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On two open problems concerning weak Roman domination in trees

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Abstract

For a graph G, let $\gamma_r(G)$, $\gamma_R(G)$ and $\gamma_{r2}(G)$ denote the weak Roman domination number, the Roman domination number and the 2-rainbow domination number, respectively. It is well-known that for every graph G, $\gamma_r(G) \leq \gamma_{r2}(G) \leq \gamma_R(G)$. In this paper, we characterize all trees T with $\gamma_r(T) = \gamma_{r2}(T)$ or $\gamma_r(T) = \gamma_R(T)$ answering two open problems posed by Chellali, Haynes and Hedetniemi [*Discrete Appl. Math.* 178 (2014), 27–32].

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1 Introduction

In this paper, G is a simple graph without isolated vertices, with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A vertex of degree one is called a *pendant vertex* or a *leaf* and its neighbour is called a *support vertex.* A *strong support vertex* is a support vertex adjacent to at least two leaves and an *end support vertex* is a support vertex having at most one non-leaf neighbor. A *pendant path* P of a graph G is an induced path such that one of the endpoints has degree one in G, and its other endpoint is the only vertex of P adjacent to some vertex in $G - P$. The *distance* between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We write P_n for the *path* of order n. A *double star* $DS_{p,q}$ is a tree containing exactly two non-pendant vertices which one is adjacent to p leaves and the other is adjacent to q leaves. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x\in A} f(x)$, and the sum $f(V(G))$ is called the *weight* $\omega(f)$ of f.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$, and the *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [9] and was inspired by the work of ReVelle and Rosing [13], Stewart [14]. It is worth mentioning that since its introduction in 2004, several new variations of Roman domination were introduced: weak Roman domination [11], 2-rainbow domination [6], Roman $\{2\}$-domination [8], maximal Roman domination [1], mixed Roman domination [2], double Roman domination [5] and recently total Roman domination [12]. Two of the previous variations will be the focus of this paper.

A 2-*rainbow dominating function* (2rDF) on a graph G is a function $f : V(G) \to \mathcal{P}(\{1, 2\})$ if for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\cup_{u\in N(v)} f(u) = \{1, 2\}$. The weight of a 2$r$DF f is defined as $\omega(f) = \sum_{v\in V(G)} |f(v)|$, and the 2-*rainbow domination number* $\gamma_{r2}(G)$ is the minimum weight of a 2rDF of G.

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function. If $V_i = \{v \in V | f(v) = i\}$ for $i \in \{0, 1, 2\}$, then f can be denoted by $f = (V_0, V_1, V_2)$. A vertex v with $f(v) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex w with $f(w) > 0$. A function f is called a *weak Roman dominating function* (WRDF) if each vertex v with $f(v) = 0$ is adjacent to a vertex w with $f(w) > 0$, such that the function f' defined by $f'(v) = 1$, $f'(w) = f(w) - 1$, and $f'(u) = f(u)$ for all $u \in V \setminus \{v, w\}$, has no undefended vertex. The weight of a WRDF is the value $f(V) = \sum_{u \in V(G)} f(u)$, and the *weak Roman domination number* $\gamma_r(G)$ is the minimum weight of a WRDF of G.

We note that a relation relating the three parameters defined above is given by the following chain of inequalities which can be found in [7]. For every graph G,

$$\gamma_r(G) \leq \gamma_{r2}(G) \leq \gamma_R(G). \tag{1}$$

Moreover, the authors [7] posed the following two problems.

Problem 1. Characterize the trees T satisfying $\gamma_r(T) = \gamma_{r2}(T)$.

Problem 2. Characterize the trees T satisfying $\gamma_r(T) = \gamma_R(T)$.

In this paper, we address these two problems by giving a constructive characterization of trees T with $\gamma_r(T) = \gamma_{r2}(T)$ or $\gamma_r(T) = \gamma_R(T)$. Before presenting our results, we mention that Alvarado, Dantas and Rautenbach [3] showed that the problem of deciding whether $\gamma_r(G) = \gamma_R(G)$ for a given graph G is NP-hard. In addition, they gave a characterization of trees T with strong equality between $\gamma_r(T)$ and $\gamma_R(T)$, that is, those trees for which every minimum WRDF is an RDF. In another paper, the same authors [4] show that it is NP-hard to decide whether $\gamma_{r2}(G) = \gamma_R(G)$ for a given connected K_4-free graph G. Clearly, because of the above, a solution of Problems 1 and 2 will be quite interesting even for the class of trees.

2 Preliminaries

In this section we provide some observations and definitions that will be useful throughout the paper.

Observation 2.1. Let H be a subgraph of a graph G. If $\gamma_r(H) = \gamma_{r2}(H)$, $\gamma_{r2}(G) \leq \gamma_{r2}(H) + s$ and $\gamma_r(G) \geq \gamma_r(H) + s$ for some non-negative integer s, then $\gamma_r(G) = \gamma_{r2}(G)$.

Proof. It follows from the assumptions and (1) that

$$\gamma_r(G) \geq \gamma_r(H) + s = \gamma_{r2}(H) + s \geq \gamma_{r2}(G) \geq \gamma_r(G),$$

and thus $\gamma_r(G) = \gamma_{r2}(G)$. □

Observation 2.2. Let H be a subgraph of a graph G. If $\gamma_r(G) = \gamma_{r2}(G)$, $\gamma_r(G) \leq \gamma_r(H) + s$ and $\gamma_{r2}(G) \geq \gamma_{r2}(H) + s$ for some non-negative integer s, then $\gamma_r(H) = \gamma_{r2}(H)$.

