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The combinatorics of rim hook tableaux

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Abstract

We give a new combinatorial proof that the coefficient of the power symmetric function p_μ in the Schur symmetric function s_λ can be expressed in terms of rim hook tableaux (the Murnaghan-Nakayama rule). Two other known identities involving the coefficient of p_μ in the homogeneous symmetric function h_λ and the coefficient of p_μ in the elementary symmetric function e_λ are proved in similar ways.

1 Introduction

Let s_λ be the Schur symmetric function corresponding to the integer partition $\lambda \vdash n$ and let p_λ be the power symmetric function corresponding to λ. Since the sets $\{s_\lambda : \lambda \vdash n\}$ and $\{p_\lambda : \lambda \vdash n\}$ are bases for symmetric functions of degree n, there are coefficients χ^λ_μ such that

$$p_\mu = \sum_{\lambda \vdash n} \chi^\lambda_\mu s_\lambda \tag{1}$$

for all $\mu \vdash n$. A theorem of Frobenius says that χ^λ_μ is the value of the irreducible character of the symmetric group S_n corresponding to λ on the conjugacy class C_μ of permutations with cycle type μ.

The Murnaghan-Nakayama rule is a method of finding the values of χ^λ_μ by understanding combinatorial objects called rim hook tableaux (also known as border strip tableaux). This rule can be found in the work of Littlewood and Richardson [2] and in the work of Murnaghan [8] before the rule was independently found by Nakayama [9].

Together with the Frobenius formula, the Murnaghan-Nakayama rule has become a standard method for finding χ^λ_μ. It appears in most treatments of symmetric functions and the representation theory of the symmetric group. There are two proofs traditionally given for the rule; one is found in either [6, 11] and the other in either [1, 10]. The latter proof uses more machinery from representation theory than the former.

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More recent proofs include a remarkable proof using labeled abaci [4, 5] and a combinatorial proof that uses the Pieri rules [3, 13]. These proofs are recounted in [7].

The Schur basis $\{s_\lambda : \lambda \vdash n\}$ is orthonormal with respect to the Hall inner product. This inner product also satisfies

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} z_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

where $z_\lambda = n!/|C_\lambda|$. Thus using Eq. (1), the coefficient of p_μ in s_λ is equal to $\langle s_\lambda, p_\mu/z_\mu \rangle = \chi^\lambda_\mu / z_\mu$, showing that Eq. (1) is equivalent to the identity

$$s_\lambda = \sum_{\mu \vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu}. \tag{2}$$

We will provide an alternative proof of Eq. (2), thereby providing an alternative route to the Murnaghan-Nakayama rule. Here we are interpreting χ^λ_μ as a signed sum of rim hook tableaux (as opposed to interpreting it as the value of an irreducible character). An impetus for this work was a question posed to the author by Jeff Remmel when they were working on [7]. This paper is dedicated to his memory.

There are several advantages to our approach:

1. Our ideas are purely combinatorial, using straightforward sign reversing involutions and bijections.

   The amount of mathematical overhead needed is minimal (other proofs of the identity require knowledge of representation theory, Littlewood-Richardson coefficients, or the Pieri rules). The elementary nature of the argument is a feature in itself.

2. The ideas we present are closely related to other identities that express the homogeneous and elementary symmetric functions in terms of the power symmetric functions. These identities also involve sums of the form $\sum_\mu \varphi(\mu) p_\mu / z_\mu$ where φ is the character of a representation of the symmetric group.

3. Our proof is valid for an infinite number of indeterminates.

   With the exception of the Pieri rule proof, the other three proofs of the rule require the "classical" definition of the Schur symmetric function given by

$$s_\lambda(x_1, \ldots, x_n) = \Delta_\lambda(x_1, \ldots, x_n)/\Delta_{(0)}(x_1, \ldots, x_n)$$

   where $\Delta_\lambda(x_1, \ldots, x_n) = \det \|x_i^{\lambda_j + n - j}\|_{i,j=1,\ldots,n}$ for any $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$ (with zero parts allowed as to make the length of λ equal to n). This definition requires that Schur functions be defined for only a finite number of variables n before the result can be extended to an infinite number of variables by letting n approach infinity.

4. Properties of the Frobenius map can be more readily understood.

   Although the identities in Eq. (1) and Eq. (2) can be found relatively easily from one another using the Hall inner product, the identity in Eq. (2) is the more natural one when it comes to developing properties of the Frobenius characteristic map. All of the other published proofs of the Murnaghan-Nakayama rule prove Eq. (1), but it may actually be more useful to have a direct proof of Eq. (2).

   In more detail, let 1_{C_μ} be the indicator function for the conjugacy class C_μ. The Frobenius characteristic map F can be defined to be the isomorphism from class functions to symmetric functions that satisfies $F(1_{C_\mu}) = p_\mu/z_\mu$. With this definition, Theorems 3.1 through 3.3 of this paper are all phrased in a way that allow for $F^{-1}(h_\lambda)$, $F^{-1}(e_\lambda)$, and $F^{-1}(s_\lambda)$ to be easily identified.

5. A combinatorial proof that $\omega(s_\lambda) = s_{\lambda'}$ can be given.

   The omega involution ω can be defined by $\omega(p_n) = (-1)^{n-1}p_n$. The identity as stated in Eq. (2) allows for a direct proof only using tableaux that $\omega(s_\lambda) = s_{\lambda'}$ where λ' is the conjugate partition to λ.

