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Globally simple Heffter arrays and orthogonal cyclic cycle decompositions

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Abstract

In this paper we introduce a particular class of Heffter arrays, called globally simple Heffter arrays, whose existence gives at once orthogonal cyclic cycle decompositions of the complete graph and of the cocktail party graph. In particular we provide explicit constructions of such decompositions for cycles of length $k \leq 10$. Furthermore, starting from our Heffter arrays we also obtain biembeddings of two k-cycle decompositions on orientable surfaces.

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1 Introduction

Arrays with particular properties are not only interesting objects *per se* but, in general, they have applications in many areas of mathematics. For these reasons, there are several types of well-studied arrays; see, for instance, [14, 16, 19, 20, 21, 24]. Here we consider Heffter arrays, introduced by Archdeacon in [2]:

Definition 1.1 An *integer Heffter array* $\mathrm{H}(m,n;h,k)$ is an $m \times n$ partially filled array such that:

(a) its entries belong to the set $\{\pm 1, \pm 2, \ldots, \pm nk\} \subset \mathbb{Z}$;

(b) no two entries agree in absolute value;

(c) each row contains h filled cells and each column contains k filled cells;

(d) the elements in every row and column sum to 0.

Trivial necessary conditions for the existence of an $\mathrm{H}(m,n;h,k)$ are $mh = nk$, $3 \leq h \leq n$ and $3 \leq k \leq m$. In this paper we will concentrate on *square* integer Heffter arrays, namely on the case $m = n$ which implies $h = k$. An $\mathrm{H}(n,n;k,k)$ will simply be denoted by $\mathrm{H}(n;k)$.

Example 1.2 The following array is an example of an $\mathrm{H}(8;7)$:

8	16		25	−27	−29	31	−24
−17	−6	23	−28	26	32	−30	
39	−10	−5	15		33	−35	−37
−38		−18	7	11	−36	34	40
−43	−45	47	−22	3	19		41
42	48	−46		−14	2	12	−44
	49	−51	−53	55	−21	1	20
9	−52	50	56	−54		−13	4

The existence problem of *square integer* Heffter arrays has been completely solved in [4, 17], where the authors proved the following theorem.

Theorem 1.3 *There exists an* $\mathrm{H}(n;k)$ *if and only if*

$$3 \leq k \leq n \quad \text{and} \quad nk \equiv 0, 3 \pmod 4.$$

If, in Definition 1.1, condition (d) is replaced by the following one:

(d′) *the elements in every row and column sum to* 0 *modulo* $2nk+1$, one speaks of a non-integer Heffter array; see [2]. The existence problem of *square non-integer* Heffter arrays has been completely solved in [3, 11, 15], proving that they exist for all $3 \leq k \leq n$.

In [2], Archdeacon showed that such arrays can be used to construct cycle decompositions of the complete graph if they satisfy an additional condition, called simplicity, which we introduce next. Let A be a finite subset of $\mathbb{Z} \setminus \{0\}$. Given an ordering $\omega = (a_1, a_2, \ldots, a_k)$ of the elements in A, let $s_i = \sum_{j=1}^{i} a_j$ be the i^{th} partial sum of A. We say that the ordering ω is *simple modulo* v if $s_b \neq s_c \pmod v$ for all $1 \leq b < c \leq k$ or, equivalently, if there is no proper subsequence of ω that sums to 0 modulo v.

With a little abuse of notation, we will identify each row (column) of an $\mathrm{H}(n;k)$ with the set of size k whose elements are the entries of the nonempty cells of such a row (column). For instance we can view the first row of the $\mathrm{H}(8;7)$ of Example 1.2 as the set $R_1 = \{8, 16, 25, -27, -29, 31, -24\}$.

Definition 1.4 An $\mathrm{H}(n;k)$ is said to be *simple* if each row and each column admits a simple ordering modulo $2nk + 1$.

Since each row and each column of an $\mathrm{H}(n;k)$ is such that $s_k = 0$ and does not contain 0 or subsets of the form $\{x, -x\}$, it is easy to see that every $\mathrm{H}(n;k)$ with $k \leq 5$ is simple.

Example 1.5 The $\mathrm{H}(8;7)$ of Example 1.2 is simple. To verify this property we need to provide an ordering for each row and each column which is simple modulo 113. One can check that the ω_i are simple orderings of the rows and the ν_i are simple orderings of the columns:

$$
\begin{array}{ll}
\omega_1 = (8, 25, 16, -27, -29, 31, -24); & \nu_1 = (8, 39, -17, -38, -43, 42, 9);\\
\omega_2 = (-17, -6, -28, 23, 26, 32, -30); & \nu_2 = (16, -6, -45, -10, 48, 49, -52);\\
\omega_3 = (39, -10, -5, 33, 15, -35, -37); & \nu_3 = (23, -5, 47, -18, -46, -51, 50);\\
\omega_4 = (-38, -18, 7, -36, 11, 34, 40); & \nu_4 = (25, -28, 15, 7, -53, -22, 56);\\
\omega_5 = (-43, -45, 47, -22, 3, 41, 19); & \nu_5 = (-27, 26, 11, 3, 55, -14, -54);\\
\omega_6 = (42, 48, -46, -14, 2, -44, 12); & \nu_6 = (-21, 32, 33, -36, 19, 2, -29);\\
\omega_7 = (20, -51, -53, 55, -21, 1, 49); & \nu_7 = (-13, -30, -35, 34, 12, 1, 31);\\
\omega_8 = (-52, 9, 50, 56, -54, -13, 4); & \nu_8 = (-37, -24, 40, 41, -44, 20, 4).
\end{array}
$$

Notice that the *natural ordering* of each row (from left to right) and of each column (from top to bottom) of the above $\mathrm{H}(8;7)$ is not simple. Clearly, larger n and k are more difficult (and tedious) is to provide explicit simple orderings for rows and columns of an $\mathrm{H}(n;k)$. Hence, we think it is reasonable to look for Heffter arrays which are simple with respect to the natural ordering of rows and columns. So we propose the following new definition.

Definition 1.6 We say that an Heffter array $\mathrm{H}(n;k)$ is *globally simple* if each row and each column is simple modulo $2nk + 1$ with respect to their natural ordering.

Clearly, the concepts of simple and globally simple Heffter array can be extended to the rectangular case $\mathrm{H}(m, n; h, k)$. A globally simple $\mathrm{H}(m, n; h, k)$ will be denoted by $\mathrm{SH}(m, n; h, k)$ and a square globally simple $\mathrm{H}(n; k)$ will be denoted by $\mathrm{SH}(n; k)$.

Example 1.7 The following is a globally simple $\mathrm{SH}(8; 7)$:

4	35	−45	46		20	−36	−24
48	−5	23	−47	−18		37	−38
−32	−10	−6	31	−41	42		16
33	−34	44	3	11	−43	−14	
	15	−28	−22	7	27	−53	54
−13		29	−30	56	1	12	−55
−49	50		19	−40	−21	2	39
9	−51	−17		25	−26	52	8

Remark 1.8 We point out that the Heffter arrays constructed in [4, 17], in general, are not globally simple. Indeed, the $\mathrm{H}(8; 7)$ of Example 1.2 was obtained according to [4, Theorem 3.12]. Furthermore, easy modifications of the existing constructions seem not to produce globally simple arrays, except for $\mathrm{SH}(n; 6)$: these arrays (see Proposition 4.1) are obtained switching the first two columns of the matrices given in [4, Theorem 2.1].

In [2] many applications of (simple) Heffter arrays are shown, in particular the relationship with orthogonal cycle decompositions of the complete graph and with biembeddings of two cycle decompositions on an orientable surface. Here, in Section 2 we show that globally simple Heffter arrays are related not only to orthogonal cyclic cycle decompositions of the complete graph, but also of the cocktail party graph. We note that very little is known about orthogonal decompositions; as far as we know, only asymptotic results have been obtained, see [9, 10]. In Section 3 we investigate the connection between Heffter arrays and biembeddings of two cycle decompositions on an orientable surface. Then in Section 4 we present direct constructions of $\mathrm{SH}(n; k)$ for $6 \leq k \leq 10$ and for any admissible n. Combining the results of Sections 2 and 4 we obtain the following theorem.

Theorem 1.9 *Let $3 \leq k \leq 10$. Then there exists a pair of orthogonal cyclic k-cycle decompositions of the complete graph of order $2nk + 1$ and of the cocktail party graph of order $2nk + 2$ for any positive integer n such that $nk \equiv 0, 3 \pmod 4$.*

We have to point out that for $3 \leq k \leq 9$, Theorem 1.9 can be obtained starting from the results of [4, 13, 17]. But, in that case, if one wants to construct the base cycles for the cycle decompositions of order $2nk + 1$ he has to find an ad hoc simple ordering for each row and each column, then he has to find other simple orderings modulo $2nk+2$. While here the cycle decompositions (both of the complete graph of order $2nk + 1$ and of the cocktail party graph of order $2nk + 2$) can be immediately written starting from the rows and columns of the arrays constructed in Section 4.

It is worth noticing that combining the previous theorem with [6, Theorem 3.3], a stronger result can be stated regarding cocktail party graphs.

Corollary 1.10 *Let $3 \leq k \leq 10$ and $n \geq 1$. Then there exists a pair of orthogonal cyclic k-cycle decompositions of the cocktail party graph of order $2nk+2$ if and only if $nk \equiv 0, 3 \pmod 4$.*

Finally, combining the results of Sections 3 and 4 we obtain the following theorem.

Theorem 1.11 *There exists a biembedding of the complete graph of order $2nk+1$ and one of the cocktail graph of order $2nk+2$ on orientable surfaces such that every face is a k-cycle, whenever $k \in \{3,5,7,9\}$, $nk \equiv 3 \pmod 4$ and $n > k$.*

2 Orthogonal cyclic cycle decompositions

We first recall some basic definitions about graph decompositions. Let Γ be a graph with v vertices. A *k-cycle decomposition* of Γ is a set $\mathcal{C}$ of k-cycles of Γ such that each edge of Γ belongs to a unique cycle of $\mathcal{C}$. If Γ is the complete graph of order v, one also speaks of a k-cycle system of order v. A k-cycle decomposition of Γ is said to be *cyclic* if it admits $\mathbb{Z}_v$ as automorphism group acting sharply transitively on the vertices. We recall the following result.

Proposition 2.1 *Let Γ be a graph with v vertices. A k-cycle decomposition $\mathcal{C}$ of Γ is sharply vertex-transitive under $\mathbb{Z}_v$ if and only if, up to isomorphisms, the following conditions hold:*

- *the set of vertices of Γ is $\mathbb{Z}_v$;*
- *for all $C = (c_1, c_2, \ldots, c_k) \in \mathcal{C}$, $C + 1 := (c_1+1, c_2+1, \ldots, c_k+1) \in \mathcal{C}$.*

Clearly, to describe a cyclic k-cycle decomposition it is sufficient to exhibit a complete system $\mathcal{B}$ of representatives for the orbits of $\mathcal{C}$ under the action of $\mathbb{Z}_v$. The elements of $\mathcal{B}$ are called *base cycles* of $\mathcal{C}$.

Here we are interested in the cases in which Γ is either the complete graph K_v whose vertex-set is $\mathbb{Z}_v$ or the cocktail party graph $K_{2t} - I$, namely the complete graph K_{2t} minus the 1-factor I whose edges are $[0,t], [1,t+1], [2,t+2], \ldots, [t-1, 2t-1]$. The problem of finding necessary and sufficient conditions for cyclic k-cycle decompositions of K_v and $K_{2t} - I$ has attracted much attention (see, for instance, [7, 25, 26] and [6, 8, 22, 23], respectively). One of most efficient tools applied for solving this problem is the *difference method*.

Definition 2.2 Let $C = (c_1, c_2, \ldots, c_k)$ be a k-cycle with vertices in $\mathbb{Z}_v$. The multiset

$$\Delta C = \{\pm(c_{h+1} - c_h) \mid 1 \leq h \leq k\},$$

where the subscripts are taken modulo k, is called the *list of differences* from C.

More generally, given a set $\mathcal{B}$ of k-cycles with vertices in $\mathbb{Z}_v$, by $\Delta\mathcal{B}$ one means the union (counting multiplicities) of all multisets ΔC, where $C \in \mathcal{B}$.

Theorem 2.3 *Let $\mathcal{B}$ be a set of k-cycles with vertices in $\mathbb{Z}_v$.*

(1) *If $\Delta\mathcal{B} = \mathbb{Z}_v \setminus \{0\}$ then $\mathcal{B}$ is a set of base cycles of a cyclic k-cycle decomposition of K_v.*

(2) *If $v = 2t$ and $\Delta\mathcal{B} = \mathbb{Z}_{2t} \setminus \{0, t\}$ then $\mathcal{B}$ is a set of base cycles of a cyclic k-cycle decomposition of $K_{2t} - I$.*

Here, we are interested in constructing pairs of orthogonal k-cycle decompositions according to the following definition.

Definition 2.4 Two k-cycle decompositions $\mathcal{C}$ and $\mathcal{C}'$ of a graph Γ are said to be *orthogonal* if for any cycle $C \in \mathcal{C}$ and any cycle $C' \in \mathcal{C}'$, C intersects C' in at most one edge.

Clearly, the same definition can be given for two arbitrary graph decompositions; see [1].

Starting from a simple $\mathrm{H}(n; k)$ it is possible to construct two orthogonal cyclic k-cycle decompositions of K_{2nk+1}, see [2, Proposition 2.1]. Firstly, we have to find a simple ordering modulo $2nk + 1$ for each row and each column. Then starting from the simple orderings of the rows we can construct a set $\mathcal{B}$ of base cycles of a cyclic k-cycle decomposition $\mathcal{C}$ of K_{2nk+1}. The vertices of the i^{th} cycle of $\mathcal{B}$ are the partial sums modulo $2nk + 1$ of the i^{th} row of $\mathrm{H}(n; k)$. Analogously, we can obtain a set of base cycles $\mathcal{B}'$ of another cyclic k-cycle decomposition $\mathcal{C}'$ of K_{2nk+1} starting from the simple orderings of the columns. The decompositions $\mathcal{C}$ and $\mathcal{C}'$ are orthogonal.