Proof. By (1) and the assumptions, we have

$$\gamma_{r2}(G) = \gamma_r(G) \leq \gamma_r(H) + s \leq \gamma_{r2}(H) + s \leq \gamma_{r2}(G)$$

and the desired result follows. □

Observation 2.3. Let H be a subgraph of a graph G. If $\gamma_r(H) = \gamma_R(H)$, $\gamma_R(G) \leq \gamma_R(H) + s$ and $\gamma_r(G) \geq \gamma_r(H) + s$ for some non-negative integer s, then $\gamma_r(G) = \gamma_R(G)$.

Proof. It follows from the assumptions and (1) that

$$\gamma_r(G) \geq \gamma_r(H) + s = \gamma_R(H) + s \geq \gamma_R(G) \geq \gamma_r(G),$$

and thus $\gamma_r(G) = \gamma_R(G)$. □

Observation 2.4. Let H be a subgraph of a graph G. If $\gamma_r(G) = \gamma_R(G)$, $\gamma_r(G) \leq \gamma_r(H) + s$ and $\gamma_R(G) \geq \gamma_R(H) + s$ for some non-negative integer s, then $\gamma_r(H) = \gamma_R(H)$.

Proof. By (1) and the assumptions, we have

$$\gamma_R(G) = \gamma_r(G) \leq \gamma_r(H) + s \leq \gamma_R(H) + s \leq \gamma_R(G)$$

and the desired result follows. □

We close this section with some definitions.

Definition 2.5. Let v be a vertex of a graph G. A function $f : V(G) \to \mathcal{P}(\{1,2\})$ is said to be an *almost* 2*-rainbow dominating function* (almost 2rDF) with respect to v, if for every vertex $x \in V(G) - \{v\}$ for which $f(x) = \emptyset$ we have $\cup_{u \in N(x)} f(u) = \{1,2\}$. Let

$$\gamma_{r2}(G; v) = \min\{\omega(f) \mid f \text{ is an almost } 2r\text{DF with respect to } v\}.$$

Observe that any 2rDF on G is an almost 2rDF with respect to any vertex of G. Therefore $\gamma_{r2}(G; v)$ is well-defined and $\gamma_{r2}(G; v) \leq \gamma_{r2}(G)$ for each $v \in V(G)$. Define $W_G^1 = \{v \in V(G) | \gamma_{r2}(G; v) = \gamma_{r2}(G)\}$.

Definition 2.6. Let v be a vertex of a graph G. A function $f : V(G) \to \{0,1,2\}$ is said to be an *almost weak Roman dominating function* (almost WRDF) with respect to v, if every vertex $x \in V(G) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) \geq 1$ such that the function $g : V(G) \to \{0,1,2\}$ defined by $g(x) = 1, g(y) = f(y) - 1$ and $g(z) = f(z)$ otherwise has no undefended vertex. Let

$$\gamma_r(G; v) = \min\{\omega(f) \mid f \text{ is an almost WRDF with respect to } v\}.$$

Observe that any WRDF on G is an almost WRDF with respect to any vertex of G. Therefore $\gamma_r(G; v)$ is well-defined and $\gamma_r(G; v) \leq \gamma_r(G)$ for each $v \in V(G)$. Define $W_G^2 = \{v \in V(G) \mid \gamma_r(G; v) = \gamma_r(G)\}$.

Definition 2.7. For a graph G and $v \in V(G)$, we say that v has property $\mathcal{P}$ in G if there exists a $\gamma_{r2}(G)$-function f such that $f(v) \neq \emptyset$. Let $W_G^3 = \{v \mid v \text{ has property } \mathcal{P} \text{ in } G\}$.

Definition 2.8. Let v be a vertex of a graph G. A function $f : V(G) \to \{0, 1, 2\}$ is said to be an *almost Roman dominating function* (almost RDF) with respect to v, if every vertex $x \in V(G) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) = 2$. Let

$$\gamma_R(G; v) = \min\{\omega(f) \mid f \text{ is an almost RDF with respect to } v\}.$$

Observe that any RDF on G is an almost RDF with respect to any vertex of G. Therefore $\gamma_R(G; v)$ is well-defined and $\gamma_R(G; v) \leq \gamma_R(G)$ for each $v \in V(G)$. Define $W^4_G = \{v \in V(G) \mid \gamma_R(G; v) = \gamma_R(G)\}$.

Definition 2.9. For a graph G and $v \in V(G)$, we say that v has property $\mathcal{Q}$ in G if there exists a $\gamma_R(G)$-function f such that $f(v) \neq 0$. Let $W^5_G = \{v \mid v \text{ has property } \mathcal{Q} \text{ in } G\}$.

3 Settlement of Problem 1

In this section we provide a constructive characterization of all trees T with $\gamma_r(T) = \gamma_{r2}(T)$. For this purpose, we define the family $\mathcal{T}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ $(m \geq 1)$ of trees such that T_1 is a path P_3, and, if $m \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{O}_1$. If $x \in V(T_i)$ and x is a strong support vertex, then T_{i+1} is obtained by adding a new vertex y attached by an edge xy.