The next section of this paper reviews the definitions needed for our work. The next section also contains a combinatorial proof that $\chi^\lambda_\mu = \chi^\lambda_\nu$ for all rearrangements ν of μ, a result first found in [12]. We are choosing to include a variant on the proof in [12] because the result is needed in our forthcoming Theorem 3.3 and because the ideas in the proof do not seem to be as well known as they could be. Our retelling is slightly different and is able to avoid invoking the Garsia-Milne involution principle.

Our main results can be found in the third section of this paper where we prove three identities involving the characters of representations of the symmetric group and the power symmetric functions in strikingly similar ways, using purely combinatorial arguments. Theorems 3.1 and 3.2 are proved for an arbitrary integer partition λ, which generalize the proofs of Theorems 2.11 and 2.12 in [7] where only the case $\lambda = (n)$ was considered.

2 Preliminaries

A tableau of shape λ is a filling of the cells in the Young diagram of the integer partition λ with positive integers. We will draw such Young diagrams using the English convention where rows weakly decrease in length reading top to bottom. The weight of a tableau T is

$$w(T) = \prod_{\text{cells } c \text{ in } T} x_{T_c}$$

where T_c is the integer in the cell c in T. A tableau T of shape $\lambda \vdash n$ is called

a. row constant if the integers in each row of T are all the same,

b. row nondecreasing if the integers in each row of T are nondecreasing when read from left to right,

c. row increasing if the integers in each row of T are increasing when read from left to right,

d. column strict if T is row nondecreasing and the integers strictly increase down each column, and

e. a tabloid if T is row increasing and if each of the integers $1, \ldots, n$ appears exactly once in T.

Important symmetric functions can be defined in terms of tableaux. For any integer partition λ, we define

a. the power symmetric function p_λ to equal $\sum w(T)$ where the sum runs over all row constant tableaux T of shape λ,

b. the homogeneous symmetric function h_λ to equal $\sum w(T)$ where the sum runs over all row nondecreasing tableaux T of shape λ,

c. the elementary symmetric function e_λ to equal $\sum w(T)$ where the sum runs over all row increasing tableaux T of shape λ, and

d. the Schur symmetric function s_λ to equal $\sum w(T)$ where the sum runs over all column strict tableaux T of shape λ.

For example, all column strict tableaux of shape $(4, 2, 1)$ with weight $x_1^3 x_2^2 x_3 x_4$ are

1	1	1	4
2	2		
3			

1	1	1	3
2	2		
4			

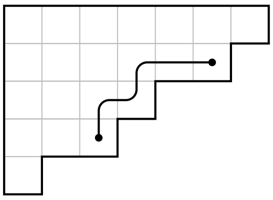
1	1	1	2
2	4		
3			

1	1	1	2
2	3		
4			

and so the coefficient of $x_1^3 x_2^2 x_3 x_4$ in $s_{(4,2,1)}$ is 4.

The goal of this paper is to provide new combinatorial proofs describing how to write the homogeneous, elementary, and Schur symmetric functions in terms of the power symmetric functions. This will involve combinatorial objects known as rim hooks and rim hook tableaux.

A rim hook of length k is a sequence of k connected cells in the Young diagram of an integer partition that begins in a cell on the southeast boundary and travels up along the southeast edge such that its removal leaves the Young diagram of a smaller integer partition. The sign of a rim hook ρ, denoted $\operatorname{sign} \rho$, is $(-1)^{(\text{the number of rows spanned by } \rho)-1}$. For example, a rim hook of length 6 with sign $(-1)^{3-1} = +1$ is shown below inside of the Young diagram of the integer partition $(7, 6, 4, 3, 1)$:

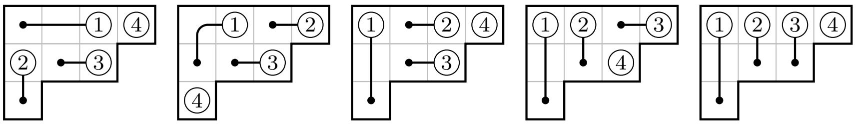


Let λ be an integer partition of n and let $\mu = (\mu_1, \dots, \mu_\ell)$ be a composition of n. A rim hook tableau of shape λ and content μ is a filling of the cells of the Young diagram of λ with rim hooks of lengths $\mu_1, \dots, \mu_\ell$ labeled with $1, \dots, \ell$ such that the removal of the last i rim hooks leaves the Young diagram of a smaller integer partition for all i. The sign of a rim hook tableau T is the product of all of the signs of the rim hooks in T. We let

$$\chi^\lambda_\mu = \sum_{\substack{T \text{ is a rim hook tableau} \\ \text{of shape } \lambda \text{ and content } \mu}} \operatorname{sign} T.$$

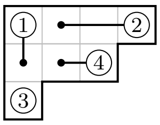
The integers χ^λ_μ are of interest because they provide the value of the irreducible character of S_n corresponding to the integer partition λ on permutations with cycle type μ. See one of [1, 6, 10] for details.

For example, all of the rim hook tableaux of shape $(4, 3, 1)$ and content $(3, 2, 2, 1)$ are given in Eq. (3).