Example 2.5 Let H be the $\mathrm{H}(8; 7)$ of Example 1.2 and consider the simple orderings ω_i and ν_i given in Example 1.5. By the partial sums of the ω_i (ν_i, respectively) in $\mathbb{Z}_{113}$ we obtain the cycles C_i (C_i', respectively):

$C_1 = (8, 33, 49, 22, -7, 24, 0)$; $C_1' = (8, 47, 30, -8, -51, -9, 0)$;
$C_2 = (-17, -23, -51, -28, -2, 30, 0)$; $C_2' = (16, 10, -35, -45, 3, 52, 0)$;
$C_3 = (39, 29, 24, 57, 72, 37, 0)$; $C_3' = (23, 18, 65, 47, 1, -50, 0)$;
$C_4 = (-38, -56, -49, -85, -74, -40, 0)$; $C_4' = (25, -3, 12, 19, -34, -56, 0)$;
$C_5 = (-43, -88, -41, -63, -60, -19, 0)$; $C_5' = (-27, -1, 10, 13, 68, 54, 0)$;
$C_6 = (42, 90, 44, 30, 32, -12, 0)$; $C_6' = (-21, 11, 44, 8, 27, 29, 0)$;
$C_7 = (20, -31, -84, -29, -50, -49, 0)$; $C_7' = (-13, -43, -78, -44, -32, -31, 0)$;
$C_8 = (-52, -43, 7, 63, 9, -4, 0)$; $C_8' = (-37, -61, -21, 20, -24, -4, 0)$.

Then $\mathcal{B} = \{C_1, \ldots, C_8\}$ and $\mathcal{B}' = \{C_1', \ldots, C_8'\}$ are two sets of base cycles of a pair of orthogonal cyclic 7-cycle decompositions of K_{113}.

Although the existence of a square integer $\mathrm{H}(n; k)$ has been completely established, the simplicity of these arrays has not been considered. In [13], we proposed the following conjecture whose validity would imply that any Heffter array is simple (other related and interesting conjectures can be found in [5]).

Conjecture 1 *Let $(G,+)$ be an abelian group. Let A be a finite subset of $G\setminus\{0\}$ such that no 2-subset $\{x,-x\}$ is contained in A and with the property that $\sum_{a\in A} a = 0$. Then there exists a simple ordering of the elements of A.*

We proved that our conjecture is true for any subset A of size less than 10. Our proof is constructive, but given an $\mathrm{H}(n;k)$ it can be long and tedious to find the required $2n$ simple orderings. This is why we came up with the idea of introducing globally simple Heffter arrays. Moreover, we will construct globally simple integer Heffter arrays $\mathrm{SH}(n;k)$ which satisfy also the following condition:

($*$) *the natural ordering of each row and column is simple modulo $2nk+2$.*

The usefulness of these arrays, which will be denoted by $\mathrm{SH}^*(n;k)$, is explained by the following proposition.

Proposition 2.6 *If there exists an $\mathrm{SH}^*(n;k)$, then there exist:*

(1) *a pair of orthogonal cyclic k-cycle decompositions of K_{2nk+1} and*

(2) *a pair of orthogonal cyclic k-cycle decompositions of $K_{2nk+2}-I$.*

PROOF: (1) This follows from previous considerations. (2) As the natural ordering of each row is simple modulo $2nk+2$, the partial sums of each row in $\mathbb{Z}_{2nk+2}$ are the vertices of a k-cycle. Let $\mathcal{B}$ be the set of the k-cycles so constructed from the rows. Since $\Delta\mathcal{B} = \mathbb{Z}_{2nk+2}\setminus\{0, nk+1\}$, in view of Theorem 2.3, $\mathcal{B}$ is a set of base cycles of a cyclic k-cycle decomposition $\mathcal{C}$ of $K_{2nk+2}-I$. Analogously, starting from the columns, we can obtain another cyclic k-cycle decomposition $\mathcal{C}'$ of $K_{2nk+2}-I$. By construction, the decompositions $\mathcal{C}$ and $\mathcal{C}'$ are orthogonal. □

Example 2.7 The following is an $\mathrm{SH}^*(10;8)$:

77	80	−78	−71	−70	−79			69	72
		−17	−20	−25	−28	26	19	18	27
5	8	13	16	−14	−7	−6	−15		
34	43			−33	−36	−41	−44	42	35
		21	24	29	32	−30	−23	−22	−31
58	51	50	59			−49	−52	−57	−60
−38	−47			37	40	45	48	−46	−39
−73	−76	74	67	66	75			−65	−68
−62	−55	−54	−63			53	56	61	64
−1	−4	−9	−12	10	3	2	11		

By the partial sums in $\mathbb{Z}_{162}$ of the natural simple orderings of the rows (columns, respectively) we obtain the cycles C_i (C_i', respectively):

$C_1 = (77, 157, 79, 8, -62, -141, -72, 0)$;
$C_2 = (-17, -37, -62, -90, -64, -45, -27, 0)$;
$C_3 = (5, 13, 26, 42, 28, 21, 15, 0)$;
$C_4 = (34, 77, 44, 8, -33, -77, -35, 0)$;
$C_5 = (21, 45, 74, 106, 76, 53, 31, 0)$;
$C_6 = (58, 109, 159, 56, 7, 117, 60, 0)$;
$C_7 = (-38, -85, -48, -8, 37, 85, 39, 0)$;
$C_8 = (-73, -149, -75, -8, 58, 133, 68, 0)$;
$C_9 = (-62, -117, -9, -72, -19, -125, -64, 0)$;
$C_{10} = (-1, -5, -14, -26, -16, -13, -11, 0)$;

$C'_1 = (77, 82, 116, 12, 136, 63, 1, 0)$;
$C'_2 = (80, 88, 131, 20, 135, 59, 4, 0)$;
$C'_3 = (-78, -95, -82, -61, -11, 63, 9, 0)$;
$C'_4 = (-71, -91, -75, -51, 8, 75, 12, 0)$;
$C'_5 = (-70, -95, -109, -142, -113, -76, -10, 0)$;
$C'_6 = (-79, -107, -114, -150, -118, -78, -3, 0)$;
$C'_7 = (26, 20, -21, -51, -100, -55, -2, 0)$;
$C'_8 = (19, 4, -40, -63, -115, -67, -11, 0)$;
$C'_9 = (69, 87, 129, 107, 50, 4, -61, 0)$;
$C'_{10} = (72, 99, 134, 103, 43, 4, -64, 0)$.

Then $\mathcal{B} = \{C_1, \ldots, C_{10}\}$ and $\mathcal{B}' = \{C'_1, \ldots, C'_{10}\}$ are two sets of base cycles of a pair of orthogonal cyclic 8-cycle decompositions of $K_{162} - I$.
Analogously, if we consider the partial sums of each row (column, respectively) in $\mathbb{Z}_{161}$, we obtain the cycles $\widetilde{C}_i$ ($\widetilde{C}'_j$, respectively):

$\widetilde{C}_i = C_i, \quad i \neq 6, 9$;
$\widetilde{C}_6 = (58, 109, 159, 57, 8, 117, 60, 0)$;
$\widetilde{C}_9 = (-62, -117, -10, -73, -20, -125, -64, 0)$;

$\widetilde{C}'_j = C_j, \quad j \neq 1, 2$;
$\widetilde{C}'_1 = (77, 82, 116, 13, 136, 63, 1, 0)$;
$\widetilde{C}'_2 = (80, 88, 131, 21, 135, 59, 4, 0)$.

Now $\widetilde{\mathcal{B}} = \{\widetilde{C}_1, \ldots, \widetilde{C}_{10}\}$ and $\widetilde{\mathcal{B}}' = \{\widetilde{C}'_1, \ldots, \widetilde{C}'_{10}\}$ are two sets of base cycles of a pair of orthogonal cyclic 8-cycle decompositions of K_{161}.

3 Biembeddings of cycle decompositions

This section is dedicated to the connection between Heffter arrays and biembeddings of two cycle decompositions on an orientable surface. We recall that an embedding of a graph with each edge on a face of size k and on a face of size h is called a *biembedding*. We point out that a biembedding is 2-colorable with the faces that are k-cycles receiving one color while those faces that are h-cycles receive the other color (see [15]).

Consider now a generic partially filled square array A of size n such that its N nonempty entries are pairwise distinct. As usual, we identify each row (column) of A with the set whose elements are the entries of the nonempty cells of such a row (column). Let ω_r (ω_c, respectively) be any ordering of the rows (columns, respectively) of A. We say that ω_r and ω_c are two *compatible* orderings if $\omega_r \circ \omega_c$ is a cycle of order N.

In particular, the following result holds:

Theorem 3.1 *Given a Heffter array $H = \mathrm{H}(m, n; h, k)$ with simple compatible orderings modulo $2nk+1$ ($2nk+2$, respectively) ω_r on the rows and ω_c on the columns of H, there exists a biembedding of K_{2nk+1} (of $K_{2nk+2} - I$, respectively) on an orientable surface such that every edge is on a simple cycle face of size k and on a simple cycle face of size h.*

PROOF: The result for complete graphs was obtained by Archdeacon [2, Sections 3 and 4] using current graphs. This construction is based on a paper of Gustin [18] that works in general for Cayley graphs, implying the result for cocktail graphs. □

Let $k \geq 1$ be an odd integer and let $A = (a_{i,j})$ be a partially filled square array of size n. We say that the element $a_{i,j}$ belongs to the diagonal D_s if $j - i \equiv s - 1 \pmod{n}$. Moreover, A is said to be cyclically k-diagonal if the nonempty cells of A are exactly those of the diagonals D_s with $s \in \{r, \ldots, r+k-1\}$ for a suitable integer $r \in \{1, \ldots, n\}$.

Example 3.2 The following partially filled array A of size $n = 9$ is cyclically 5-diagonal (with $r = 8$):

38	39	40					36	37
42	43	44	45					41
1	2	3	4	5				
	6	7	8	9	10			
		11	12	13	14	15		
			16	17	18	19	20	
				21	22	23	24	25
30					26	27	28	29
34	35					31	32	33

Given a cyclically k-diagonal array A whose nonempty cells belong to the diagonals $D_r, D_{r+1}, \ldots, D_{r+k-1}$, we can relabel its elements setting $b_{i,j} = a_{i-r+1,j}$, where the indices are considered modulo n in such a way that they belong to the set $\{1, \ldots, n\}$. We obtain a partially filled array B of size n which is still cyclically k-diagonal but with nonempty diagonals $D_1, \ldots, D_k$. We call such B the standard form of A.

Note that this procedure has no influence on any orderings ω_r and ω_c of the rows and of the columns of A, respectively.

Example 3.3 Starting from the array A of Example 3.2 we get the following B:

1	2	3	4	5				
	6	7	8	9	10			
		11	12	13	14	15		
			16	17	18	19	20	
				21	22	23	24	25
30					26	27	28	29
34	35					31	32	33
38	39	40					36	37
42	43	44	45					41

Proposition 3.4 *Let k be an odd integer and let A be a cyclically k-diagonal partially filled square array of size $n \geq k$ such that its nonempty entries are pairwise distinct. If $\gcd(n, k-1) = 1$, then there exist two compatible orderings ω_r and ω_c of the rows and the columns of A.*

Proof: It is not restrictive to consider A written in the standard form, so that its nonempty entries are the diagonals $D_1, \dots, D_k$. Let ω_r be the natural ordering of the rows of A from left to right and let ω_c be the natural ordering of the columns of A from top to bottom for the first $n-1$ columns, and from bottom to top for the last column, namely:

$$\begin{aligned}
\omega_r &= (a_{1,1}, a_{1,2}, \dots, a_{1,k})(a_{2,2}, a_{2,3} \dots, a_{2,k+1}) \cdots (a_{n,n}, a_{n,1} \dots, a_{n,k-1}),\\
\omega_c &= (a_{n-k+2,1}, a_{n-k+3,1}, \dots, a_{n,1}, a_{1,1})(a_{n-k+3,2}, a_{n-k+4,2}, \dots, a_{n,2}, a_{1,2}, a_{2,2}) \cdots\\
&\quad (a_{n,k-1}, a_{1,k-1}, a_{2,k-1}, \dots, a_{k-1,k-1})(a_{1,k}, a_{2,k}, \dots, a_{k,k})\\
&\quad (a_{2,k+1}, a_{3,k+1}, \dots, a_{k+1,k+1}) \cdots (a_{n-k,n-1}, a_{n-k+1,n-1}, \dots, a_{n-1,n-1})\\
&\quad (a_{n,n}, a_{n-1,n}, \dots, a_{n-k+1,n}).
\end{aligned}$$

Then $\omega_r \circ \omega_c$ moves cyclically from left to right and goes down $n-1$ times and up once. Setting $t = n-k+1$, we obtain that

$$\omega_r \circ \omega_c = (D'_2, D'_4, \dots, D'_{k-1}, D'_1, D'_3, \dots, D'_{k-2}, a_{t,k+t-1}, a_{2t,k+2t-1}, \dots, a_{nt,k+nt-1}),$$

where the indices are considered modulo n, and D'_s is the sequence

$$a_{n-s+1,n}, a_{n-s+2,1}, \dots, a_{n,s-1}, a_{1,s}, a_{2,s+1}, \dots, a_{n-s,n-1}.$$

Note that, for $s \in \{1, \dots, k-1\}$, the elements of D'_s are exactly the ones of the diagonal D_s and hence are pairwise distinct. The last n elements $a_{t,k+t-1}, a_{2t,k+2t-1}, \dots, a_{nt,k+nt-1}$ belong to the diagonal D_k and are pairwise distinct since $\gcd(n,t) = 1$. Therefore $\omega_r \circ \omega_c$ is a cycle of order nk. □