Operation $\mathcal{O}_2$. If $x \in W^3_{T_i}$, then T_{i+1} is obtained by adding a path P_2 attached by an edge joining x and a leaf of P_2.

Operation $\mathcal{O}_3$. If $x \in W^1_{T_i} \cap W^2_{T_i}$, then T_{i+1} is obtained by adding a path P_3 attached by an edge joining x and the central vertex of P_3.

Operation $\mathcal{O}_4$. If $x \in V(T_i)$ is not a support vertex and is adjacent to a strong support vertex of T_i, then T_{i+1} is obtained by adding a new vertex y attached by an edge xy.

Operation $\mathcal{O}_5$. If $x \in V(T_i)$, then T_{i+1} is obtained by adding a star $K_{1,3}$ attached by an edge joining x and a leaf of $K_{1,3}$.

Lemma 3.1. If T_i is a tree with $\gamma_r(T_i) = \gamma_{r2}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$.

Proof. Clearly $\gamma_r(T_{i+1}) = \gamma_r(T_i)$ and $\gamma_{r2}(T_{i+1}) = \gamma_{r2}(T_i)$i, and thus $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$. □

Lemma 3.2. If T_i is a tree with $\gamma_r(T_i) = \gamma_{r2}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_2$ add a path $P_2 = yz$ and join x to y. Since $x \in W^3_{T_i}$, let f be a $\gamma_{r2}(T_i)$-function such that $f(x) \neq \emptyset$. Then f can be extended to a $2r$DF of T_{i+1} by assigning $\emptyset$ to y and $\{1\}$ (or $\{2\}$) to z, implying that $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i)+1$. Now let g be a $\gamma_r(T_{i+1})$-function. If $g(y) = 2$, then clearly $g(x) = 0$ and thus the function $h : V(T_i) \to \{0, 1, 2\}$ defined by $h(x) = 1$ and $h(u) = g(u)$ otherwise, is a WRDF of T_i. Hence $\gamma_r(T_i) \leq \omega(h) \leq \gamma_r(T_{i+1}) - 1$. If $g(y) \in \{0, 1\}$, then either $g(x) > 0$ or can be defended by one of its neighbors in T_i, and thus the restriction of g to T_i yields a WRDF of T_i. Hence $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 1$. By Observation 2.1, we obtain $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$. □

Lemma 3.3. If T_i is a tree with $\gamma_r(T_i) = \gamma_{r2}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_3$ add a path yzw and the edge xz. Then $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i)+2$ since any $\gamma_{r2}(T_i)$-function f can be extended to a $2r$DF of T_{i+1} by assigning $\{1, 2\}$ to z and $\emptyset$ to y and w. Now let g be a $\gamma_r(T_{i+1})$-function. Clearly we may assume that $g(z) \in \{0, 2\}$. If $g(z) = 0$, then $g(y) = g(w) = 1$ and so the restriction of g to T_i is a WRDF of T_i, yielding $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 2$. Hence we assume that $g(z) = 2$. Then the restriction of g to T_i is an almost WRDF of T_i with respect to x and since $x \in W^2_{T_i}$, we conclude that $\gamma_r(T_{i+1}) \geq \gamma_r(T_i; x) + 2 = \gamma_r(T_i) + 2$. Now the result follows by Observation 2.1. □

Lemma 3.4. If T_i is a tree with $\gamma_r(T_i) = \gamma_{r2}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_4$, then $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_4$ add a vertex y and the edge xy, and let z be the strong support vertex of T_i adjacent to x. Clearly, $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 1$ since any $\gamma_{r2}(T_i)$-function f can be extended to a $2r$DF of T_{i+1} by assigning $\{1\}$ to y. Now let g be a $\gamma_r(T_{i+1})$-function. Then $g(z) = 2$ an so $g(x) \in \{0, 1\}$. If $g(x) = 0$, then the restriction of g to T_i is a WRDF of T_i implying that $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 1$. If $g(x) = 1$, then $g(y) = 0$ and thus reassigning the values 0 and 1 to x and y instead of 1 and 0, respectively, brings us back to the previous situation, and so $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 1$. Now by Observation 2.1, we obtain $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$. □

Lemma 3.5. If T_i is a tree with $\gamma_r(T_i) = \gamma_{r2}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_5$, then $\gamma_r(T_{i+1}) = \gamma_{r2}(T_{i+1})$.

Proof. Let Operation $\mathcal{O}_5$ add a star $K_{1,3}$ centered at z and the edge xy, where y is a leaf of $K_{1,3}$. Clearly, $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 2$. Let g be a $\gamma_r(T_{i+1})$-function. Without loss of generality, we may assume that $g(z) = 2$. It follows that $g(u) = 0$ for every $u \in N(z)$ and thus the restriction of g to T_i is a WRDF of T_i. Hence $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 2$, and the desired result follows from Observation 2.1. □

We recall the following proposition from [10].

Proposition 3.6. Let G be a connected graph. If there is a path $v_3v_2v_1$ in G with $\deg(v_2) = 2$ and $\deg(v_1) = 1$, then G has a $\gamma_{r2}(G)$-function f such that $|f(v_1)| = 1$, $|f(v_3)| \geq 1$ and $f(v_1) \neq f(v_3)$.