(3)

The signs of these rim hook tableaux are -1, -1, $+1$, -1, and $+1$, and so $\chi^{(4,3,1)}_{(3,2,2,1)} = -1$. As another example, we have $\chi^{(4,3,1)}_{(2,3,1,2)} = -1$ because the only rim hook tableau of shape $(4, 3, 1)$ and content $(2, 3, 1, 2)$ is shown in Eq. (4).



(4)

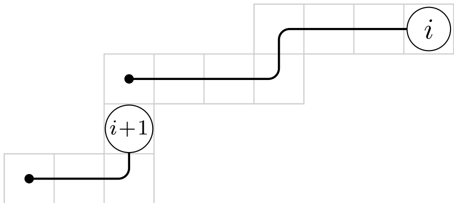
The next theorem shows that it is no coincidence that $\chi^{(4,3,1)}_{(3,2,2,1)} = \chi^{(4,3,1)}_{(2,3,1,2)}$. The combinatorial arguments given in the proof are essentially the same as those described in [12].

Theorem 2.1. *If λ is an integer partition and μ and ν are compositions of n that are rearrangements of one another, then $\chi^\lambda_\mu = \chi^\lambda_\nu$.*

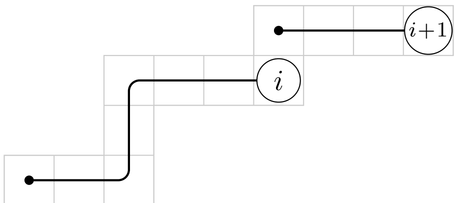
Proof. The result will follow if we can show that the order of two consecutive rim hooks can be interchanged. Then, by interchanging consecutive parts repeatedly, the composition μ can be turned into any rearrangement ν of μ. Therefore we take μ

to be the composition $(\mu_1, \ldots, \mu_i, \mu_{i+1}, \ldots, \mu_\ell)$ and take ν to be the composition μ with μ_i and μ_{i+1} interchanged. We will prove that $\chi^\lambda_\mu = \chi^\lambda_\nu$ by first defining a sign reversing involution I on the set of rim hook tableaux of shape λ and content μ.

Let T be a rim hook tableau of shape λ and content μ and let C be the cells in T that are occupied by rim hooks i and $i+1$. If C is itself a single rim hook, then let T' be the rim hook tableau found by switching the positions of rim hooks i and $i+1$ in T within C. For example, suppose rim hooks i and $i+1$ appear in T as shown below.

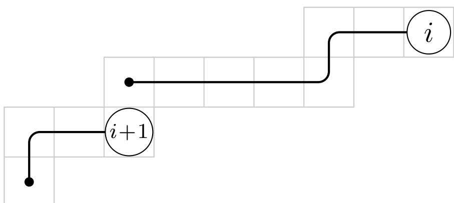


Then rim hooks i and $i+1$ appear in T' as shown here:

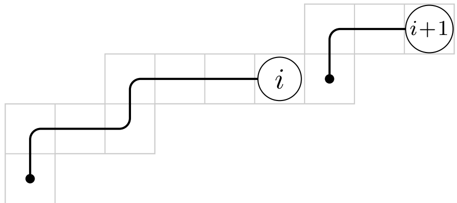


Rim hook $i+1$ lies outside of the rim hook i in T. By outside, we mean that every cell in rim hook $i+1$ is entirely south of rim hook i or that rim hook $i+1$ is entirely east of rim hook i. If rim hook $i+1$ still lies outside of rim hook i in T', then set $I(T) = T'$. If this does not happen (as depicted above) or if C is not a single rim hook, set $I(T) = T$.

Suppose $I(T) \neq T$ and suppose that rim hook $i+1$ appears before rim hook i when reading left to right in T. Since rim hook $i+1$ lies outside of rim hook i in T, the last cell (the most north and east cell) of rim hook $i+1$ must appear below the first cell (the most south and west cell) of rim hook i in T. This situation is depicted below.



Rim hook $i+1$ also appears outside of rim hook i in $I(T)$, which means that the last cell of rim hook i must appear to the left of the first cell in rim hook $i+1$ in $I(T)$. This situation in $I(T)$ is depicted below.



The gap between the rim hooks i and $i+1$ in T spans two rows while the gap between the rim hooks i and $i+1$ in $I(T)$ does not. The sign of a rim hook changes by (-1) every time a rim hook has a vertical line segment, and since the total number of cells in C is constant, this means that $\operatorname{sign} T \neq \operatorname{sign} I(T)$. Therefore I is a sign reversing involution and

$$\chi^\lambda_\mu = \sum_{\substack{T \text{ is a rim hook tableau of shape } \lambda \\ \text{and content } \mu \text{ such that } I(T)=T}} \operatorname{sign} T.$$

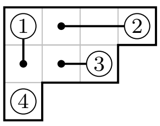
By applying the same involution I to rim hooks i and $i+1$ found in rim hook tableaux of shape λ but with content ν, we also have

$$\chi^\lambda_\nu = \sum_{\substack{T \text{ is a rim hook tableau of shape } \lambda \\ \text{and content } \nu \text{ such that } I(T)=T}} \operatorname{sign} T.$$

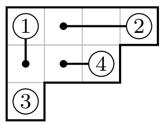
Therefore to complete the proof we will define a sign preserving bijection B from the set of rim hook tableaux T with shape λ, content μ, and $I(T)=T$ to the set of rim hook tableaux T' with shape λ, content ν, and $I(T')=T'$. This bijection will be defined on a case by case basis.

