To do so, for each x, y, z, we replace the instance of $x\bar{m}y\bar{m}z$ in L_i with the interval $[x\bar{m}y\bar{m}z, x\omega y\omega z)$.

Then, if $u \in A \cup B_1 \cup B_2$ and $xk_1yk_2z \in D$ with $k_1, k_2 \geqslant \bar{\bar{m}}$, and $u \not\leqslant xk_1yk_2z$, then we have $u \not\leqslant x\bar{m}y\bar{m}z$. Thus $x\bar{m}y\bar{m}z$ is less than u in some L_i, and thus xk_1yk_2z is less than u in $\hat{L}_i$. This completes the proof. $\square$

With Propositions 4.5 and 4.7 established, we note that the proof of Theorem 1.2 is complete, given the remarks at the beginning of Section 4.

5 Concluding Remarks

Theorems 1.1 and 1.2 characterize the finite dimensional downsets in the posets of integer partitions and compositions, respectively. There are several similar contexts in which the analogous questions have yet to be considered. One such context is the poset of permutations under the permutation pattern order. We refer to the second author's survey [16] for more information on this order. A related example is the poset of set partitions, first studied by Klazar [7–9] and Sagan [12]. Another natural context would be the generalized subword order over an arbitrary poset P, a context where McNamara and Sagan [10] have recently determined the Möbius function. Indeed, even the special case of words over a two-element antichain appears to be untouched.

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