Example 3.5 Considering the array B of Example 3.3 we have the following ordering for the rows:

$$\begin{aligned}
\omega_r = {} & (1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20)\\
& (21,22,23,24,25)(26,27,28,29,30)(31,32,33,34,35)(36,37,38,39,40)\\
& (41,42,43,44,45);
\end{aligned}$$

and the following ordering for the column:

$$\begin{aligned}
\omega_c = {} & (30,34,38,42,1)(35,39,43,2,6)(40,44,3,7,11)(45,4,8,12,16)\\
& (5,9,13,17,21)(10,14,18,22,26)(15,19,23,27,31)(20,24,28,32,36)\\
& (41,37,33,29,25).
\end{aligned}$$

Hence,

$$\begin{aligned}
\omega_r \circ \omega_c = {} & (37,42,2,7,12,17,22,27,32,29,34,39,44,4,9,14,19,24,41,1,6,11,16,\\
& 21,26,31,36,33,38,43,3,8,13,18,23,28,25,5,30,10,35,15,40,20,45).
\end{aligned}$$

Proposition 3.6 *Let A be a cyclically 7-diagonal partially filled square array of odd size $n \geq 7$, such that its nonempty entries are pairwise distinct. Then there exist two compatible orderings ω_r and ω_c of the rows and the columns of A.*

PROOF: It is not restrictive to consider A written in the standard form, so that its nonempty entries are the diagonals $D_1, \dots, D_7$. Let ω_r be the natural ordering of the rows of A from left to right and let ω_c be the natural ordering of the columns of A from top to bottom for the first $n-4$ columns, and from bottom to top for the last 4 columns, that is:

$$\begin{array}{rcl} \omega_r & = & (a_{1,1}, a_{1,2}, \dots, a_{1,7})(a_{2,2}, a_{2,3} \dots, a_{2,8}) \cdots (a_{n,n}, a_{n,1} \dots, a_{n,6}), \\ \omega_c & = & (a_{n-5,1}, a_{n-4,1}, \dots, a_{n,1}, a_{1,1})(a_{n-4,2}, a_{n-3,2}, \dots, a_{n,2}, a_{1,2}, a_{2,2}) \cdots \\ & & (a_{n,6}, a_{1,6}, a_{2,6}, \dots, a_{6,6})(a_{1,7}, a_{2,7}, \dots, a_{7,7})(a_{2,8}, a_{3,8}, \dots, a_{8,8}) \cdots \\ & & (a_{n-10,n-4}, a_{n-9,n-4}, \dots, a_{n-4,n-4})(a_{n-3,n-3}, a_{n-4,n-3}, \dots, a_{n-9,n-3}) \\ & & (a_{n-2,n-2}, a_{n-3,n-2}, \dots, a_{n-8,n-2})(a_{n-1,n-1}, a_{n-2,n-1}, \dots, a_{n-7,n-1}) \\ & & (a_{n,n}, a_{n-1,n}, \dots, a_{n-6,n}). \end{array}$$

Then $\omega_r \circ \omega_c$ moves cyclically from left to right and goes down $n-4$ times and up four times. It can be showed that $\omega_r \circ \omega_c$ is a cycle of order $7n$. However, since the proof depends on the residue class of n modulo 6, we present here only the case $n \equiv 3 \pmod 6$, i.e. the case not covered by Proposition 3.4 (for $n = 9$ it suffices an easy direct check, so we also assume $n > 9$).

For $s = 1, \dots, 6$, consider the sequences

$$D'_s = \quad a_{n-s+1,n}, a_{n-s+2,1}, \dots, a_{n,s-1}, a_{1,s}, a_{2,s+1}, \dots, a_{n-3-s,n-4}$$

and

$$E_s = \begin{cases} a_{n-9+s,n-3+s}, a_{n-9+s-6,n-3+s-6}, \dots, a_{s,6+s}, a_{n-6+s,s} & \text{if } s = 1,2,3; \\ a_{n-15+s,n-9+s}, a_{n-15+s-6,n-9+s-6}, \dots, a_{s,6+s}, a_{n-6+s,s} & \text{if } s = 4,5,6. \end{cases}$$

Then, it is easy to see that

$$\begin{array}{rcl} \omega_r \circ \omega_c & = & (D'_4, a_{n-8,n-3}, a_{n-2,n-2}, a_{n-3,n-1}, D'_5, E_6, D'_2, a_{n-6,n-3}, a_{n-7,n-2}, a_{n-1,n-1}, \\ & & D'_3, a_{n-7,n-3}, E_1, E_4, a_{n-3,n-2}, a_{n-4,n-1}, D'_6, a_{n-3,n-3}, a_{n-4,n-2}, a_{n-5,n-1}, \\ & & E_3, a_{n-4,n-3}, a_{n-5,n-2}, a_{n-6,n-1}, D'_1, a_{n-5,n-3}, a_{n-6,n-2}, E_2, E_5, a_{n-2,n-1}). \end{array}$$

Since the elements of D'_s are those of $D_s \setminus \{a_{n-2-s,n-3}, a_{n-1-s,n-2}, a_{n-s,n-1}\}$ for all $s = 1, \dots, 6$ and the elements of $E_1 \cup \ldots \cup E_6$ are those of D_7, it follows that $\omega_r \circ \omega_c$ is a cycle of order $7n$. □

4 Direct constructions of $\mathrm{SH}^*(n;k)$ for $k \leq 10$

In this section, we provide direct constructions of $\mathrm{SH}^*(n;k)$ for $6 \leq k \leq 10$ and for any n satisfying the necessary conditions of Theorem 1.3.

Clearly, the main task is to check the simplicity of each row and each column. A little help is given by noticing that, from Definition 1.1, the i^{th} partial sum s_i is different from s_{i+1} and from s_{i+2} both modulo $2nk+1$ and modulo $2nk+2$, where the subscripts are taken modulo k. So, if $k = 3,4,5$ then every ordering of any row and column of an $H = \mathrm{H}(n;k)$ is simple both modulo $2nk+1$ and modulo $2nk+2$, and hence H is an $\mathrm{SH}^*(n;k)$. We recall that, for these values of k, explicit constructions of $\mathrm{H}(n;k)$ have been described in [4, 17]. So, we start with the case $k=6$.

We also fix some notation. Given a row or a column A of a partially filled array, we denote by $\|A\|$ the list of the absolute values of the nonempty entries of A and by $\mathcal{S}(A)$ the sequence of the partial sums of A with respect to the natural ordering (ignoring the empty cells). More generally, if $A_1,\ldots,A_r$ are rows (or columns), by $\|\cup_{i=1}^r A_i\|$ we mean the union $\cup_{i=1}^r \|A_i\|$. Furthermore, $\square^t$ means a sequence of t empty cells.

Proposition 4.1 *Let $n\geq 6$ be even. Then, there exists an* $\mathrm{SH}^*(n;6)$.

PROOF: Let H be the $n\times n$ partially filled array whose rows R_t are as follows:

$$\begin{array}{rcl}
R_1 &=& (5,-1,2,-7,-9,10,\square^{n-6}),\\
R_2 &=& (-4,3,-6,8,11,-12,\square^{n-6}),\\
R_{3+2i} &=& (\square^{2+2i},-13-12i,17+12i,14+12i,-19-12i,-21-12i,22+12i,\square^{n-8-2i}),\\
R_{4+2i} &=& (\square^{2+2i},15+12i,-16-12i,-18-12i,20+12i,23+12i,-24-12i,\square^{n-8-2i}),\\
R_{n-3} &=& (-14+6n,15-6n,\square^{n-6},23-6n,-19+6n,-22+6n,17-6n),\\
R_{n-2} &=& (12-6n,-13+6n,\square^{n-6},-21+6n,20-6n,18-6n,-16+6n),\\
R_{n-1} &=& (5-6n,-10+6n,3-6n,-2+6n,\square^{n-6},11-6n,-7+6n),\\
R_n &=& (-4+6n,6-6n,-1+6n,-6n,\square^{n-6},-9+6n,8-6n),
\end{array}$$

where $i=0,\ldots,\frac{n-8}{2}$. Note that every row contains exactly 6 filled cells. Also, it is easy to see that $\|R_{2h+1}\cup R_{2h+2}\|=\{1+12h,\ldots,12+12h\}$ for all $h=0,\ldots,\frac{n-2}{2}$. Hence, H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$\begin{array}{rcl}
\mathcal{S}(R_1) &=& (5,4,6,-1,-10,0),\\
\mathcal{S}(R_2) &=& (-4,-1,-7,1,12,0),\\
\mathcal{S}(R_{3+2i}) &=& (-13-12i,4,18+12i,-1,-22-12i,0),\\
\mathcal{S}(R_{4+2i}) &=& (15+12i,-1,-19-12i,1,24+12i,0),\\
\mathcal{S}(R_{n-3}) &=& (-14+6n,1,24-6n,5,-17+6n,0),\\
\mathcal{S}(R_{n-2}) &=& (12-6n,-1,-22+6n,-2,16-6n,0),\\
\mathcal{S}(R_{n-1}) &=& (5-6n,-5,-2-6n,-4,7-6n,0),\\
\mathcal{S}(R_n) &=& (-4+6n,2,1+6n,1,-8+6n,0).
\end{array}$$

Clearly, each row sums to 0. In view of previous considerations, in order to prove that each row is simple modulo $\nu\in\{12n+1,12n+2\}$ it suffices to prove that $s_i\not\equiv s_{i+3} \pmod{\nu}$ for $i=1,2,3$. From the definition of H we obtain the following expression of the columns:

$$\begin{array}{rcl}
C_1 &=& (5,-4,\square^{n-6},-14+6n,12-6n,5-6n,-4+6n)^T,\\
C_2 &=& (-1,3,\square^{n-6},15-6n,-13+6n,-10+6n,6-6n)^T,\\
C_3 &=& (2,-6,-13,15,\square^{n-6},3-6n,-1+6n)^T,\\
C_4 &=& (-7,8,17,-16,\square^{n-6},-2+6n,-6n)^T,\\
C_{5+2j} &=& (\square^{2j},-9-12j,11+12j,14+12j,-18-12j,-25-12j,27+12j,\square^{n-6-2j})^T,\\
C_{6+2j} &=& (\square^{2j},10+12j,-12-12j,-19-12j,20+12j,29+12j,-28-12j,\square^{n-6-2j})^T,
\end{array}$$

where $j = 0, \ldots, \frac{n-6}{2}$. We observe that each column contains exactly 6 filled cells, then condition (c) of Definition 1.1 is satisfied. The lists of the partial sums of the columns are

$$\begin{array}{rcl} \mathcal{S}(C_1) & = & (5, 1, -13+6n, -1, 4-6n, 0), \\ \mathcal{S}(C_2) & = & (-1, 2, 17-6n, 4, -6+6n, 0), \\ \mathcal{S}(C_3) & = & (2, -4, -17, -2, 1-6n, 0), \\ \mathcal{S}(C_4) & = & (-7, 1, 18, 2, 6n, 0), \\ \mathcal{S}(C_{5+2j}) & = & (-9-12j, 2, 16+12j, -2, -27-12j, 0), \\ \mathcal{S}(C_{6+2j}) & = & (10+12j, -2, -21-12j, -1, 28+12j, 0). \end{array}$$

Since every column sums to 0, also condition (d) of Definition 1.1 holds and so H is an $\mathrm{H}(n; 6)$. Finally, for each column one can check that $s_i \not\equiv s_{i+3} \pmod{\nu}$, for $i = 1, 2, 3$, where $\nu \in \{12n+1, 12n+2\}$. So we conclude that H is a globally simple $\mathrm{SH}(n; 6)$ that also satisfies condition $(*)$, namely H is an $\mathrm{SH}^*(n; 6)$. □

Example 4.2 Following the construction illustrated in the proof of Proposition 4.1 for $n = 8$, we obtain

$\mathrm{SH}^*(8; 6) =$

5	−1	2	−7	−9	10		
−4	3	−6	8	11	−12		
		−13	17	14	−19	−21	22
		15	−16	−18	20	23	−24
34	−33			−25	29	26	−31
−36	35			27	−28	−30	32
−43	38	−45	46			−37	41
44	−42	47	−48			39	−40

Proposition 4.3 *Let $n \equiv 0 \pmod 4$ and $n \geq 8$. Then, there exists an* $\mathrm{SH}^*(n; 7)$.