Case 1: Rim hooks i and $i+1$ do not share a border. Since either rim hook i can be considered to lie outside of rim hook $i+1$ or vice versa, we define $B(T)$ to be the tableau found by switching the labels of i and $i+1$ in T.

For example, if $i=3$, the bijection B sends the rim hook tableau



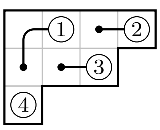
to the rim hook tableau



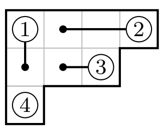
.

Case 2: The cells C form a single rim hook. Define $B(T)$ to be the tableau found by switching the labels of i and $i+1$ in T' where T' is the rim hook tableau described in the definition of the involution I. Since $I(T)=T$, rim hook $i+1$ lies outside of i in $B(T)$, so $B(T)$ is indeed a rim hook tableau with content ν. It follows that $I(B(T))=B(T)$ and, in this case, that B is a bijection.

For example, if $i=1$, the bijection B sends the rim hook tableau



to the rim hook tableau



.

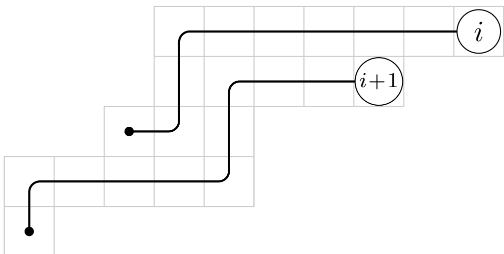
The next two paragraphs will show that the signs of T and $B(T)$ are the same.

If rim hook $i+1$ appears before rim hook i when reading left to right in T, then the last cell of rim hook $i+1$ must appear below the first cell of rim hook i in T. Since $I(T) = T$, rim hook $i+1$ does not lie outside of rim hook i in T'. Therefore the last cell of rim hook i must appear below the first cell of rim hook $i+1$ in T'. This verifies that $\operatorname{sign} T = \operatorname{sign} B(T)$ if rim hook $i+1$ appears before rim hook i in T.

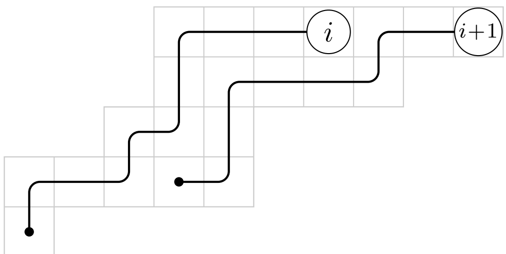
If rim hook i appears before rim hook $i+1$ in T, then the last cell of rim hook i must appear to the left of the first cell of rim hook $i+1$ in T. Since $I(T) = T$, rim hook $i+1$ does not lie outside of rim hook i in T', implying that the last cell of rim hook $i+1$ must also appear to the left of the first cell of rim hook i in T'. This verifies that $\operatorname{sign} T = \operatorname{sign} B(T)$ if rim hook i appears before rim hook $i+1$ in T.

We have now shown that B is sign preserving in this case.

Case 3: Not Case 1 *and not Case* 2. Rim hooks i and $i+1$ share a border of length 2 or longer in T. Rim hook $i+1$ must trace the outside border of C, staring from the southwest end, as shown below.



Define $B(T)$ to be the rim hook tableau T created by drawing the rim hook i to follow along the outside cells of C instead of rim hook $i+1$ following along the outside cells C and then switching the i and $i+1$ labels. Because rim hooks i and $i+1$ are consecutively placed within T, this will always be possible. For example, the image of the above pair of rim hooks is shown below.



Given $B(T)$, we can easily reconstruct T, showing that B is a bijection. The sign of the rim hook tableau changes exactly twice when turning T into $B(T)$: once at the first cell where rim hooks i and $i+1$ border each other and once at the last cell where rim hooks i and $i+1$ border each other. Therefore the signs of T and $B(T)$ are the same.

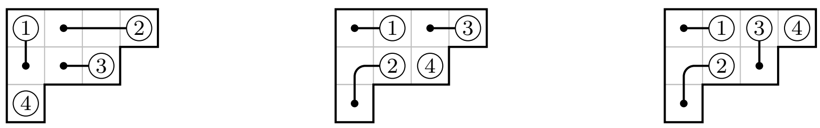
This completes our description of the sign reversing involution I and the bijection B, finishing the proof. $\square$

To illustrate the involution I and the bijection B in the proof of Theorem 2.1, suppose we wish to show

$$\chi^{(4,3,1)}_{(3,2,2,1)} = \chi^{(4,3,1)}_{(2,3,1,2)}$$

by starting with a rim hook tableau with content $(3, 2, 2, 1)$, switching rim hooks 1 and 2, and then switching rim hooks 3 and 4.

In switching rim hooks 1 and 2, the involution I pairs the first and third rim hook tableaux as displayed in Eq. (3). The bijection B sends the second, fourth, and fifth rim hook tableaux in Eq. (3) to these three rim hook tableaux:



Then, to switch the rim hooks 3 and 4, the involution I pairs the last two of the above rim hook tableaux before the bijection B sends the remaining rim hook tableau to the rim hook tableau in Eq. (4).