Now we are ready to prove the main result of this section.

Theorem 3.7. Let T be a tree of order $n \geq 3$. Then $\gamma_r(T) = \gamma_{r2}(T)$ if and only if $T \in \mathcal{T}$.

Proof. First we prove the sufficiency. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that T_1 is P_3, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the aforementioned Operations.

We proceed by induction on the number of operations applied to construct T. If $k = 1$, then $T = P_3$ and $\gamma_r(P_3) = \gamma_{r2}(P_3) = 2$. Suppose that the result is true for each tree $T' \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_r(T') = \gamma_{r2}(T')$. Since $T = T_k$ is obtained from T' by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ or $\mathcal{O}_5$, we conclude from Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 that $\gamma_r(T) = \gamma_{r2}(T)$.

Now we prove the necessity. Let T be a tree with $\gamma_r(T) = \gamma_{r2}(T)$. We use an induction on the order n of T. If $n = 3$, then the only tree T of order 3 with $\gamma_r(T) = \gamma_{r2}(T)$ is P_3 that belongs to $\mathcal{T}$. Let $n \geq 4$ and let the statement hold for all trees T' of order less than n and $\gamma_r(T') = \gamma_{r2}(T')$. Let T be a tree of order n with $\gamma_r(T) = \gamma_{r2}(T)$ and let f be a $\gamma_r(T)$-function. If $\operatorname{diam}(T) = 2$, then T is a star belongs to $\mathcal{T}$ since it can be obtained from P_3 by applying Operation $\mathcal{O}_1$. If $\operatorname{diam}(T) = 3$, then T is a double star $DS_{p,q}$ $(q \geq p \geq 1)$ different from a path P_4 (since $2 = \gamma_r(P_4) < \gamma_{r2}(P_4) = 3$). Hence $q \geq 2$. If $p = 1$, then $T \in \mathcal{T}$ because it is obtained from P_3 by applying first Operation $\mathcal{O}_2$, and then Operation $\mathcal{O}_1$ so that the support vertex can have any number of leaves. If $p \geq 2$, then $T \in \mathcal{T}$ because it is obtained from P_3 by applying first Operation $\mathcal{O}_3$, and then Operation $\mathcal{O}_1$ so that the support vertices can have any number of leaves. Henceforth we may assume that $\operatorname{diam}(T) \geq 4$.

Let $v_1v_2 \ldots v_k$, with $k \geq 5$, be a diametral path in T such that $\deg(v_2)$ is as large as possible and root T at v_k. If $\deg_T(v_2) \geq 4$, then clearly $\gamma_r(T) = \gamma_r(T - v_1)$ and $\gamma_{r2}(T) = \gamma_{r2}(T - v_1)$, implying that $\gamma_r(T - v_1) = \gamma_{r2}(T - v_1)$. By the induction hypothesis on $T - v_1$, we have $T - v_1 \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ because it is obtained from $T - v_1$ by using Operation $\mathcal{O}_1$. Hence we can assume that $\deg_T(v_2) \leq 3$. We consider two cases.

Case 1. $\deg_T(v_2) = 3$. Consider the following subcases.
Subcase 1.1. v_3 has at least one child, say y, with depth 1. Clearly $\deg_T(y) \in \{2, 3\}$. Let $T' = T - T_{v_2}$. If $\deg_T(y) = 2$, then clearly, the restriction of any $\gamma_{r2}(T)$-function g satisfying the condition of Proposition 3.6, to T' is a $2r$DF of T' of weight $\gamma_{r2}(T) - 2$. If $\deg_T(y) = 3$, then there is a $\gamma_{r2}(T)$-function that assigns the set $\{1, 2\}$ to v_2 and y, and so the restriction of such a $\gamma_{r2}(T)$-function to T' is a $2r$DF of T' of weight $\gamma_{r2}(T) - 2$. In each case, we obtain $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$. Moreover, if h is a $\gamma_r(T')$-function, then it can be extended to a WRDF of T by assigning a 2 to v_2 and a 0

to its leaves yielding that $\gamma_r(T) \leq \gamma_r(T') + 2$. By Observation 2.2, we deduce that $\gamma_r(T') = \gamma_{r2}(T')$, and thus $\gamma_{r2}(T) = \gamma_{r2}(T')+2$ and $\gamma_r(T) = \gamma_r(T')+2$. By induction on T', we have $T' \in \mathcal{T}$. Next we shall show that $v_3 \in W^1_{T'} \cap W^2_{T'}$. Clearly, if $v_3 \notin W^1_{T'}$, then $\gamma_{r2}(T'; v_3) < \gamma_{r2}(T')$, and so any minimum almost $2r$DF of T' with respect to v_3 can be extended to a $2r$DF of T by assigning the sets $\{1, 2\}$ and $\emptyset$ to v_2 and its leaves, respectively. Hence $\gamma_{r2}(T) \leq \gamma_{r2}(T'; v_3) + 2 < \gamma_{r2}(T') + 2$, a contradiction. Hence $v_3 \in W^1_{T'}$. Likewise, if $v_3 \notin W^2_{T'}$, then $\gamma_r(T) \leq \gamma_r(T'; v_3)+2 < \gamma_r(T')+2$, which leads to a contradiction. Hence $v_3 \in W^2_{T'}$ and therefore $v_3 \in W^1_{T'} \cap W^2_{T'}$. Consequently, $T \in \mathcal{T}$ since it is obtained from T' by using Operation $\mathcal{O}_3$.