Proof: An $\mathrm{SH}^*(8; 7)$ and an $\mathrm{SH}^*(12; 7)$ are given in [12]. So, assume $n = 4a \geq 16$ and let H be the $n \times n$ array whose rows R_t are the following:

$$\begin{array}{rcl} R_1 & = & (-5a, \square^{2a-8}, 3-24a, \square, -3+16a, \square^3, -1-2a, \square^{2a-6}, -4+24a, \square, 4-16a, \\ & & \square, 1+7a, \square), \\ R_2 & = & (\square, -2-6a, \square^{2a-8}, 5-28a, \square, -5+20a, \square^3, 2, \square^{2a-6}, -6+28a, \square, 6-20a, \\ & & \square, 6a), \\ R_3 & = & (2+7a, \square, 1-5a, \square^{2a-8}, 7-24a, \square, -7+16a, \square^3, -3-2a, \square^{2a-6}, -8+24a, \\ & & \square, 8-16a, \square), \\ R_4 & = & (\square, -1+6a, \square, -3-6a, \square^{2a-8}, 9-28a, \square, -9+20a, \square^3, 4, \square^{2a-6}, -10+28a, \\ & & \square, 10-20a), \\ R_5 & = & (12-16a, \square, 3+7a, \square, 2-5a, \square^{2a-8}, 11-24a, \square, -11+16a, \square^3, -5-2a, \\ & & \square^{2a-6}, -12+24a, \square), \\ R_6 & = & (\square, 14-20a, \square, -2+6a, \square, -4-6a, \square^{2a-8}, 13-28a, \square, -13+20a, \square^3, 6, \\ & & \square^{2a-6}, -14+28a), \\ R_{7+2i} & = & (\square^{2i}, -16+24a-4i, \square, 16-16a+4i, \square, 4+7a+i, \square, 3-5a+i, \square^{2a-8}, \\ & & 15-24a+4i, \square, -15+16a-4i, \square^3, -7-2a-2i, \square^{2a-6-2i}), \end{array}$$

$$\begin{array}{rcl}
R_{8+2i} & = & (\square^{1+2i}, -18+28a-4i, \square, 18-20a+4i, \square, -3+6a-i, \square, -5-6a-i, \square^{2a-8},\\
& & 17-28a+4i, \square, -17+20a-4i, \square^3, 8+2i, \square^{2a-7-2i}),\\
R_{2a-1} & = & (\square^{2a-8}, 28a, \square, -20a, \square, 8a, \square, -12a, \square^{2a-8}, -1-20a, \square, 1+12a, \square^3, 4a, \square^2),\\
R_{2a} & = & (\square^{2a-7}, -2+24a, \square, 2-16a, \square, 1+5a, \square, -9a, \square^{2a-8}, 1-24a, \square, -1+16a,\\
& & \square^3, -1+4a, \square),\\
R_{2a+1} & = & (\square^{2a-6}, -4+28a, \square, 4-20a, \square, 10a, \square, -1-10a, \square^{2a-8}, 3-28a, \square, -3+20a,\\
& & \square^3, 1),\\
R_{2a+2} & = & (-2-2a, \square^{2a-6}, -6+24a, \square, 6-16a, \square, 1+11a, \square, 1-9a, \square^{2a-8}, 5-24a,\\
& & \square, -5+16a, \square^3),\\
R_{2a+3} & = & (\square, 3, \square^{2a-6}, -8+28a, \square, 8-20a, \square, -1+10a, \square, -2-10a, \square^{2a-8}, 7-28a,\\
& & \square, -7+20a, \square^2),\\
R_{2a+4} & = & (\square^2, -4-2a, \square^{2a-6}, -10+24a, \square, 10-16a, \square, 2+11a, \square, 2-9a, \square^{2a-8},\\
& & 9-24a, \square, -9+16a, \square),\\
R_{2a+5} & = & (\square^3, 5, \square^{2a-6}, -12+28a, \square, 12-20a, \square, -2+10a, \square, -3-10a, \square^{2a-8},\\
& & 11-28a, \square, -11+20a),\\
R_{2a+6} & = & (-13+16a, \square^3, -6-2a, \square^{2a-6}, -14+24a, \square, 14-16a, \square, 3+11a, \square, 3-9a,\\
& & \square^{2a-8}, 13-24a, \square),\\
R_{2a+7} & = & (\square, -15+20a, \square^3, 7, \square^{2a-6}, -16+28a, \square, 16-20a, \square, -3+10a, \square, -4-10a,\\
& & \square^{2a-8}, 15-28a),\\
R_{2a+8+2i} & = & (\square^{2i}, 17-24a+4i, \square, -17+16a-4i, \square^3, -8-2a-2i, \square^{2a-6},\\
& & -18+24a-4i, \square, 18-16a+4i, \square, 4+11a+i, \square, 4-9a+i, \square^{2a-8-2i}),\\
R_{2a+9+2i} & = & (\square^{1+2i}, 19-28a+4i, \square, -19+20a-4i, \square^3, 9+2i, \square^{2a-6}, -20+28a-4i,\\
& & \square, 20-20a+4i, \square, -4+10a-i, \square, -5-10a-i, \square^{2a-9-2i}),\\
R_{4a} & = & (\square^{2a-8}, 1-28a, \square, -1+20a, \square^3, 2a, \square^{2a-6}, -2+28a, \square, 2-20a,\\
& & \square, 1+4a, \square, -1-6a),
\end{array}$$

where $i = 0, \ldots, a-5$. Note that each row contains exactly 7 elements. It is not hard to see that

$$\begin{array}{rcl}
\left\| \cup_{i=0}^{a-2} R_{2+2i} \cup \cup_{j=0}^{a} R_{2a-1+2j} \cup R_{4a} \right\| & = & \{1, \ldots, 2a\} \cup \{4a, 4a+1\} \cup \{5a+2, \ldots, 7a\} \cup\\
& & \{8a\} \cup \{9a+1, \ldots, 11a\} \cup \{12a, 12a+1\} \cup\\
& & \{16a, \ldots, 20a+1\} \cup \{24a, \ldots, 28a\},\\
\left\| \cup_{i=0}^{a-2} R_{1+2i} \cup \cup_{j=0}^{a-1} R_{2a+2j} \right\| & = & \{2a+1, \ldots, 4a-1\} \cup \{4a+2, \ldots, 5a+1\} \cup \{7a+1, \ldots,\\
& & 8a-1\} \cup \{8a+1, \ldots, 9a\} \cup \{11a+1, \ldots, 12a-1\} \cup\\
& & \{12a+2, \ldots, 16a-1\} \cup \{20a+2, \ldots, 24a-1\}.
\end{array}$$

Hence, H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$\begin{array}{rcl}
\mathcal{S}(R_1) & = & (-5a, 3-29a, -13a, -1-15a, -5+9a, -1-7a, 0),\\
\mathcal{S}(R_2) & = & (-2-6a, 3-34a, -2-14a, -14a, -6+14a, -6a, 0),\\
\mathcal{S}(R_3) & = & (2+7a, 3+2a, 10-22a, 3-6a, -8a, -8+16a, 0),\\
\mathcal{S}(R_4) & = & (-1+6a, -4, 5-28a, -4-8a, -8a, -10+20a, 0),\\
\mathcal{S}(R_5) & = & (12-16a, 15-9a, 17-14a, 28-38a, 17-22a, 12-24a, 0),\\
\mathcal{S}(R_6) & = & (14-20a, 12-14a, 8-20a, 21-48a, 8-28a, 14-28a, 0)
\end{array}$$

$$\begin{aligned}
\mathcal{S}(R_{7+2i}) &= (-16+24a-4i, 8a, 4+15a+i, 7+10a+2i, 22-14a+6i, 7+2a+2i, 0),\\
\mathcal{S}(R_{8+2i}) &= (-18+28a-4i, 8a, -3+14a-i, -8+8a-2i, 9-20a+2i, -8-2i, 0),\\
\mathcal{S}(R_{2a-1}) &= (28a, 8a, 16a, 4a, -1-16a, -4a, 0),\\
\mathcal{S}(R_{2a}) &= (-2+24a, 8a, 1+13a, 1+4a, 2-20a, 1-4a, 0),\\
\mathcal{S}(R_{2a+1}) &= (-4+28a, 8a, 18a, -1+8a, 2-20a, -1, 0),\\
\mathcal{S}(R_{2a+2}) &= (-2-2a, -8+22a, -2+6a, -1+17a, 8a, 5-16a, 0),\\
\mathcal{S}(R_{2a+3}) &= (3, -5+28a, 3+8a, 2+18a, 8a, 7-20a, 0),\\
\mathcal{S}(R_{2a+4}) &= (-4-2a, -14+22a, -4+6a, -2+17a, 8a, 9-16a, 0),\\
\mathcal{S}(R_{2a+5}) &= (5, -7+28a, 5+8a, 3+18a, 8a, 11-20a, 0),\\
\mathcal{S}(R_{2a+6}) &= (-13+16a, -19+14a, -33+38a, -19+22a, -16+33a, -13+24a, 0),\\
\mathcal{S}(R_{2a+7}) &= (-15+20a, -8+20a, -24+48a, -8+28a, -11+38a, -15+28a, 0),\\
\mathcal{S}(R_{2a+8+2i}) &= (17-24a+4i, -8a, -8-10a-2i, -26+14a-6i, -8-2a-2i,\\
&\qquad -4+9a-i, 0),\\
\mathcal{S}(R_{2a+9+2i}) &= (19-28a+4i, -8a, 9-8a+2i, -11+20a-2i, 9+2i, 5+10a+i, 0),\\
\mathcal{S}(R_{4a}) &= (1-28a, -8a, -6a, -2+22a, 2a, 1+6a, 0).
\end{aligned}$$

Clearly, each row sums to 0. By a direct check (keeping in mind previous considerations on partial sums) one can see that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct modulo $14n+1$ and modulo $14n+2$ for any $n \equiv 0 \pmod 4$. From the definition of H we obtain the following expression of the columns:

$$\begin{aligned}
C_1 &= (-5a, \square, 2+7a, \square, 12-16a, \square, -16+24a, \square^{2a-6}, -2-2a, \square^3, -13+16a, \square,\\
&\qquad 17-24a, \square^{2a-8})^T,\\
C_{2+2i} &= (\square^{1+2i}, -2-6a-i, \square, -1+6a-i, \square, 14-20a+4i, \square, -18+28a-4i, \square^{2a-6},\\
&\qquad 3+2i, \square^3, -15+20a-4i, \square, 19-28a+4i, \square^{2a-9-2i})^T,\\
C_{3+2h} &= (\square^{2+2h}, 1-5a+h, \square, 3+7a+h, \square, 16-16a+4h, \square, -20+24a-4h, \square^{2a-6},\\
&\qquad -4-2a-2h, \square^3, -17+16a-4h, \square, 21-24a+4h, \square^{2a-10-2h})^T,\\
C_{2a-7} &= (\square^{2a-8}, -4-4a, \square, -2+8a, \square, -4-12a, \square, 28a, \square^{2a-6}, 6-4a, \square^3, 3+12a,\\
&\qquad \square, 1-28a)^T,\\
C_{2a-6} &= (3-24a, \square^{2a-8}, 2-7a, \square, 3+5a, \square, -2-16a, \square, -2+24a, \square^{2a-6}, -5+2a,\\
&\qquad \square^3, 1+16a, \square)^T,\\
C_{2a-5} &= (\square, 5-28a, \square^{2a-8}, -3-4a, \square, -1+8a, \square, -20a, \square, -4+28a, \square^{2a-6}, 4-4a,\\
&\qquad \square^3, -1+20a)^T,\\
C_{2a-4} &= (-3+16a, \square, 7-24a, \square^{2a-8}, 1-7a, \square, 2+5a, \square, 2-16a, \square, -6+24a, \square^{2a-6},\\
&\qquad -3+2a, \square^3)^T,\\
C_{2a-3} &= (\square, -5+20a, \square, 9-28a, \square^{2a-8}, -2-4a, \square, 8a, \square, 4-20a, \square, -8+28a, \square^{2a-6},\\
&\qquad 2-4a, \square^2)^T,\\
C_{2a-2} &= (\square^2, -7+16a, \square, 11-24a, \square^{2a-8}, -7a, \square, 1+5a, \square, 6-16a, \square, -10+24a,\\
&\qquad \square^{2a-6}, -1+2a, \square)^T,\\
C_{2a-1} &= (\square^3, -9+20a, \square, 13-28a, \square^{2a-8}, -12a, \square, 10a, \square, 8-20a, \square, -12+28a,\\
&\qquad \square^{2a-6}, 2a)^T,
\end{aligned}$$

$$\begin{aligned}
C_{2a+2j} &= (\square^{2j}, -1-2a-2j, \square^3, -11+16a-4j, \square, 15-24a+4j, \square^{2a-8}, -9a+j, \square,\\
&\quad 1+11a+j, \square, 10-16a+4j, \square, -14+24a-4j, \square^{2a-6-2j})^T,\\
C_{2a+1+2j} &= (\square^{1+2j}, 2+2j, \square^3, -13+20a-4j, \square, 17-28a+4j, \square^{2a-8}, -1-10a-j,\\
&\quad \square, -1+10a-j, \square, 12-20a+4j, \square, -16+28a-4j, \square^{2a-7-2j})^T,\\
C_{4a-6} &= (\square^{2a-6}, 5-4a, \square^3, 1+12a, \square, 3-28a, \square^{2a-8}, -3-8a, \square, -2+12a, \square,\\
&\quad -2-12a, \square, -2+28a)^T,\\
C_{4a-5} &= (-4+24a, \square^{2a-6}, -4+2a, \square^3, -1+16a, \square, 5-24a, \square^{2a-8}, 2-11a, \square,\\
&\quad 2+9a, \square, -16a, \square)^T,\\
C_{4a-4} &= (\square, -6+28a, \square^{2a-6}, 3-4a, \square^3, -3+20a, \square, 7-28a, \square^{2a-8}, -2-8a, \square,\\
&\quad -1+12a, \square, 2-20a)^T,\\
C_{4a-3} &= (4-16a, \square, -8+24a, \square^{2a-6}, -2+2a, \square^3, -5+16a, \square, 9-24a, \square^{2a-8}, 1-11a,\\
&\quad \square, 1+9a, \square)^T,\\
C_{4a-2} &= (\square, 6-20a, \square, -10+28a, \square^{2a-6}, 4a, \square^3, -7+20a, \square, 11-28a, \square^{2a-8}, -1-8a,\\
&\quad \square, 1+4a)^T,\\
C_{4a-1} &= (1+7a, \square, 8-16a, \square, -12+24a, \square^{2a-6}, -1+4a, \square^3, -9+16a, \square, 13-24a,\\
&\quad \square^{2a-8}, -11a, \square)^T,\\
C_{4a} &= (\square, 6a, \square, 10-20a, \square, -14+28a, \square^{2a-6}, 1, \square^3, -11+20a, \square, 15-28a,\\
&\quad \square^{2a-8}, -1-6a)^T,
\end{aligned}$$

for $i = 0, \dots, a-5$, $h = 0, \dots, a-6$ and $j = 0, \dots, a-4$. Note that each column contains exactly 7 elements, hence H satisfies also condition (c). One can check that the partial sums for the columns are the following:

$$\begin{aligned}
\mathcal{S}(C_1) &= (-5a, 2+2a, 14-14a, -2+10a, -4+8a, -17+24a, 0),\\
\mathcal{S}(C_{2+2i}) &= (-2-6a-i, -3-2i, 11-20a+2i, -7+8a-2i, -4+8a, -19+28a-4i, 0),\\
\mathcal{S}(C_{3+2h}) &= (1-5a+h, 4+2a+2h, 20-14a+6h, 10a+2h, -4+8a, -21+24a-4h, 0),\\
\mathcal{S}(C_{2a-7}) &= (-4-4a, -6+4a, -10-8a, -10+20a, -4+16a, -1+28a, 0),\\
\mathcal{S}(C_{2a-6}) &= (3-24a, 5-31a, 8-26a, 6-42a, 4-18a, -1-16a, 0),\\
\mathcal{S}(C_{2a-5}) &= (5-28a, 2-32a, 1-24a, 1-44a, -3-16a, 1-20a, 0),\\
\mathcal{S}(C_{2a-4}) &= (-3+16a, 4-8a, 5-15a, 7-10a, 9-26a, 3-2a, 0),\\
\mathcal{S}(C_{2a-3}) &= (-5+20a, 4-8a, 2-12a, 2-4a, 6-24a, -2+4a, 0),\\
\mathcal{S}(C_{2a-2}) &= (-7+16a, 4-8a, 4-15a, 5-10a, 11-26a, 1-2a, 0),\\
\mathcal{S}(C_{2a-1}) &= (-9+20a, 4-8a, 4-20a, 4-10a, 12-30a, -2a, 0),\\
\mathcal{S}(C_{2a+2j}) &= (-1-2a-2j, -12+14a-6j, 3-10a-2j, 3-19a-j, 4-8a,\\
&\quad 14-24a+4j, 0),\\
\mathcal{S}(C_{2a+1+2j}) &= (2+2j, -11+20a-2j, 6-8a+2j, 5-18a+j, 4-8a, 16-28a+4j, 0),\\
\mathcal{S}(C_{4a-6}) &= (5-4a, 6+8a, 9-20a, 6-28a, 4-16a, 2-28a, 0),\\
\mathcal{S}(C_{4a-5}) &= (-4+24a, -8+26a, -9+42a, -4+18a, -2+7a, 16a, 0),\\
\mathcal{S}(C_{4a-4}) &= (-6+28a, -3+24a, -6+44a, 1+16a, -1+8a, -2+20a, 0),\\
\mathcal{S}(C_{4a-3}) &= (4-16a, -4+8a, -6+10a, -11+26a, -2+2a, -1-9a, 0),\\
\mathcal{S}(C_{4a-2}) &= (6-20a, -4+8a, -4+12a, -11+32a, 4a, -1-4a, 0),\\
\mathcal{S}(C_{4a-1}) &= (1+7a, 9-9a, -3+15a, -4+19a, -13+35a, 11a, 0),\\
\mathcal{S}(C_{4a}) &= (6a, 10-14a, -4+14a, -3+14a, -14+34a, 1+6a, 0).
\end{aligned}$$

Note that each column sums to 0 and so condition (d) is satisfied, hence H is an H$(n;7)$. By a direct check one can verify that the elements of each $\mathcal{S}(C_t)$ are pairwise distinct modulo $14n+1$ and modulo $14n+2$ for any $n \equiv 0 \pmod 4$. Thus, for these values of n, H is an SH$^*(n;7)$. □

Proposition 4.4 *Let* $n \equiv 1 \pmod 4$ *and* $n \geq 9$. *Then, there exists an* SH$^*(n;7)$.

PROOF: Assume $n = 4a+1 \geq 9$ and let H be the $n \times n$ array whose rows R_t are the following:

$$\begin{array}{rcl}
R_1 &=& (-1-4a, -5-16a, -2-7a, 5+12a, \square^{4a-6}, 6+28a, 3+11a, -6-24a),\\
R_2 &=& (-5-18a, -4a, -7-22a, -3-8a, 6+14a, \square^{4a-6}, 6+26a, 3+12a),\\
R_3 &=& (3+9a, -6-20a, -1+2a, -7-20a, -2-11a, 6+12a, \square^{4a-6}, 7+28a),\\
R_{4+4i} &=& (\square^{4i}, 7+26a+2i, 2+8a-i, -6-18a-2i, 1-4a+2i, -8-22a-2i,\\
&& -3-4a-i, 7+14a+2i, \square^{4a-6-4i}),\\
R_{5+4i} &=& (\square^{1+4i}, 7+24a+2i, 3+5a+i, -6-16a-2i, -2+2a-2i, -8-20a-2i,\\
&& -1-7a+i, 7+12a+2i, \square^{4a-7-4i}),\\
R_{6+4i} &=& (\square^{2+4i}, 8+26a+2i, 2+12a-i, -7-18a-2i, 2-4a+2i, -9-22a-2i,\\
&& -4-8a-i, 8+14a+2i, \square^{4a-8-4i}),\\
R_{7+4j} &=& (\square^{3+4j}, 8+24a+2j, 4+9a+j, -7-16a-2j, -3+2a-2j, -9-20a-2j,\\
&& -1-11a+j, 8+12a+2j, \square^{4a-9-4j}),\\
R_{4a-1} &=& (4+14a, \square^{4a-6}, 4+26a, 2+10a, -3-18a, 1, -5-22a, -3-10a),\\
R_{4a} &=& (-2-5a, 4+12a, \square^{4a-6}, 5+28a, 3+7a, -4-20a, -1-2a, -5-20a),\\
R_{4a+1} &=& (-6-22a, -2-4a, 5+14a, \square^{4a-6}, 5+26a, 2+6a, -4-18a, -2a),
\end{array}$$

where $i = 0, \ldots, a-2$ and $j = 0, \ldots, a-3$. Note that each row contains exactly 7 elements. One can see that

$$\begin{array}{rcl}
\cup_{i=-1}^{1} \|R_{2+i} \cup R_{4a+i}\| &=& \{1, 2a-1, 2a, 2a+1, 4a, 4a+1, 4a+2, 5a+2, 6a+2, 7a+2,\\
&& 7a+3, 8a+3, 9a+3, 10a+2, 10a+3, 11a+2, 11a+3\} \cup\\
&& \{12a+3, \ldots, 12a+6\} \cup \{14a+4, 14a+5, 14a+6, 16a+5,\\
&& 18a+3, 18a+4, 18a+5\} \cup \{20a+4, \ldots, 20a+7\} \cup \{22a+5,\\
&& 22a+6, 22a+7, 24a+6, 26a+4, 26a+5, 26a+6, 28a+5,\\
&& 28a+6, 28a+7\},\\
\left\|\cup_{i=0}^{4a-6} R_{4+i}\right\| &=& \{2, \ldots, 2a-2\} \cup \{2a+2, \ldots, 4a-1\} \cup \{4a+3, \ldots, 5a+1\} \cup\\
&& \{5a+3, \ldots, 6a+1\} \cup \{6a+3, \ldots, 7a+1\} \cup \{7a+4, \ldots, 8a+2\} \cup\\
&& \{8a+4, \ldots, 9a+2\} \cup \{9a+4, \ldots, 10a+1\} \cup \{10a+4, \ldots, 11a+1\} \cup\\
&& \{11a+4, \ldots, 12a+2\} \cup \{12a+7, \ldots, 14a+3\} \cup \{14a+7, \ldots, 16a+4\} \cup\\
&& \{16a+6, \ldots, 18a+2\} \cup \{18a+6, \ldots, 20a+3\} \cup \{20a+8, \ldots, 22a+4\} \cup\\
&& \{22a+8, \ldots, 24a+5\} \cup \{24a+7, \ldots, 26a+3\} \cup \{26a+7, \ldots, 28a+4\}.
\end{array}$$

Hence, H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$\begin{array}{rcl}
\mathcal{S}(R_1) &=& (-1-4a, -6-20a, -8-27a, -3-15a, 3+13a, 6+24a, 0),\\
\mathcal{S}(R_2) &=& (-5-18a, -5-22a, -12-44a, -15-52a, -9-38a, -3-12a, 0),\\
\mathcal{S}(R_3) &=& (3+9a, -3-11a, -4-9a, -11-29a, -13-40a, -7-28a, 0),\\
\mathcal{S}(R_{4+4i}) &=& (7+26a+2i, 9+34a+i, 3+16a-i, 4+12a+i, -4-10a-i, -7-14a-2i, 0),
\end{array}$$

$$
\begin{aligned}
\mathcal{S}(R_{5+4i}) &= (7+24a+2i, 10+29a+3i, 4+13a+i, 2+15a-i, -6-5a-3i,\\
&\quad -7-12a-2i, 0),\\
\mathcal{S}(R_{6+4i}) &= (8+26a+2i, 10+38a+i, 3+20a-i, 5+16a+i, -4-6a-i, -8-14a-2i, 0),\\
\mathcal{S}(R_{7+4j}) &= (8+24a+2j, 12+33a+3j, 5+17a+j, 2+19a-j, -7-a-3j,\\
&\quad -8-12a-2j, 0),\\
\mathcal{S}(R_{4a-1}) &= (4+14a, 8+40a, 10+50a, 7+32a, 8+32a, 3+10a, 0),\\
\mathcal{S}(R_{4a}) &= (-2-5a, 2+7a, 7+35a, 10+42a, 6+22a, 5+20a, 0),\\
\mathcal{S}(R_{4a+1}) &= (-6-22a, -8-26a, -3-12a, 2+14a, 4+20a, 2a, 0).
\end{aligned}
$$

Note that each row sums to 0. By a direct check one can verify that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct modulo $14n+1$ and modulo $14n+2$ for any $n \equiv 1 \pmod 4$. From the definition of H we obtain the following expression of the columns:

$$
\begin{aligned}
C_1 &= (-1-4a, -5-18a, 3+9a, 7+26a, \square^{4a-6}, 4+14a, -2-5a, -6-22a)^T,\\
C_2 &= (-5-16a, -4a, -6-20a, 2+8a, 7+24a, \square^{4a-6}, 4+12a, -2-4a)^T,\\
C_3 &= (-2-7a, -7-22a, -1+2a, -6-18a, 3+5a, 8+26a, \square^{4a-6}, 5+14a)^T,\\
C_{4+4i} &= (\square^{4i}, 5+12a+2i, -3-8a-i, -7-20a-2i, 1-4a+2i, -6-16a-2i,\\
&\quad 2+12a-i, 8+24a+2i, \square^{4a-6-4i})^T,\\
C_{5+4i} &= (\square^{1+4i}, 6+14a+2i, -2-11a+i, -8-22a-2i, -2+2a-2i, -7-18a-2i,\\
&\quad 4+9a+i, 9+26a+2i, \square^{4a-7-4i})^T,\\
C_{6+4i} &= (\square^{2+4i}, 6+12a+2i, -3-4a-i, -8-20a-2i, 2-4a+2i, -7-16a-2i,\\
&\quad 1+8a-i, 9+24a+2i, \square^{4a-8-4i})^T,\\
C_{7+4j} &= (\square^{3+4j}, 7+14a+2j, -1-7a+j, -9-22a-2j, -3+2a-2j, -8-18a-2j,\\
&\quad 4+5a+j, 10+26a+2j, \square^{4a-9-4j})^T,\\
C_{4a-1} &= (6+28a, \square^{4a-6}, 3+16a, -3-6a, -5-24a, 1, -4-20a, 2+6a)^T,\\
C_{4a} &= (3+11a, 6+26a, \square^{4a-6}, 3+14a, -2-9a, -5-22a, -1-2a, -4-18a)^T,\\
C_{4a+1} &= (-6-24a, 3+12a, 7+28a, \square^{4a-6}, 4+16a, -3-10a, -5-20a, -2a)^T,
\end{aligned}
$$

where $i = 0, \dots, a-2$ and $j = 0, \dots, a-3$. Note that each column contains exactly 7 elements, hence H satisfies also condition (c). One can check that the partial sums for the columns are the following:

$$
\begin{aligned}
\mathcal{S}(C_1) &= (-1-4a, -6-22a, -3-13a, 4+13a, 8+27a, 6+22a, 0),\\
\mathcal{S}(C_2) &= (-5-16a, -5-20a, -11-40a, -9-32a, -2-8a, 2+4a, 0),\\
\mathcal{S}(C_3) &= (-2-7a, -9-29a, -10-27a, -16-45a, -13-40a, -5-14a, 0),\\
\mathcal{S}(C_{4+4i}) &= (5+12a+2i, 2+4a+i, -5-16a-i, -4-20a+i, -10-36a-i,\\
&\quad -8-24a-2i, 0),\\
\mathcal{S}(C_{5+4i}) &= (6+14a+2i, 4+3a+3i, -4-19a+i, -6-17a-i, -13-35a-3i,\\
&\quad -9-26a-2i, 0),\\
\mathcal{S}(C_{6+4i}) &= (6+12a+2i, 3+8a+i, -5-12a-i, -3-16a+i, -10-32a-i,\\
&\quad -9-24a-2i, 0),\\
\mathcal{S}(C_{7+4j}) &= (7+14a+2j, 6+7a+3j, -3-15a+j, -6-13a-j, -14-31a-3j,\\
&\quad -10-26a-2j, 0),\\
\mathcal{S}(C_{4a-1}) &= (6+28a, 9+44a, 6+38a, 1+14a, 2+14a, -2-6a, 0),
\end{aligned}
$$

$$\begin{aligned}\mathcal{S}(C_{4a}) &= (3+11a, 9+37a, 12+51a, 10+42a, 5+20a, 4+18a, 0),\\ \mathcal{S}(C_{4a+1}) &= (-6-24a, -3-12a, 4+16a, 8+32a, 5+22a, 2a, 0).\end{aligned}$$

Since each column sums to 0, condition (d) holds and hence H is an $\mathrm{H}(n;7)$. By a direct check one can verify that the elements of each $\mathcal{S}(C_t)$ are pairwise distinct modulo $14n+1$ and modulo $14n+2$ for any $n \equiv 1 \pmod 4$. Thus, for these values of n, H is an $\mathrm{SH}^*(n;7)$. □

Example 4.5 Let $n = 9$, by the construction given in the proof of Proposition 4.4, we obtain the following $\mathrm{SH}^*(9;7)$:

−9	−37	−16	29			62	25	−54
−41	−8	−51	−19	34			58	27
21	−46	3	−47	−24	30			63
59	18	−42	−7	−52	−11	35		
	55	13	−38	2	−48	−15	31	
		60	26	−43	−6	−53	−20	36
32			56	22	−39	1	−49	−23
−12	28			61	17	−44	−5	−45
−50	−10	33			57	14	−40	−4

Proposition 4.6 *Let $n \geq 8$ be even. Then, there exists an* $\mathrm{SH}^*(n;8)$.