3 Combinatorial proofs of identities involving the power symmetric functions

The symmetric group acts on the set of tabloids of shape $\lambda \vdash n$ by using $\sigma \in S_n$ to permute the elements in T and by then sorting the integers in each row of T into increasing order. Given $\sigma \in S_n$, let $\varphi^\lambda(\sigma)$ be the number of tabloids T of shape λ that satisfy $\sigma T = T$. For example, $\varphi^{(3,2,1)}\big((1\ 5)(2\ 6)\big) = 4$; here are the 4 tabloids of shape $(3, 2, 1)$ that are fixed by $(1\ 5)(2\ 6)$:

1	3	5
2	6	
4		

2	3	6
1	5	
4		

1	4	5
2	6	
3		

2	4	6
1	5	
3		

The function φ^λ is constant on the conjugacy class C_μ containing the permutations with cycle type μ, so we let φ^λ_μ denote the value of φ^λ on C_μ. Furthermore, if S_λ is the Young subgroup of S_n and 1 is the trivial representation of S_λ, then φ^λ is the character of the induced representation $1{\uparrow}^{S_n}_{S_\lambda}$. See [1, 10] for details.

Theorem 3.1. *We have* $h_\lambda = \displaystyle\sum_{\mu \vdash n} \varphi^\lambda_\mu \frac{p_\mu}{z_\mu}$.

Proof. Multiplying through by $n!$, we will prove

$$n! h_\lambda = \sum_{\mu \vdash n} |C_\mu| \varphi^\lambda_\mu p_\mu = \sum_{\sigma \in S_n} \varphi^\lambda(\sigma)\, p_{\mu(\sigma)} \tag{5}$$

where $\mu(\sigma)$ denotes the cycle type of σ.

Count the right hand side of Eq. (5) by first selecting a permutation $\sigma \in S_n$.

To account for the $p_{\mu(\sigma)}$ term, assign an integer c to each cycle of σ. Inspired by the use of the power symmetric functions in Pólya theory, this c will be called the color of the cycle. Choosing a color for each cycle is the same as choosing a row-constant tableau with shape $\mu(\sigma)$, where we associate each cycle with a row and fill each cell in a given row with the color c of the corresponding cycle.

Lastly, select a tabloid T of shape λ such that $\sigma T = T$. The choice of σ, the colors c for each cycle, and the tabloid T account for all terms on the right hand side of Eq. (5).

Write σ in cycle notation such that the maximum integer in each cycle appears first. Arrange the colored cycles of σ such that smaller colors appear first and, if two cycles have the same color, write the cycle with the smaller maximum element first. Since $\sigma T = T$, the integers in each cycle of σ must appear in a single row of T. Place the cycles of σ using this order into the rows of the Young diagram of λ to indicate which row of T contains the integers in each cycle. Let $\mathcal{U}$ be the set of objects created in this way.

For example, if $\lambda = (7, 4, 3)$ and σ is the permutation

$$(7\ \ 6)_1\ \ (10)_1\ \ (12\ \ 3\ \ 9)_1\ \ (1)_2\ \ (5\ \ 2)_2\ \ (13\ \ 4\ \ 11)_2\ \ (14\ \ 8)_3$$

where the color of each cycle is denoted by the subscript, then one $U \in \mathcal{U}$ is shown in Eq. (6).

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
(10)_1 & (12 & 3 & 9)_1 & (1)_2 & (14 & 8)_3 \\
\hline
(7 & 6)_1 & (5 & 2)_2 \\
\cline{1-4}
(13 & 4 & 11)_2 \\
\cline{1-3}
\end{array} \tag{6}$$

If we define the weight of $U \in \mathcal{U}$ to be

$$w(U) = \prod_{i=1}^{n} x_{(\text{the color of the cycle containing } i \text{ in } U)},$$

then by construction, the right side of Eq. (5) is equal to $\sum_{U \in \mathcal{U}} w(U)$. In the above example, $w(U) = x_1^6 x_2^6 x_3^2$.

Given a $U \in \mathcal{U}$, create a permutation τ by listing the integers in U without parentheses, reading each row from left to right beginning with the top row. Create a tableau T' by recording the color of each cell in a tableau of shape λ. In this way, each $U \in \mathcal{U}$ is in a natural 1–1 correspondence with a pair of the form (τ, T') where $\tau \in S_n$ and T' is a tableau of shape λ with nondecreasing rows. For example, the above U corresponds to (τ, T') where

$$\tau = 10\ \ 12\ \ 3\ \ 9\ \ 1\ \ 14\ \ 8\ \ 7\ \ 6\ \ 5\ \ 2\ \ 13\ \ 4\ \ 11$$

and the corresponding T' is shown here:

T' =

1	1	1	1	2	3	3
1	1	2	2			
2	2	2				

These pairs are counted by $n!h_\lambda$, the left hand side of Eq. (5), as needed.

This weight preserving correspondence between $U \in \mathcal{U}$ and pairs (τ, T') is a bijection because the placement of the parentheses in σ can be reconstructed. Indeed, after τ is placed into the Young diagram of shape λ, there is a unique way to insert pairs of parentheses into each row so that the maximum element in each cycle appears first and cycles of the same color are sorted in increasing order according to maximum element.