Subcase 1.2. v_3 is a support vertex. Assume first that v_3 has at least three leaves. Let T' be the tree obtained from T by removing a leaf neighbor of v_3. Note that v_3 remains a strong support vertex in T', and so one can check that $\gamma_r(T) = \gamma_r(T')$ and $\gamma_{r2}(T) = \gamma_{r2}(T')$. It follows that $\gamma_r(T') = \gamma_{r2}(T')$, and the induction on T' implies that $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Hence we can assume that v_3 has either one or two leaves.

Suppose that v_3 is adjacent to two leaves. Let $T' = T - T_{v_2}$. Obviously, $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$ and $\gamma_r(T) \leq \gamma_r(T') + 2$. Using the fact that $\gamma_r(T) = \gamma_{r2}(T)$, the previous inequalities imply that

$$\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2 \geq \gamma_r(T') + 2 \geq \gamma_r(T),$$

and thus $\gamma_{r2}(T) = \gamma_{r2}(T')+2$, $\gamma_r(T) = \gamma_r(T')+2$ and $\gamma_r(T') = \gamma_{r2}(T')$. By induction on T', we obtain that $T' \in \mathcal{T}$. Now using a similar argument to that used in Subcase 1.1 we can see that $v_3 \in W^1_{T'} \cap W^2_{T'}$. Therefore $T \in \mathcal{T}$ since it is obtained from T' by using Operation $\mathcal{O}_3$.

Finally, suppose that v_3 is adjacent to exactly one leaf, say w. Note in that case $\deg_T(v_3) = 3$. Let $T' = T - \{w\}$ and let g be a $\gamma_{r2}(T)$-function. Without loss of generality, we may assume that $|g(v_3)| \neq 1$. We also note that $g(v_2) = \{1, 2\}$, since v_2 has two leaves. Now if $g(v_3) = \emptyset$, then clearly $|g(w)| = 1$, and thus the restriction of g to T' is a $2r$DF of T' implying that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 1$. If $g(v_3) = \{1, 2\}$, then clearly $g(w) = g(v_4) = \emptyset$, and so the function $g' : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $g'(v_3) = \emptyset$, $g'(v_4) = \{1\}$ and $g'(x) = g(x)$ otherwise, is a $2r$DF of T' yielding also $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 1$. On the other hand, the inequality $\gamma_r(T) \leq \gamma_r(T') + 1$ follows from the fact that any $\gamma_r(T')$-function can be extended to a WRDF of T by assigning a 1 to w. Now by Observation 2.2, we deduce that $\gamma_r(T') = \gamma_{r2}(T')$. Using the induction on T', it follows that $T' \in \mathcal{T}$ which implies that $T \in \mathcal{T}$ since T can be obtained from T' by applying Operation $\mathcal{O}_4$.

Subcase 1.3. $\deg_T(v_3) = 2$. Let $T' = T - T_{v_3}$. Note that T' has order $n' \geq 2$, since $\text{diam}(T) \geq 4$. Moreover, $n' \neq 2$ for otherwise T is a tree of order 6 with $\gamma_r(T) = 3 < \gamma_{r2}(T) = 4$. Hence we assume that $n' \geq 3$. On the other hand, it is a simple matter to see that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$. Also, $\gamma_r(T) \leq \gamma_r(T') + 2$ since any $\gamma_r(T')$-function can be extended to a WRDF of T by assigning a 2 to v_2 and a 0 to every $u \in N(v_2)$. According to Observation 2.2, we obtain that $\gamma_r(T') = \gamma_{r2}(T')$. By induction on T', we have $T \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ because it is obtained from T' by applying Operation $\mathcal{O}_5$.

Case 2. $\deg(v_2) = 2$. Let $T' = T - T_{v_2}$ and let g be a $\gamma_{r2}(T)$-function. Without loss of generality, we may assume that $g(v_2) = \emptyset$ and $g(v_1) = \{1\}$ and $2 \in g(v_3)$. Thus the restriction of g to T' is a $2r$DF yielding $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 1$. On the other hand, we also have $\gamma_r(T) \leq \gamma_r(T') + 1$. From the assumption $\gamma_{r2}(T) = \gamma_r(T)$ and Observation 2.2, we conclude that $\gamma_r(T') = \gamma_{r2}(T')$ and thus $\gamma_{r2}(T) = \gamma_{r2}(T') + 1$. By induction on T', we have $T' \in \mathcal{T}$. Using the fact that $\gamma_{r2}(T) = \gamma_{r2}(T') + 1$, we deduce that $v_3 \in W^3_{T'}$. Therefore $T \in \mathcal{T}$ because it is obtained from T' by applying Operation $\mathcal{O}_2$. □

4 Settlement of Problem 2

In this section we provide a constructive characterization of all trees T with $\gamma_r(T) = \gamma_R(T)$. For this purpose, we define the family $\mathcal{F}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ $(m \geq 1)$ of trees such that T_1 is a path P_3, and, if $m \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{T}_1$. If $x \in V(T_i)$ and x is a strong support vertex, then T_{i+1} is obtained by adding a new vertex y attached by an edge xy.