PROOF: An $\mathrm{SH}^*(8;8)$ can be found in [12]. So, assume $n \geq 10$.
Case 1. $n \equiv 0, 2 \pmod 6$. Let H be the $n \times n$ array whose rows R_t are the following:

$$\begin{aligned}R_1 &= (-3+8n, 8n, 1-8n, 10-8n, 9-8n, 2-8n, \square^{n-8}, -11+8n, -8+8n),\\ R_{2+2i} &= (\square^{2+2i}, -17-16i, -20-16i, -25-16i, -28-16i, 27+16i, 18+16i, 19+16i,\\ &\quad 26+16i, \square^{n-10-2i}),\\ R_{3+2j} &= (\square^{2j}, 5+16j, 8+16j, 13+16j, 16+16j, -15-16j, -6-16j, -7-16j,\\ &\quad -14-16j, \square^{n-8-2j}),\\ R_{n-6} &= (-45+8n, -38+8n, \square^{n-8}, 47-8n, 44-8n, 39-8n, 36-8n, -37+8n, -46+8n),\\ R_{n-4} &= (-21+8n, -30+8n, -29+8n, -22+8n, \square^{n-8}, 31-8n, 28-8n, 23-8n, 20-8n),\\ R_{n-3} &= (41-8n, 34-8n, \square^{n-8}, -43+8n, -40+8n, -35+8n, -32+8n, 33-8n, 42-8n),\\ R_{n-2} &= (7-8n, 4-8n, -5+8n, -14+8n, -13+8n, -6+8n, \square^{n-8}, 15-8n, 12-8n),\\ R_{n-1} &= (17-8n, 26-8n, 25-8n, 18-8n, \square^{n-8}, -27+8n, -24+8n, -19+8n, -16+8n),\\ R_n &= (-1, -4, -9, -12, 11, 2, 3, 10, \square^{n-8}),\end{aligned}$$

where $i = 0, \ldots, \frac{n-10}{2}$ and $j = 0, \ldots, \frac{n-8}{2}$. Note that each row has exactly 8 filled cells. It is easy to see that $\|R_3 \cup R_n\| = \{1, \ldots, 16\}$, $\|R_1 \cup R_{n-2}\| = \{8n-15, \ldots, 8n\}$, and $\|R_{2h} \cup R_{2h+3}\| = \{1+16h, \ldots, 16+16h\}$, for all $h = 1, \ldots, \frac{n-4}{2}$. Hence, conditions (a) and (b) of Definition 1.1 hold. Now, we list the partial sums for each row. We have

$$\begin{aligned}\mathcal{S}(R_1) &= (-3+8n, -3+16n, -2+8n, 8, 17-8n, 19-16n, 8-8n, 0),\\ \mathcal{S}(R_{2+2i}) &= (-17-16i, -37-32i, -62-48i, -90-64i, -63-48i, -45-32i, -26-16i, 0),\\ \mathcal{S}(R_{3+2j}) &= (5+16j, 13+32j, 26+48j, 42+64j, 27+48j, 21+32j, 14+16j, 0),\\ \mathcal{S}(R_{n-6}) &= (-45+8n, -83+16n, -36+8n, 8, 47-8n, 83-16n, 46-8n, 0),\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(R_{n-4}) &= (-21+8n, -51+16n, -80+24n, -102+32n, -71+24n, -43+16n,\\
&\quad -20+8n, 0),\\
\mathcal{S}(R_{n-3}) &= (41-8n, 75-16n, 32-8n, -8, -43+8n, -75+16n, -42+8n, 0),\\
\mathcal{S}(R_{n-2}) &= (7-8n, 11-16n, 6-8n, -8, -21+8n, -27+16n, -12+8n, 0),\\
\mathcal{S}(R_{n-1}) &= (17-8n, 43-16n, 68-24n, 86-32n, 59-24n, 35-16n, 16-8n, 0),\\
\mathcal{S}(R_n) &= (-1, -5, -14, -26, -15, -13, -10, 0).
\end{aligned}$$

It is not so hard to see that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct both modulo $16n+1$ and modulo $16n+2$ for any even integer n. From the definition of H we obtain the following expression of the columns:

$$\begin{aligned}
C_1 &= (-3+8n, \square, 5, \square^{n-10}, -45+8n, \square, -21+8n, 41-8n, 7-8n, 17-8n, -1)^T,\\
C_2 &= (8n, \square, 8, \square^{n-10}, -38+8n, \square, -30+8n, 34-8n, 4-8n, 26-8n, -4)^T,\\
C_3 &= (1-8n, -17, 13, \square, 21, \square^{n-10}, -29+8n, \square, -5+8n, 25-8n, -9)^T,\\
C_4 &= (10-8n, -20, 16, \square, 24, \square^{n-10}, -22+8n, \square, -14+8n, 18-8n, -12)^T,\\
C_5 &= (9-8n, -25, -15, -33, 29, \square, 37, \square^{n-10}, -13+8n, \square, 11)^T,\\
C_6 &= (2-8n, -28, -6, -36, 32, \square, 40, \square^{n-10}, -6+8n, \square, 2)^T,\\
C_7 &= (\square, 27, -7, -41, -31, -49, 45, \square, 53, \square^{n-10}, 3)^T,\\
C_8 &= (\square, 18, -14, -44, -22, -52, 48, \square, 56, \square^{n-10}, 10)^T,\\
C_{9+2i} &= (\square^{1+2i}, 19+16i, \square, 43+16i, -23-16i, -57-16i, -47-16i, -65-16i,\\
&\quad 61+16i, \square, 69+16i, \square^{n-11-2i})^T,\\
C_{10+2i} &= (\square^{1+2i}, 26+16i, \square, 34+16i, -30-16i, -60-16i, -38-16i, -68-16i,\\
&\quad 64+16i, \square, 72+16i, \square^{n-11-2i})^T,\\
C_{n-1} &= (-11+8n, \square^{n-10}, -61+8n, \square, -37+8n, 57-8n, 23-8n, 33-8n, 15-8n,\\
&\quad -19+8n, \square)^T,\\
C_n &= (-8+8n, \square^{n-10}, -54+8n, \square, -46+8n, 50-8n, 20-8n, 42-8n, 12-8n,\\
&\quad -16+8n, \square)^T,
\end{aligned}$$

where $i = 0, \dots, \frac{n-12}{2}$. Each column has 8 filled cells, hence H satisfies also condition (c). The partial sums for the columns are the following:

$$\begin{aligned}
\mathcal{S}(C_1) &= (-3+8n, 2+8n, -43+16n, -64+24n, -23+16n, -16+8n, 1, 0),\\
\mathcal{S}(C_2) &= (8n, 8+8n, -30+16n, -60+24n, -26+16n, -22+8n, 4, 0),\\
\mathcal{S}(C_3) &= (1-8n, -16-8n, -3-8n, 18-8n, -11, -16+8n, 9, 0),\\
\mathcal{S}(C_4) &= (10-8n, -10-8n, 6-8n, 30-8n, 8, -6+8n, 12, 0),\\
\mathcal{S}(C_5) &= (9-8n, -16-8n, -31-8n, -64-8n, -35-8n, 2-8n, -11, 0),\\
\mathcal{S}(C_6) &= (2-8n, -26-8n, -32-8n, -68-8n, -36-8n, 4-8n, -2, 0),\\
\mathcal{S}(C_7) &= (27, 20, -21, -52, -101, -56, -3, 0),\\
\mathcal{S}(C_8) &= (18, 4, -40, -62, -114, -66, -10, 0),\\
\mathcal{S}(C_{9+2i}) &= (19+16i, 62+32i, 39+16i, -18, -65-16i, -130-32i, -69-16i, 0),\\
\mathcal{S}(C_{10+2i}) &= (26+16i, 60+32i, 30+16i, -30, -68-16i, -136-32i, -72-16i, 0),\\
\mathcal{S}(C_{n-1}) &= (-11+8n, -72+16n, -109+24n, -52+16n, -29+8n, 4, 19-8n, 0),\\
\mathcal{S}(C_n) &= (-8+8n, -62+16n, -108+24n, -58+16n, -38+8n, 4, 16-8n, 0).
\end{aligned}$$

Since each column sums to 0, also condition (d) is satisfied. Hence H is an integer Heffter array. Then, again by a direct check, one can see that the elements of each

$\mathcal{S}(C_t)$ are always pairwise distinct modulo $16n+1$ for any even n. As, by hypothesis, $n \equiv 0$ or $2 \pmod 6$, the partial sums are distinct also modulo $16n+2$. Thus, for these values of n, H is an $\mathrm{SH}^*(n;8)$.

Case 2. $n \equiv 4 \pmod 6$. Let H be the $n \times n$ array whose rows R_t are the following:

$$\begin{array}{rcl}
R_1 &=& (-3+8n, 8n, 2-8n, 9-8n, 10-8n, 1-8n, \square^{n-8}, -11+8n, -8+8n),\\
R_{2+2i} &=& (\square^{2+2i}, -17-16i, -20-16i, -25-16i, -28-16i, 26+16i, 19+16i, 18+16i,\\
&& 27+16i, \square^{n-10-2i}),\\
R_{3+2j} &=& (\square^{2j}, 5+16j, 8+16j, 13+16j, 16+16j, -14-16j, -7-16j, -6-16j,\\
&& -15-16j, \square^{n-8-2j}),\\
R_{n-6} &=& (-46+8n, -37+8n, \square^{n-8}, 47-8n, 44-8n, 39-8n, 36-8n, -38+8n, -45+8n),\\
R_{n-3} &=& (42-8n, 33-8n, \square^{n-8}, -43+8n, -40+8n, -35+8n, -32+8n, 34-8n, 41-8n),\\
R_{n-2} &=& (7-8n, 4-8n, -6+8n, -13+8n, -14+8n, -5+8n, \square^{n-8}, 15-8n, 12-8n),\\
R_{n-1} &=& (18-8n, 25-8n, 26-8n, 17-8n, \square^{n-8}, -27+8n, -24+8n, -19+8n, -16+8n),\\
R_n &=& (-1, -4, -9, -12, 10, 3, 2, 11, \square^{n-8}),
\end{array}$$

where $i = 0, \ldots, \frac{n-10}{2}$ and $j = 0, \ldots, \frac{n-8}{2}$. Firstly, we note that each row of H has 8 filled cells. Then, one can check that $\|R_3 \cup R_n\| = \{1, \ldots, 16\}$, $\|R_1 \cup R_{n-2}\| = \{8n-15, \ldots, 8n\}$, and $\|R_{2h} \cup R_{2h+3}\| = \{1+16h, \ldots, 16+16h\}$, for all $h = 1, \ldots, \frac{n-4}{2}$. Hence, H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$\begin{array}{rcl}
\mathcal{S}(R_1) &=& (-3+8n, -3+16n, -1+8n, 8, 18-8n, 19-16n, 8-8n, 0),\\
\mathcal{S}(R_{2+2i}) &=& (-17-16i, -37-32i, -62-48i, -90-64i, -64-48i, -45-32i, -27-16i, 0),\\
\mathcal{S}(R_{3+2j}) &=& (5+16j, 13+32j, 26+48j, 42+64j, 27+48j, 21+32j, 14+16j, 0),\\
\mathcal{S}(R_{n-6}) &=& (-46+8n, -83+16n, -36+8n, 8, 47-8n, 83-16n, 45-8n, 0),\\
\mathcal{S}(R_{n-4}) &=& (-22+8n, -51+16n, -81+24n, -102+32n, -71+24n, -43+16n,\\
&& -20+8n, 0),\\
\mathcal{S}(R_{n-3}) &=& (42-8n, 75-16n, 32-8n, -8, -43+8n, -75+16n, -41+8n, 0),\\
\mathcal{S}(R_{n-2}) &=& (7-8n, 11-16n, 5-8n, -8, -22+8n, -27+16n, -12+8n, 0),\\
\mathcal{S}(R_{n-1}) &=& (18-8n, 43-16n, 69-24n, 86-32n, 59-24n, 35-16n, 16-8n, 0),\\
\mathcal{S}(R_n) &=& (-1, -5, -14, -26, -16, -13, -11, 0).
\end{array}$$