This unique way is to locate the maximum integer in each row in τ with color c, to place that integer and everything to its right that also has color c in one cycle, and to iterate. As an example, if

$$\tau = 6 \quad 3 \quad 10 \quad 12 \quad 11 \quad 1 \quad 7 \quad 14 \quad 13 \quad 2 \quad 4 \quad 9 \quad 5 \quad 8$$

and T' is as shown below,

T' =

1	1	2	2	2	3	3
2	2	2	2			
1	1	1				

then the $U \in \mathcal{U}$ corresponding to the pair (τ, T') is equal to this object:

(6	$3)_1$	$(10)_2$	(12	$11)_2$	$(1)_3$	$(7)_3$
(14	13	2	$4)_2$			
(9	5	$8)_1$				

This completes the proof. □

Let sign_μ denote the sign of any permutation σ with cycle type μ.

Theorem 3.2. *We have* $e_\lambda = \sum_{\mu \vdash n} \text{sign}_\mu \, \varphi^\lambda_\mu \frac{p_\mu}{z_\mu}$.

Proof. This identity can be written as

$$n!e_\lambda = \sum_{\sigma \in S_n} \text{sign}(\sigma)\varphi^\lambda(\sigma)p_{\mu(\sigma)}. \tag{7}$$

Let $\mathcal{U}$ be the same set of objects created in the proof of Theorem 3.1. Define the sign of $U \in \mathcal{U}$ to be the sign of the underlying permutation. It follows that the right hand side of Eq. (7) is equal to $\sum_{U \in \mathcal{U}} \text{sign}(U)\, w(U)$.

We now define a sign reversing, weight preserving involution I on $\mathcal{U}$. Take $U \in \mathcal{U}$. Each cell in U has a color given by the color of the cycle to which it belongs. Let

c be the smallest color that appears in two cells within a single row. If no such c exists, define $I(U) = U$. Otherwise, in the highest row of U that contains two cells colored c, let m be the largest integer colored c and let m' be the second largest integer colored c.

If m and m' are in the same cycle, then create $I(U)$ by

1. cutting that cycle into two cycles, one starting with m and one starting with m', and then
2. listing the newly created cycle that starts with m' before the cycle starting with m.

If m and m' are not in the same cycle, then create $I(U)$ by

1. writing the cycle containing m' after the cycle containing m, and then
2. concatenating the two cycles, creating one cycle starting with m.

For example, if $U \in \mathcal{U}$ is as shown in Eq. (6), then $I(U)$ is shown here:

(12	3	9	$10)_1$	$(1)_2$	(14	$8)_3$
(7	$6)_1$	(5	$2)_2$			
(13	4	$11)_2$				

The function I is a weight preserving involution. If U is not a fixed point, then the total number of cycles in U changes by one and so $\operatorname{sign} U \neq \operatorname{sign} I(U)$. The fixed points under I cannot have a repeated color in any row, and so, in a similar manner as in the proof of Theorem 3.1, fixed points correspond to pairs (τ, T') where $\tau \in S_n$ and T' is a tableau with strictly increasing rows. These pairs correspond to $n!e_\lambda$, as needed. ☐

Those familiar with the representation theory of the symmetric group might recognize that the coefficient $\operatorname{sign}_\mu \varphi_\mu^\lambda$ in the statement of Theorem 3.2 is the character of the representation $\operatorname{sign} \otimes 1\!\uparrow_{S_\lambda}^{S_n}$. See [1, 10] for details.

Theorem 3.3. *Eq. (2) is true. Restated, we have*

$$n!s_\lambda = \sum_{\sigma \in S_n} \chi^\lambda_{\mu(\sigma)} p_{\mu(\sigma)} \tag{8}$$

for all $\lambda \vdash n$.

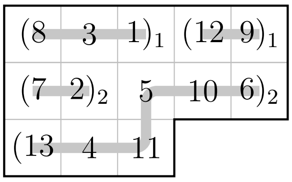
Proof. To account for the choice of $\sigma \in S_n$ and the $p_{\mu(\sigma)}$ on the right hand side of Eq. (8), select a colored permutation with cycles colored in the same way as in Theorem 3.1. Arrange the colored cycles in the same way as in Theorem 3.1 with smaller colors first and then sorted by smallest maximum element. One such σ is

$$(8\ \ 3\ \ 1)_1\ \ (12\ \ 9)_1\ \ (7\ \ 2)_2\ \ (13\ \ 4\ \ 11\ \ 5\ \ 10\ \ 6)_2$$

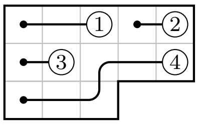
where the color of each cycle is its subscript.

To account for the $\chi^{\lambda}_{\mu(\sigma)}$ term in Eq. (8), select a rim hook tableau of shape λ and content ν where ν is the composition giving the lengths of the colored cycles in σ (using Theorem 2.1). Write down each colored cycle on top of the corresponding rim hook in the rim hook tableau.

For example, if $\lambda = (5, 5, 3)$ and σ is as displayed above, then $\nu = (3, 2, 2, 6)$ and one object created by this process is given here:



In the above object we have not drawn the integer labels on the rim hooks because these labels can be deduced from the colors and the maximum integer in each cycle. For example, the underlying rim hook tableau with the correctly labeled rim hooks for the object shown above is:



Let $\mathcal{T}$ be the set of all objects created in this way. The weight of $T \in \mathcal{T}$ is

$$w(T) = \prod_{i=1}^{n} x_{(\text{the color of the cycle containing } i \text{ in } T)}$$

and the sign of T is the sign of the underlying rim hook tableau. The cells of a given color in any given $T \in \mathcal{T}$ form a skew-shape. For example, the weight of the $T \in \mathcal{T}$ shown above is $x_1^5 x_2^8$ and the sign is -1. The right side of Eq. (8) is equal to $\sum_{T \in \mathcal{T}} \operatorname{sign}(T)\, w(T)$ by construction.