Operation $\mathcal{T}_2$. If $x \in W^5_{T_i}$, then T_{i+1} is obtained by adding a path P_2 attached by an edge joining x and a leaf of P_2.

Operation $\mathcal{T}_3$. If $x \in W^2_{T_i} \cap W^4_{T_i}$, then T_{i+1} is obtained by adding a path P_3 attached by an edge joining x and the central vertex of P_3.

Operation $\mathcal{T}_4$. If $x \in V(T_i)$ is not a support vertex and is adjacent to a strong support vertex of T_i, then T_{i+1} is obtained by adding a new vertex y attached by an edge xy.

Operation $\mathcal{T}_5$. If $x \in V(T_i)$, then T_{i+1} is obtained by adding a star $K_{1,3}$ attached by an edge joining x and a leaf of $K_{1,3}$.

In the rest of the paper, we shall prove that for any tree T of order $n \geq 3$, $\gamma_r(T) = \gamma_R(T)$ if and only if $T \in \mathcal{F}$.

It worth mentioning that if T is a tree with $\gamma_r(T) = \gamma_R(T)$, then (1) implies that $\gamma_r(T) = \gamma_{r2}(T)$, and thus by Theorem 3.7, $T \in \mathcal{T}$. However, not every tree $T \in \mathcal{T}$ satisfies $\gamma_r(T) = \gamma_R(T)$. This can be seen by the path P_5, where $P_5 \in \mathcal{T}$ but $3 = \gamma_r(P_5) < \gamma_R(P_5) = 4$.

We will use the following lemmas.

Lemma 4.1. If T_i is a tree with $\gamma_r(T_i) = \gamma_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_1$, then $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$.

Proof. Clearly $\gamma_r(T_{i+1}) = \gamma_r(T_i)$ and $\gamma_R(T_{i+1}) = \gamma_R(T_i)$, and thus $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$. □

Lemma 4.2. If T_i is a tree with $\gamma_r(T_i) = \gamma_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_2$, then $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$.

Proof. Let $\mathcal{T}_2$ add a path $P_2 = yz$ and join x to y. Since $x \in W^5_{T_i}$, let f be a $\gamma_R(T_i)$-function such that $f(x) \neq 0$. Clearly, if $f(x) = 2$, then $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 1$. Hence assume that $f(x) = 1$. Then the function f' defined by $f'(z) = f'(x) = 0$, $f'(y) = 2$ and $f'(u) = f(u)$ for every $u \in V(T_i) - \{x\}$ is an RDF of T_{i+1} and thus $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 1$. In any case, $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 1$. Now let g be a $\gamma_r(T_{i+1})$-function. If $g(y) = 2$, then clearly $g(x) = 0$ and so the function $h : V(T_i) \to \{0, 1, 2\}$ defined by $h(x) = 1$ and $h(u) = g(u)$ otherwise, is a WRDF of T_i implying that $\gamma_r(T_i) \leq \omega(h) = \gamma_r(T_{i+1}) - 1$. If $g(y) = 0$ or 1, then x is defended by some vertex of $N[x] - y$, and so the restriction of g to T_i yields a WRDF of T_i and so $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 1$. By Observation 2.3, we obtain $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$. □

Lemma 4.3. If T_i is a tree with $\gamma_r(T_i) = \gamma_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_3$, then $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$.

Proof. Let $\mathcal{T}_3$ add a path yzw and the edge xz. Then $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$ since any $\gamma_R(T_i)$-function can be extended to an RDF of T_{i+1} by assigning a 2 to z and a 0 to y and w. Now let g be a $\gamma_r(T_{i+1})$-function. If $g(z) = 2$, then the restriction of g to T_i is an almost WRDF of T_i with respect to x and since $x \in W^2_{T_i}$, we deduce that $\gamma_r(T_{i+1}) \geq \gamma_r(T_i; x) + 2 = \gamma_r(T_i) + 2$. If $g(z) = 0$ then $g(y) = g(w) = 1$ and clearly the restriction of g to T_i is a WRDF of T_i, implying that $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 2$. The case $g(z) = 1$ is ignored since we can construct a $\gamma_r(T_{i+1})$-function that assigns a 2 to z by using the positive weight assigned to y or z. Now the desired result follows by Observation 2.3. □

Lemma 4.4. If T_i is a tree with $\gamma_r(T_i) = \gamma_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_4$, then $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$.

Proof. Let $\mathcal{T}_4$ add a vertex y and the edge xy. Obviously, $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 1$. Now let g be a $\gamma_r(T_{i+1})$-function. Note that we can assume that the strong support vertex adjacent to x in T_i is assigned a 2. Now, if $g(x) = 0$, then $g(y) = 1$ and the restriction of g to T_i is a WRDF of T_i implying that $\gamma_r(T_{i+1}) \geq \gamma_r(T_i)+1$. If $g(x) > 0$, then we can restrict the function g to T_i by assigning to x the value $g(x)-1$, yielding $\gamma_r(T_{i+1}) \geq \gamma_r(T_i) + 1$. Using Observation 2.3, the desired result follows. □

Lemma 4.5. If T_i is a tree with $\gamma_r(T_i) = \gamma_R(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_5$, then $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$.