Note that each row sums to 0. It is not hard to check that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct modulo $16n+1$ and modulo $16n+2$, for any even n. From the definition of H we obtain the following expression of the columns:

$$\begin{array}{rcl}
C_1 &=& (-3+8n, \square, 5, \square^{n-10}, -46+8n, \square, -22+8n, 42-8n, 7-8n, 18-8n, -1)^T,\\
C_2 &=& (8n, \square, 8, \square^{n-10}, -37+8n, \square, -29+8n, 33-8n, 4-8n, 25-8n, -4)^T,\\
C_3 &=& (2-8n, -17, 13, \square, 21, \square^{n-10}, -30+8n, \square, -6+8n, 26-8n, -9)^T,\\
C_4 &=& (9-8n, -20, 16, \square, 24, \square^{n-10}, -21+8n, \square, -13+8n, 17-8n, -12)^T,\\
C_5 &=& (10-8n, -25, -14, -33, 29, \square, 37, \square^{n-10}, -14+8n, \square, 10)^T,\\
C_6 &=& (1-8n, -28, -7, -36, 32, \square, 40, \square^{n-10}, -5+8n, \square, 3)^T,\\
C_7 &=& (\square, 26, -6, -41, -30, -49, 45, \square, 53, \square^{n-10}, 2)^T,\\
C_8 &=& (\square, 19, -15, -44, -23, -52, 48, \square, 56, \square^{n-10}, 11)^T,
\end{array}$$

$$
\begin{aligned}
C_{9+2i} &= (\square^{1+2i}, 18+16i, \square, 42+16i, -22-16i, -57-16i, -46-16i, -65-16i,\\
&\quad 61+16i, \square, 69+16i, \square^{n-11-2i})^T,\\
C_{10+2i} &= (\square^{1+2i}, 27+16i, \square, 35+16i, -31-16i, -60-16i, -39-16i, -68-16i,\\
&\quad 64+16i, \square, 72+16i, \square^{n-11-2i})^T,\\
C_{n-1} &= (-11+8n, \square^{n-10}, -62+8n, \square, -38+8n, 58-8n, 23-8n, 34-8n,\\
&\quad 15-8n, -19+8n, \square)^T,\\
C_{n} &= (-8+8n, \square^{n-10}, -53+8n, \square, -45+8n, 49-8n, 20-8n, 41-8n,\\
&\quad 12-8n, -16+8n, \square)^T,
\end{aligned}
$$

where $i = 0, \ldots, \frac{n-12}{2}$. Since also each column has exactly 8 filled cells, then condition (c) is satisfied. Now we list the partial sums for the columns:

$$
\begin{aligned}
\mathcal{S}(C_1) &= (-3+8n, 2+8n, -44+16n, -66+24n, -24+16n, -17+8n, 1, 0),\\
\mathcal{S}(C_2) &= (8n, 8+8n, -29+16n, -58+24n, -25+16n, -21+8n, 4, 0),\\
\mathcal{S}(C_3) &= (2-8n, -15-8n, -2-8n, 19-8n, -11, -17+8n, 9, 0),\\
\mathcal{S}(C_4) &= (9-8n, -11-8n, 5-8n, 29-8n, 8, -5+8n, 12, 0),\\
\mathcal{S}(C_5) &= (10-8n, -15-8n, -29-8n, -62-8n, -33-8n, 4-8n, -10, 0),\\
\mathcal{S}(C_6) &= (1-8n, -27-8n, -34-8n, -70-8n, -38-8n, 2-8n, -3, 0),\\
\mathcal{S}(C_7) &= (26, 20, -21, -51, -100, -55, -2, 0),\\
\mathcal{S}(C_8) &= (19, 4, -40, -63, -115, -67, -11, 0),\\
\mathcal{S}(C_{9+2i}) &= (18+16i, 60+32i, 38+16i, -19, -65-16i, -130-32i, -69-16i, 0),\\
\mathcal{S}(C_{10+2i}) &= (27+16i, 62+32i, 31+16i, -29, -68-16i, -136-32i, -72-16i, 0),\\
\mathcal{S}(C_{n-1}) &= (-11+8n, -73+16n, -111+24n, -53+16n, -30+8n, 4, 19-8n, 0),\\
\mathcal{S}(C_n) &= (-8+8n, -61+16n, -106+24n, -57+16n, -37+8n, 4, 16-8n, 0).
\end{aligned}
$$

Each column sums to 0, hence H satisfies also condition (d). So, H is an integer Heffter array. Finally, again by a direct check, one can see that, since $n \equiv 4 \pmod 6$, the elements of each $\mathcal{S}(C_t)$ are pairwise distinct both modulo $16n+1$ and modulo $16n+2$. Thus, H is an $\mathrm{SH}^*(n;8)$ for any $n \equiv 4 \pmod 6$. □

We point out that the $\mathrm{SH}^*(10;8)$ given in Example 2.7 has been obtained following the proof of Proposition 4.6.

Proposition 4.7 *Let $n \geq 9$ be odd. Then, there exists an* $\mathrm{SH}^*(n;8)$.

Proof: If $n = 9, 11, 13, 15, 17, 19$ an $\mathrm{SH}^*(n;8)$ can be found in [12]. Let now $n \geq 21$ and let $a = n-9$, obviously a is an even integer and $a \geq 12$. Let H be the $(a+9)\times(a+9)$ array whose first a rows are the ones of the $a \times a$ array constructed in Proposition 4.6 with nine empty cells at the end and the last nine rows are the following:

$$
\begin{aligned}
R_{a+1} &= (\square^a, 3+8a, -61-8a, -20-8a, -19-8a, 68+8a, \square, 44+8a, 36+8a, -51-8a),\\
R_{a+2} &= (\square^a, 30+8a, \square, 38+8a, -46-8a, -23-8a, -13-8a, 71+8a, -63-8a, 6+8a),\\
R_{a+3} &= (\square^{a+1}, 43+8a, 2+8a, -10-8a, -50-8a, 67+8a, 35+8a, -27-8a, -60-8a),\\
R_{a+4} &= (\square^a, -48-8a, -16-8a, 65+8a, \square, 41+8a, -58-8a, -26-8a, 9+8a, 33+8a),\\
R_{a+5} &= (\square^a, -12-8a, 70+8a, 29+8a, 37+8a, 5+8a, -22-8a, -53-8a, -54-8a, \square),
\end{aligned}
$$

$$\begin{array}{rcl} R_{a+6} & = & (\square^a, 66+8a, 34+8a, \square, 1+8a, -59-8a, -49-8a, -17-8a, -18-8a, 42+8a), \\ R_{a+7} & = & (\square^a, -21-8a, -52-8a, -11-8a, 28+8a, \square, 4+8a, -62-8a, 45+8a, 69+8a), \\ R_{a+8} & = & (\square^a, 39+8a, 7+8a, -56-8a, -55-8a, -14-8a, 31+8a, \square, 72+8a, -24-8a), \\ R_{a+9} & = & (\square^a, -57-8a, -25-8a, -47-8a, 64+8a, 32+8a, 40+8a, 8+8a, \square, -15-8a). \end{array}$$

Note that these rows have exactly 8 filled cells. Also $\|\cup_{h=1}^{9} R_{a+h}\| = \{8a+1, \ldots, 8a+72\}$. Hence H satisfies conditions (a) and (b) of Definition 1.1. Clearly, the lists of the partial sums of the first a rows of H are the same written in the proof of Proposition 4.6. So, we list only the partial sums for the last nine rows:

$$\begin{array}{rcl} \mathcal{S}(R_{a+1}) & = & (3+8a, -58, -78-8a, -97-16a, -29-8a, 15, 51+8a, 0), \\ \mathcal{S}(R_{a+2}) & = & (30+8a, 68+16a, 22+8a, -1, -14-8a, 57, -6-8a, 0), \\ \mathcal{S}(R_{a+3}) & = & (43+8a, 45+16a, 35+8a, -15, 52+8a, 87+16a, 60+8a, 0), \\ \mathcal{S}(R_{a+4}) & = & (-48-8a, -64-16a, 1-8a, 42, -16-8a, -42-16a, -33-8a, 0), \\ \mathcal{S}(R_{a+5}) & = & (-12-8a, 58, 87+8a, 124+16a, 129+24a, 107+16a, 54+8a, 0), \\ \mathcal{S}(R_{a+6}) & = & (66+8a, 100+16a, 101+24a, 42+16a, -7+8a, -24, -42-8a, 0), \\ \mathcal{S}(R_{a+7}) & = & (-21-8a, -73-16a, -84-24a, -56-16a, -52-8a, -114-16a, -8a-69, 0), \\ \mathcal{S}(R_{a+8}) & = & (39+8a, 46+16a, -10+8a, -65, -79-8a, -48, 24+8a, 0), \\ \mathcal{S}(R_{a+9}) & = & (-57-8a, -82-16a, -129-24a, -65-16a, -33-8a, 7, 15+8a, 0). \end{array}$$

Note that each row sums to 0. By a long and direct verification one can see that the elements of each $\mathcal{S}(R_t)$, $1 \leq t \leq a+9$, are pairwise distinct both modulo $16(a+9)+1$ and modulo $16(a+9)+2$.

Now, since the first a cells of each row R_{a+h}, $1 \leq h \leq 9$, are empty, the first a columns of H are the ones of the $a \times a$ array defined by Proposition 4.6 with nine empty cells at the end. Also, the last nine columns are the following:

$$\begin{array}{rcl} C_{a+1} & = & (\square^a, 3+8a, 30+8a, \square, -48-8a, -12-8a, 66+8a, -21-8a, 39+8a, -57-8a)^T, \\ C_{a+2} & = & (\square^a, -61-8a, \square, 43+8a, -16-8a, 70+8a, 34+8a, -52-8a, 7+8a, -25-8a)^T, \\ C_{a+3} & = & (\square^a, -20-8a, 38+8a, 2+8a, 65+8a, 29+8a, \square, -11-8a, -56-8a, -47-8a)^T, \\ C_{a+4} & = & (\square^a, -19-8a, -46-8a, -10-8a, \square, 37+8a, 1+8a, 28+8a, -55-8a, 64+8a)^T, \\ C_{a+5} & = & (\square^a, 68+8a, -23-8a, -50-8a, 41+8a, 5+8a, -59-8a, \square, -14-8a, 32+8a)^T, \\ C_{a+6} & = & (\square^{a+1}, -13-8a, 67+8a, -58-8a, -22-8a, -49-8a, 4+8a, 31+8a, 40+8a)^T, \\ C_{a+7} & = & (\square^a, 44+8a, 71+8a, 35+8a, -26-8a, -53-8a, -17-8a, -62-8a, \square, 8+8a)^T, \\ C_{a+8} & = & (\square^a, 36+8a, -63-8a, -27-8a, 9+8a, -54-8a, -18-8a, 45+8a, 72+8a, \square)^T, \\ C_{a+9} & = & (\square^a, -51-8a, 6+8a, -60-8a, 33+8a, \square, 42+8a, 69+8a, -24-8a, -15-8a)^T. \end{array}$$

Since also these columns have exactly 8 filled cells, H satisfies condition (c). Obviously, the lists of the partial sums of the first a columns of H are the same written in the proof of Proposition 4.6. So, as done for the rows, we list only the partial sums for the last nine columns:

$$\begin{array}{rcl} \mathcal{S}(C_{a+1}) & = & (3+8a, 33+16a, -15+8a, -27, 39+8a, 18, 57+8a, 0), \\ \mathcal{S}(C_{a+2}) & = & (-61-8a, -18, -34-8a, 36, 70+8a, 18, 25+8a, 0), \\ \mathcal{S}(C_{a+3}) & = & (-20-8a, 18, 20+8a, 85+16a, 114+24a, 103+16a, 47+8a, 0), \\ \mathcal{S}(C_{a+4}) & = & (-19-8a, -65-16a, -75-24a, -38-16a, -37-8a, -9, -64-8a, 0), \\ \mathcal{S}(C_{a+5}) & = & (68+8a, 45, -5-8a, 36, 41+8a, -18, -32-8a, 0), \end{array}$$

$$\begin{array}{rcl}
\mathcal{S}(C_{a+6}) & = & (-13-8a, 54, -4-8a, -26-16a, -75-24a, -71-16a, -40-8a, 0),\\
\mathcal{S}(C_{a+7}) & = & (44+8a, 115+16a, 150+24a, 124+16a, 71+8a, 54, -8-8a, 0),\\
\mathcal{S}(C_{a+8}) & = & (36+8a, -27, -54-8a, -45, -99-8a, -117-16a, -72-8a, 0),\\
\mathcal{S}(C_{a+9}) & = & (-51-8a, -45, -105-8a, -72, -30+8a, 39+16a, 15+8a, 0).
\end{array}$$

Each column sums to 0, so also condition (d) is satisfied. Hence H is an integer Heffter array. Finally, again by a direct check, one can see that the elements of each $\mathcal{S}(C_t)$, $1 \leq t \leq a+9$, are pairwise distinct both modulo $16(a+9)+1$ and modulo $16(a+9)+2$. Thus, H is an $\mathrm{SH}^*(n;8)$. □

Proposition 4.8 *Let* $n \equiv 0 \pmod 4$ *and* $n \geq 12$. *Then, there exists an* $\mathrm{SH}^*(n;9)$.