We now define a sign reversing and weight preserving involution I on $\mathcal{T}$. Scan the cells of $T \in \mathcal{T}$ from left to right and from top to bottom, looking for the first occurrence of either

Case A: a cell x in the same rim hook as the cell immediately below x, or

Case B: a cell colored c that is immediately above the terminal cell in a c colored rim hook.

The example T displayed above is in Case A because the cell containing 5 and the cell below the 5 are both contained in the same rim hook.

Suppose we are in Case A and let x be the first cell that is in the same rim hook as the cell below x. Let m be the maximum integer in the rim hook containing x. Define $I(T)$ to be the element in $\mathcal{T}$ created by following these instructions:

1. Cut the rim hook containing x at the down step between x and the cell below x. This ends the rim hook below x at the cell below x.

2. Let C be the cells that are in the same row as x, that are the same color as x, and belong to a rim hook that has a maximum integer no bigger than m. Erase all rim hooks and parentheses in C.

3. Reinsert parentheses into C, thereby creating cycles and rim hooks, in the unique manner that forces the maximum element in each cycle to appear first and forces cycles to be sorted in increasing order according to maximum element (in the same way as found at the end of the proof of Theorem 3.1).

For example, if T is the object displayed above, then $I(T)$ is here:

(8	3	$1)_1$	(12	$9)_1$
(7	2	$5)_2$	(10	$6)_2$
(13	4	$11)_2$		

If T is in Case A, then $I(T)$ will be in Case B because the cell x that was in the same rim hook as the cell below x is now a cell colored c that is immediately below the terminal cell in a c colored rim hook.

No cell preceding x (when cells in T are read left to right and from top to bottom) can fit into either Case A or Case B because the involution I only changes cells to the right of x, in rows below x, and in C. Cells in C will not fall into Case A because rim hooks in C are contained in one row. Cells in C will not fall into Case B because the first terminal cell of a rim hook in a row below C is the cell below x.

Now suppose T is in Case B and let x be the first cell colored c that is immediately above a terminal cell in a c colored rim hook. Suppose the maximum integer in the rim hook below x is m. Define $I(T)$ to be the element in $\mathcal{T}$ created by following these instructions:

1. Let C be the cells that are in the same row as x, that are the same color as x, and that are in a rim hook with a maximum integer no bigger than m. Erase all rim hooks and parentheses in C.

2. Extend the rim hook below x to include x and the cells in C to the right of cell x.

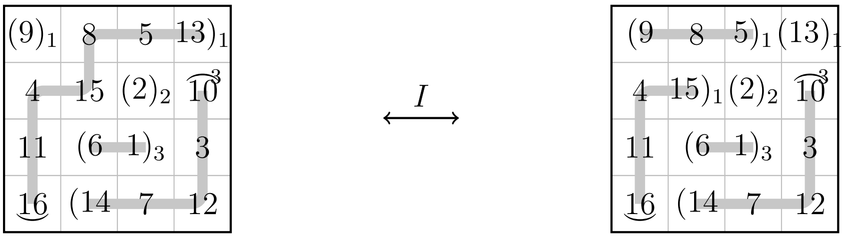
3. Reinsert parentheses into the remaining cells in C, thereby creating cycles and rim hooks, in the unique manner that forces the maximum element in each cycle to appear first and forces cycles to be sorted in increasing order according to maximum element (in the same way as found at the end of the proof of Theorem 3.1).

If $T \in \mathcal{T}$ is in Case B, then $I(T)$ will be in Case A because the cell x that was the same color as the rim hook below x is now in the same rim hook as the cell below x.

No cell preceding x will fit into Case A or Case B since $I(T)$ differs from T in cells that are to the right of x, in rows below x, and in C. Cells in C will not fall into Case A or Case B for similar reasons as outlined above.

If $T \in \mathcal{T}$ is not in either Case A or Case B, then we define $I(T) = T$. The function I on a Case B object is the defined to be inverse function to I on a Case A object, making I an involution.

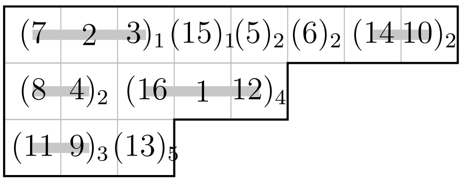
This completes our description of the involution I. As a second example, I pairs these two elements in $\mathcal{T}$:



If T is not a fixed point, then I changes the sign of T because I removes exactly one vertical segment in Case A and introduces exactly one vertical segment into a rim hook in Case B. The involution is weight preserving.

Fixed points under I cannot have any rim hooks that span two or more rows because otherwise we would be in Case A. Thus all fixed points have sign $+1$. Fixed points cannot have a cell with color c above another cell with color c because, since all the rim hooks are flat, we would be in Case B.