Proof. Let $\mathcal{T}_5$ add a star $K_{1,3}$ centered at z and the edge xy, where y is a leaf of $K_{1,3}$. Clearly, $\gamma_R(T_{i+1}) \leq \gamma_R(T_i) + 2$. Let g be a $\gamma_r(T_{i+1})$-function. Note that $g(z) = 2$. If $g(y) = 0$, then the restriction of g to T_i is a WRDF of T_i of weight $\gamma_r(T_{i+1}) - 2$. If $g(x) = 1$, then we can restrict the function g to T_i by assigning 1 to x, yielding a WRDF of T_i of weight $\gamma_r(T_{i+1})-2$. In any case, $\gamma_r(T_{i+1}) \geq \gamma_r(T_i)+2$. By Observation 2.3, we obtain $\gamma_r(T_{i+1}) = \gamma_R(T_{i+1})$. □

Now we are ready to prove the main result of this section.

Theorem 4.6. *Let T be a tree of order $n \geq 3$. Then $\gamma_r(T) = \gamma_R(T)$ if and only if $T \in \mathcal{F}$.*

Proof. First we prove the sufficiency. Let $T \in \mathcal{F}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that T_1 is P_3, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the aforementioned Operations.

We proceed by induction on the number of operations applied to construct T. If $k = 1$, then $T = P_3$ and $\gamma_r(P_3) = \gamma_R(P_3) = 2$. .Suppose that the result is true for each tree of $\mathcal{F}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By induction on T', we have $\gamma_r(T') = \gamma_{r2}(T')$. Since $T = T_k$ is obtained from T' by one of the Operations $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, $\mathcal{T}_4$ and $\mathcal{T}_5$, we conclude from Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5 that $\gamma_r(T) = \gamma_R(T)$.

Now we prove the necessity. Let T be a tree with $\gamma_r(T) = \gamma_R(T)$. We proceed by induction on n. If $n = 3$, then $T = P_3$ and clearly $P_3 \in \mathcal{F}$. Let $n \geq 4$ and assume that for every tree T' of order n', with $3 \leq n' < n$ such that $\gamma_r(T') = \gamma_R(T')$, we have $T' \in \mathcal{F}$. Let T be a tree of order n with $\gamma_r(T) = \gamma_R(T)$. If $\mathrm{diam}(T) = 2$, then T is a star that belongs to $\mathcal{F}$ since it can be obtained from P_3 by applying Operation $\mathcal{T}_1$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{p,q}$ ($q \geq p \geq 1$) different from a path P_4 (since $\gamma_r(P_4) < \gamma_R(P_4)$). Hence $q \geq 2$. If $p = 1$, then $T \in \mathcal{F}$ because it is obtained from P_3 by applying first Operation $\mathcal{T}_2$, and then Operations $\mathcal{T}_1$. If $p \geq 2$, then $T \in \mathcal{F}$ because it is obtained from P_3 by applying first Operation $\mathcal{T}_3$, and then Operation $\mathcal{T}_1$ so that the support vertices can have any number of leaves. Henceforth we assume that $\mathrm{diam}(T) \geq 4$.

Let $v_1 v_2 \ldots v_k$ ($k \geq 5$) be a diametral path in T such that $\deg(v_2)$ is as large as possible and root T at v_k. If $\deg_T(v_2) \geq 4$, then $\gamma_r(T) = \gamma_r(T - v_1)$ and $\gamma_R(T) = \gamma_R(T - v_1)$ and thus $\gamma_r(T - v_1) = \gamma_R(T - v_1)$. By induction on $T - v_1$, we have $T - v_1 \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it is obtained from $T - v_1$ by using Operation $\mathcal{T}_1$. Hence we assume that $\deg_T(v_2) \in \{2, 3\}$. We consider two cases.

Case 1. $\deg_T(v_2) = 3$. We consider the following subcases.
Subcase 1.1. v_3 has at least one child besides v_2, say u_2, which is a support vertex. Let $T' = T - T_{v_2}$. First Suppose that g is a $\gamma_R(T)$-function with a maximum number of vertices assigned a 2. Then either $g(u_2) = 2$ or $g(v_3) > 0$ and the leaf neighbor of u_2 is assigned a positive value. In any case, the restriction of g to T' is an RDF of T', and thus $\gamma_R(T) \geq \gamma_R(T') + 2$. On the other hand, we have $\gamma_r(T) \leq \gamma_r(T') + 2$. By Observation 2.4, we obtain $\gamma_r(T') = \gamma_R(T')$. It follows that $\gamma_R(T) = \gamma_R(T') + 2$ and $\gamma_r(T) = \gamma_r(T') + 2$. Moreover, since $\gamma_r(T') = \gamma_R(T')$, by induction on T', we have $T' \in \mathcal{F}$. In the next we shall show that $v_3 \in W^2_{T'} \cap W^4_{T'}$. It is a simple matter to see that $v_3 \in W^4_{T'}$, and hence we only show that $v_3 \in W^2_{T'}$. Suppose, to the contrary, that $v_3 \notin W^2_{T'}$, and let h be a minimum almost WRDF of T' with respect to v_3. Then h can be extended to WRDF of T by assigning a 2 to v_2 and a 0 to its leaves, which implies that $\gamma_r(T) \leq \gamma_r(T'; v_3) + 2 < \gamma_r(T') + 2$, a contradiction. Hence $v_3 \in W^2_{T'}$ and therefore $v_3 \in W^2_{T'} \cap W^4_{T'}$. Consequently, $T \in \mathcal{F}$ because it can be obtained from

T' by applying Operation $\mathcal{T}_3$.

Subcase 1.2. v_3 is a support vertex. We first assume that v_3 has at least three leaves. Let T' be the tree obtained from T by deleting a leaf neighbor of v_3. Hence v_3 remains a strong support vertex in T', and thus $\gamma_r(T) = \gamma_r(T')$, and $\gamma_R(T) = \gamma_R(T')$. It follows that $\gamma_r(T') = \gamma_R(T')$, and so $T' \in \mathcal{F}$. Therefore $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{T}_1$. Hence we can assume that v_3 is a support vertex with at most two leaves.

Suppose that v_3 is adjacent to two leaves. Let $T' = T - T_{v_2}$. Then $\gamma_R(T) \geq \gamma_R(T') + 2$ and $\gamma_r(T) \leq \gamma_r(T') + 2$. It follows that

$$\gamma_r(T) = \gamma_R(T) \geq \gamma_R(T') + 2 \geq \gamma_r(T') + 2 \geq \gamma_r(T),$$

and thus $\gamma_R(T) = \gamma_R(T') + 2$, $\gamma_r(T) = \gamma_r(T') + 2$ and $\gamma_r(T') = \gamma_R(T')$. By induction on T', we obtain that $T' \in \mathcal{F}$. Using the same argument as in Subcase 1.1, we can show that $v_3 \in W^2_{T'} \cap W^4_{T'}$. Therefore $T \in \mathcal{F}$ since it can be obtained from T' by using Operation $\mathcal{T}_3$.

Suppose now that v_3 is adjacent to exactly one leaf, say w. Seeing the previous cases, we have $\deg_T(v_3) = 3$. Let $T' = T - \{w\}$. Clearly, $\gamma_r(T) \leq \gamma_r(T') + 1$. Let g be a $\gamma_R(T)$-function. We may assume that $g(v_2) = 2$ and thus $g(v_3) \neq 1$. If $g(v_3) = 0$, then clearly $g(w) = 1$ and the restriction of g to T' is an RDF of T' implying that $\gamma_R(T) \geq \gamma_R(T') + 1$. If $g(v_3) = 2$, then clearly $g(w) = g(v_4) = 0$, and so the function $g' : V(T') \to \{0, 1, 2\}$ defined by $g'(v_3) = 0$, $g'(v_4) = 1$ and $g'(u) = g(u)$ otherwise, is an RDF of T' yielding $\gamma_R(T) \geq \gamma_R(T') + 1$. By Observation 2.4, we have $\gamma_r(T') = \gamma_R(T')$ and so $T' \in \mathcal{F}$. Therefore $T \in \mathcal{F}$ since it can be obtained from T' by using Operation $\mathcal{T}_4$.

Subcase 1.3. $\deg_T(v_3) = 2$. Let $T' = T - T_{v_3}$. Using the facts that $\mathrm{diam}(T) \geq 4$ and $\gamma_r(T) = \gamma_R(T)$ one can see that T' has order at least three. Since there is a $\gamma_R(T)$-function g that assigns a 2 to v_2 and a 0 to every neighbor of v_2, the restriction of g to T' yields $\gamma_R(T) \geq \gamma_R(T') + 2$. Also, $\gamma_r(T) \leq \gamma_r(T') + 2$. By Observation 2.4, $\gamma_r(T') = \gamma_R(T')$ and thus $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it can be obtained from T' by applying Operation $\mathcal{T}_5$.

Case 2. $\deg_T(v_2) = 2$. Let $T' = T - T_{v_2}$. Clearly $\gamma_r(T) \leq \gamma_r(T') + 1$. Let g be a $\gamma_R(T)$-function with maximum number of vertices assigned a 2. The choice of g implies that $g(v_2) \in \{2, 0\}$. If $g(v_2) = 2$, then the function $h : V(T') \to \{0, 1, 2\}$ defined by $h(v_3) = \min\{2, g(v_3) + 1\}$ and $h(u) = g(u)$ otherwise, is an RDF of T' implying that $\gamma_R(T) \geq \gamma_R(T') + 1$. If $g(v_2) = 0$, then we must have $g(v_1) = 1$ (else we can change the assignments of v_1 and v_2 to be in the previous situation). Hence $g(v_3) = 2$ and the restriction of g to T' yields also $\gamma_R(T) \geq \gamma_R(T') + 1$. It follows that $\gamma_r(T) = \gamma_R(T) \geq \gamma_R(T') + 1 \geq \gamma_r(T') + 1 \geq \gamma_r(T)$ and thus we have equality throughout this inequality chain. In particular, $\gamma_r(T') = \gamma_R(T')$ and $\gamma_R(T) = \gamma_R(T') + 1$. By induction on T', we have $T' \in \mathcal{F}$. Also, $\gamma_R(T) = \gamma_R(T') + 1$ implies that $v_3 \in W^5_{T'}$ (according to the restriction of g to T'). It follows that $T \in \mathcal{F}$ because it is obtained from T' by applying Operation $\mathcal{T}_2$. □

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