One example of a fixed point is here:



Create a permutation τ by listing the integers in a given fixed point without parentheses, reading each row from left to right beginning with the top row. Create a tableau T' by recording the color of each cell in a tableau of shape λ. In this way, a fixed point under I is in a natural 1–1 correspondence with a pair of the form (τ, T') where $\tau \in S_n$ and T' is a column strict tableau of shape λ. The ordering of the rim hooks with larger colors outside smaller colors guarantees that T' is column strict.

For example, the above fixed point corresponds to (τ, T') where σ is

$$\sigma = 7 \;\; 2 \;\; 3 \;\; 15 \;\; 5 \;\; 6 \;\; 14 \;\; 10 \;\; 8 \;\; 4 \;\; 16 \;\; 1 \;\; 12 \;\; 11 \;\; 9 \;\; 13$$

and the column strict tableau shown here:

$$T' = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 4 & 4 & 4 \\ \cline{1-5} 3 & 3 & 5 \\ \cline{1-3} \end{array}$$

The fixed point corresponding to the pair (τ, T') can be reconstructed in the same manner as found at the end of the proof of Theorem 3.1.

These pairs are counted by $n!s_\lambda$, the left hand side of Eq. (8), as needed. □

4 Final remarks

The proof of Theorem 3.3 is valid when the integer partition λ is replaced with a skew shape λ/α, giving a combinatorial proof only using tableaux that

$$s_{\lambda/\alpha} = \sum_{\mu \vdash |\lambda/\alpha|} \chi_\mu^{\lambda/\alpha} \frac{p_\mu}{z_\mu}. \tag{9}$$

The identity in Eq. (9) allows for a combinatorial proof that $\omega(s_{\lambda/\alpha}) = s_{\lambda'/\alpha'}$. Indeed, if a rim hook ρ of length k in a rim hook tableau has i horizontal steps and j vertical steps, then $i + j = k - 1$ and the sign of ρ is $(-1)^j$. The sign of the conjugated rim hook ρ' is

$$\operatorname{sign} \rho' = (-1)^i = (-1)^{k-1}(-1)^j = \operatorname{sign} \sigma \operatorname{sign} \rho,$$

where σ is a permutation with cycle type (k). This implies that

$$\operatorname{sign}_\mu \chi_\mu^{\lambda/\alpha} = \chi_\mu^{\lambda'/\alpha'}$$

where sign_μ is the sign of a permutation with cycle type μ. Therefore, if ω is the involution on the ring of symmetric functions defined by $\omega(p_n) = (-1)^{n-1}p_n$, then using Theorem 3.3 we have

$$\begin{aligned} \omega(s_{\lambda/\alpha}) &= \sum_{\mu \vdash |\lambda/\alpha|} \chi_\mu^{\lambda/\alpha} \frac{\omega(p_\mu)}{z_\mu} \\ &= \sum_{\mu \vdash |\lambda/\alpha|} \operatorname{sign}_\mu \chi_\mu^{\lambda/\alpha} \frac{p_\mu}{z_\mu} \\ &= \sum_{\mu \vdash |\lambda/\alpha|} \chi_\mu^{\lambda'/\alpha'} \frac{p_\mu}{z_\mu} \\ &= s_{\lambda'/\alpha'}, \end{aligned}$$

giving a combinatorial proof of why $\omega(s_{\lambda/\alpha}) = s_{\lambda'/\alpha'}$ completely in terms of tableaux.

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References

[1] G. D. James, *The representation theory of the symmetric groups*, Lec. Notes in Math. Vol. 682, Springer, Berlin, 1978.

[2] D. E. Littlewood and A. R. Richardson, Group characters and algebra, *Phil. Trans. Royal Soc. A* (London) 233 (1934), 99–141.

[3] J. LoBue and J. B. Remmel, A Murnaghan-Nakayama rule for generalized Demazure atoms, *Discrete Math. Theor. Comput. Sci. Proc., AS* (2013), 969–980.

[4] N. A. Loehr, Abacus proofs of Schur function identities, *SIAM J. Discrete Math.* 24(4) (2010), 1356–1370.

[5] N. A. Loehr, *Bijective combinatorics*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2011.

[6] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Classic Texts in the Physical Sciences, Clarendon Press, Oxford University Press, New York, second ed., 2015.

[7] A. Mendes and J. Remmel, *Counting with symmetric functions*, Developments in Mathematics Vol. 43, Springer, Cham, 2015.

[8] F. D. Murnaghan, The Characters of the Symmetric Group, *Amer. J. Math.* 59(4) (1937), 739–753.

[9] T. Nakayama, On some modular properties of irreducible representations of symmetric groups. II, *Jap. J. Math.* 17 (1941), 411–423.

[10] B. E. Sagan, *The symmetric group*, *Graduate Texts in Math.* Vol. 203, Springer-Verlag, New York, second ed., 2001.

[11] R. P. Stanley, *Enumerative combinatorics, Vol. 2*, *Cambridge Studies in Advanced Math.* Vol. 62, Cambridge University Press, Cambridge, 1999.

[12] D. W. Stanton and D. E. White, A Schensted algorithm for rim hook tableaux, *J. Combin. Theory Ser. A* 40(2) (1985), 211–247.

[13] J. L. Tiefenbruck, *Combinatorial Properties of Quasisymmetric Schur Functions and Generalized Demazure Atoms*, Thesis (Ph.D.)—University of California, San Diego, (2015)

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