PROOF: Let $n = 4a$ and let H be the $n \times n$ array whose rows R_t are defined as follows:

$$\begin{array}{rcl}
R_1 & = & (-2+12a, 2-4a, -2+16a, -4+36a, 2-36a, \square^{2a-5}, -1-12a, 3-12a,\\
& & \square^{2a-4}, 5-36a, -3+36a),\\
R_2 & = & (1+20a, 1-16a, 3-4a, -5+20a, 36a, -2-20a, \square^{2a-5}, -4+12a, 5-12a,\\
& & \square^{2a-4}, 1-36a),\\
R_{3+2i} & = & (\square^{2i}, -3-20a-8i, 5+20a+8i, -2-12a-i, 4-4a+2i, -3+16a-i,\\
& & 4+20a+8i, -6-20a-8i, \square^{2a-5}, -6+12a-4i, 7-12a+4i, \square^{2a-4-2i}),\\
R_{4+2j} & = & (\square^{2j+1}, -7-20a-8j, 9+20a+8j, -16a-j, 5-4a+2j, -6+20a-j,\\
& & 8+20a+8j, -10-20a-8j, \square^{2a-5}, -8+12a-4j, 9-12a+4j, \square^{2a-5-2j}),\\
R_{2a} & = & (2-8a, \square^{2a-4}, 9-28a, -7+28a, 2-17a, -3+20a, -1+17a, -8+28a,\\
& & 6-28a, \square^{2a-5}, -12a),\\
R_{2a+1+2j} & = & (\square^{2j}, -1+4a+4j, -4a-4j, \square^{2a-4}, 5-28a-8j, -3+28a+8j, 2-15a+j,\\
& & -2+2a-2j, 1+13a+j, -4+28a+8j, 2-28a-8j, \square^{2a-5-2j}),\\
R_{2a+2+2j} & = & (\square^{1+2j}, 1+4a+4j, -2-4a-4j, \square^{2a-4}, 1-28a-8j, 1+28a+8j,\\
& & 4-19a+j, -3+2a-2j, 17a+j, 28a+8j, -2-28a-8j, \square^{2a-6-2j}),\\
R_{4a-3} & = & (18-36a, \square^{2a-5}, -9+8a, 8-8a, \square^{2a-4}, 21-36a, -19+36a, -14a, 2,\\
& & -1+14a, -20+36a),\\
R_{4a-2} & = & (-16+36a, 14-36a, \square^{2a-5}, -7+8a, 6-8a, \square^{2a-4}, 17-36a, -15+36a,\\
& & 2-18a, 1-2a, -2+20a),\\
R_{4a-1} & = & (-8a, -12+36a, 10-36a, \square^{2a-5}, -5+8a, 20a, \square^{2a-4}, 13-36a, -11+36a,\\
& & 1-12a, 4-8a),\\
R_{4a} & = & (1, -4+20a, -8+36a, 6-36a, \square^{2a-5}, 1-20a, 3-8a, \square^{2a-4}, 9-36a,\\
& & -7+36a, -1+8a),
\end{array}$$

where $i = 0, \ldots, a-2$ and $j = 0, \ldots, a-3$; hence, each row has 9 filled cells. Since

$$\begin{array}{rcl}
\left\| R_1 \cup R_2 \cup \cup_{t=4a-3}^{4a} R_t \right\| & = & \{1, 2, 2a-1, 4a-3, 4a-2\} \cup \{8a-9, \ldots, 8a-3\} \cup\\
& & \{8a-1, 8a\} \cup \{12a-5, \ldots, 12a-1\} \cup \{12a+1, 14a-1,\\
& & 14a, 16a-2, 16a-1, 18a-2, 20a-5, 20a-4\} \cup\\
& & \{20a-2, \ldots, 20a+2\} \cup \{36a-21, \ldots, 36a\},
\end{array}$$

$$\begin{aligned}\cup_{j=0}^{a-3}\|R_{3+2j}\cup R_{4+2j}\| &= \{2a+1,\dots,4a-4\}\cup\{8a+3,\dots,12a-6\}\cup\{12a+2,\dots,\\&\quad 13a-1\}\cup\{15a,\dots,16a-3\}\cup\{16a,\dots,17a-3\}\cup\{19a-3,\\&\quad \dots,20a-6\}\cup\{20a+3,\dots,28a-14\},\\ \|R_{2a-1}\cup R_{2a}\| &= \{2a,8a-2,8a+1,8a+2,12a,13a,15a-1,17a-2,17a-1,20a-3\}\cup\\&\quad \{28a-13,\dots,28a-6\},\\ \cup_{j=0}^{a-3}\|R_{2a+1+2j}\cup R_{2a+2+2j}\| &= \{3,\dots,2a-2\}\cup\{4a-1,\dots,8a-10\}\cup\{13a+1,\dots,\\&\quad 14a-2\}\cup\{14a+1,\dots,15a-2\}\cup\{17a,\dots,18a-3\}\cup\\&\quad \{18a-1,\dots,19a-4\}\cup\{28a-5,\dots,36a-22\},\end{aligned}$$

H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$\begin{aligned}\mathcal{S}(R_1) &= (-2+12a,8a,-2+24a,-6+60a,-4+24a,-5+12a,-2,3-36a,0),\\ \mathcal{S}(R_2) &= (1+20a,2+4a,5,20a,56a,-2+36a,-6+48a,-1+36a,0),\\ \mathcal{S}(R_{3+2i}) &= (-3-20a-8i,2,-12a-i,4-16a+i,1,5+20a+8i,-1,-7+12a-4i,0),\\ \mathcal{S}(R_{4+2j}) &= (-7-20a-8j,2,2-16a-j,7-20a+j,1,9+20a+8j,-1,\\&\quad -9+12a-4j,0),\\ \mathcal{S}(R_{2a}) &= (2-8a,11-36a,4-8a,6-25a,3-5a,2+12a,-6+40a,12a,0),\\ \mathcal{S}(R_{2a+1+2j}) &= (-1+4a+4j,-1,4-28a-8j,1,3-15a+j,1-13a-j,2,-2+28a+8j,0),\\ \mathcal{S}(R_{2a+2+2j}) &= (1+4a+4j,-1,-28a-8j,1,5-19a+j,2-17a-j,2,2+28a+8j,0),\\ \mathcal{S}(R_{4a-3}) &= (18-36a,9-28a,17-36a,38-72a,19-36a,19-50a,21-50a,\\&\quad 20-36a,0),\\ \mathcal{S}(R_{4a-2}) &= (-16+36a,-2,-9+8a,-3,14-36a,-1,1-18a,2-20a,0),\\ \mathcal{S}(R_{4a-1}) &= (-8a,-12+28a,-2-8a,-7,-7+20a,6-16a,-5+20a,-4+8a,0),\\ \mathcal{S}(R_{4a}) &= (1,-3+20a,-11+56a,-5+20a,-4,-1-8a,8-44a,1-8a,0).\end{aligned}$$

By a long direct calculation, the reader can check that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct both modulo $72a+1$ and modulo $72a+2$ and in particular each row sums to 0. From the definition of H we obtain the following expression of the columns:

$$\begin{aligned}C_1 &= (-2+12a,1+20a,-3-20a,\square^{2a-4},2-8a,-1+4a,\square^{2a-5},18-36a,\\&\quad -16+36a,-8a,1)^T,\\ C_2 &= (2-4a,1-16a,5+20a,-7-20a,\square^{2a-4},-4a,1+4a,\square^{2a-5},14-36a,\\&\quad -12+36a,-4+20a)^T,\\ C_3 &= (-2+16a,3-4a,-2-12a,9+20a,-11-20a,\square^{2a-4},-2-4a,3+4a,\\&\quad \square^{2a-5},10-36a,-8+36a)^T,\\ C_4 &= (-4+36a,-5+20a,4-4a,-16a,13+20a,-15-20a,\square^{2a-4},-4-4a,\\&\quad 5+4a,\square^{2a-5},6-36a)^T,\\ C_5 &= (2-36a,36a,-3+16a,5-4a,-3-12a,17+20a,-19-20a,\square^{2a-4},\\&\quad -6-4a,7+4a,\square^{2a-5})^T,\\ C_{6+2h} &= (\square^{1+2h},-2-20a-8h,4+20a+8h,-6+20a-h,6-4a+2h,-1-16a-h,\\&\quad 21+20a+8h,-23-20a-8h,\square^{2a-4},-8-4a-4h,9+4a+4h,\square^{2a-6-2h})^T,\\ C_{7+2h} &= (\square^{2+2h},-6-20a-8h,8+20a+8h,-4+16a-h,7-4a+2h,-4-12a-h,\\&\quad 25+20a+8h,-27-20a-8h,\square^{2a-4},-10-4a-4h,11+4a+4h,\square^{2a-7-2h})^T,\\ C_{2a} &= (\square^{2a-5},22-28a,-20+28a,-3+19a,-2a,2-17a,-3+28a,1-28a,\\&\quad \square^{2a-4},20a,1-20a)^T,\end{aligned}$$

$$\begin{aligned}
C_{2a+1} &= (-1-12a, \square^{2a-5}, 18-28a, -16+28a, -1+15a, -3+20a, 2-15a,\\
&\quad 1+28a, -3-28a, \square^{2a-4}, 3-8a)^T,\\
C_{2a+2+2i} &= (\square^{2i}, 3-12a+4i, -4+12a-4i, \square^{2a-5}, 14-28a-8i, -12+28a+8i,\\
&\quad -1+17a+i, -2+2a-2i, 4-19a+i, 5+28a+8i, -7-28a-8i, \square^{2a-4-2i})^T,\\
C_{2a+3+2j} &= (\square^{1+2j}, 5-12a+4j, -6+12a-4j, \square^{2a-5}, 10-28a-8j, -8+28a+8j,\\
&\quad 1+13a+j, -3+2a-2j, 3-15a+j, 9+28a+8j, -11-28a-8j, \square^{2a-5-2j})^T,\\
C_{4a-1} &= (5-36a, \square^{2a-4}, -3-8a, 2+8a, \square^{2a-5}, 26-36a, -24+36a, -1+14a,\\
&\quad 1-2a, 1-12a, -7+36a)^T,\\
C_{4a} &= (-3+36a, 1-36a, \square^{2a-4}, -1-8a, -12a, \square^{2a-5}, 22-36a, -20+36a,\\
&\quad -2+20a, 4-8a, -1+8a)^T,
\end{aligned}$$

where $h = 0, \dots, a-4$, $i = 0, \dots, a-2$ and $j = 0, \dots, a-3$. Note that each column has 9 filled cells and so condition (c) holds. We have:

$$\begin{aligned}
\mathcal{S}(C_1) &= (-2+12a, -1+32a, -4+12a, -2+4a, -3+8a, 15-28a, -1+8a, -1, 0),\\
\mathcal{S}(C_2) &= (2-4a, 3-20a, 8, 1-20a, 1-24a, 2-20a, 16-56a, 4-20a, 0),\\
\mathcal{S}(C_3) &= (-2+16a, 1+12a, -1, 8+20a, -3, -5-4a, -2, 8-36a, 0),\\
\mathcal{S}(C_4) &= (-4+36a, -9+56a, -5+52a, -5+36a, 8+56a, -7+36a, -11+32a,\\
&\quad -6+36a, 0),\\
\mathcal{S}(C_5) &= (2-36a, 2, -1+16a, 4+12a, 1, 18+20a, -1, -7-4a, 0),\\
\mathcal{S}(C_{6+2h}) &= (-2-20a-8h, 2, -4+20a-h, 2+16a+h, 1, 22+20a+8h, -1,\\
&\quad -9-4a-4h, 0),\\
\mathcal{S}(C_{7+2h}) &= (-6-20a-8h, 2, -2+16a-h, 5+12a+h, 1, 26+20a+8h, -1,\\
&\quad -11-4a-4h, 0),\\
\mathcal{S}(C_{2a}) &= (22-28a, 2, -1+19a, -1+17a, 1, -2+28a, -1, -1+20a, 0),\\
\mathcal{S}(C_{2a+1}) &= (-1-12a, 17-40a, 1-12a, 3a, -3+23a, -1+8a, 36a, -3+8a, 0),\\
\mathcal{S}(C_{2a+2+2i}) &= (3-12a+4i, -1, 13-28a-8i, 1, 17a+i, -2+19a-i, 2, 7+28a+8i, 0),\\
\mathcal{S}(C_{2a+3+2j}) &= (5-12a+4j, -1, 9-28a-8j, 1, 2+13a+j, -1+15a-j, 2,\\
&\quad 11+28a+8j, 0),\\
\mathcal{S}(C_{4a-1}) &= (5-36a, 2-44a, 4-36a, 30-72a, 6-36a, 5-22a, 6-24a, 7-36a, 0),\\
\mathcal{S}(C_{4a}) &= (-3+36a, -2, -3-8a, -3-20a, 19-56a, -1-20a, -3, 1-8a, 0).
\end{aligned}$$

Since every column sums to 0, condition (d) is satisfied and so H is an $\mathrm{H}(4a;9)$. Also in this case, the elements of each $\mathcal{S}(C_t)$ are pairwise distinct both modulo $72a+1$ and modulo $72a+2$. We conclude that H is an $\mathrm{SH}^*(4a;9)$. □

Example 4.9 By the proof of Proposition 4.8, we obtain the following $\mathrm{SH}^*(12;9)$:

34	−10	46	104	−106		−37	−33			−103	105
61	−47	−9	55	108	−62		32	−31			−107
−63	65	−38	−8	45	64	−66		30	−29		
	−67	69	−48	−7	54	68	−70		28	−27	
		−71	73	−39	−6	44	72	−74		26	−25
−22			−75	77	−49	57	50	76	−78		−36
11	−12			−79	81	−43	4	40	80	−82	
	13	−14			−83	85	−53	3	51	84	−86
−90		15	−16			−87	89	−42	2	41	88
92	−94		17	−18			−91	93	−52	−5	58
−24	96	−98		19	60			−95	97	−35	−20
1	56	100	−102		−59	−21			−99	101	23

Proposition 4.10 *Let $n \equiv 3 \pmod 4$ and $n \geq 11$. Then, there exists an $\mathrm{SH}^*(n;9)$.*

PROOF: An $\mathrm{SH}^*(11;9)$ can be found in [12]. So, we assume $n \geq 15$. We split the proof into two cases.
Case 1. Let $n = 8a+3$ and H be the $n \times n$ array whose rows R_t are defined as follows:

$$\begin{aligned}
R_1 &= (2+8a, 23+64a, -9-24a, -24-64a, 13+40a, \square^{8a-6}, -14-40a, -17-48a,\\
&\quad 8+16a, 18+48a),\\
R_2 &= (16+40a, -1-6a, 27+72a, -7-20a, -22-56a, 12+31a, \square^{8a-6}, -12-25a,\\
&\quad -21-56a, 8+20a),\\
R_3 &= (4+8a, 20+48a, 4a, 25+64a, -5-8a, -26-64a, 13+26a, \square^{8a-6}, -12-30a,\\
&\quad -19-48a),\\
R_4 &= (-17-40a, 6+12a, 18+40a, -1+2a, 23+56a, -7-12a, -24-56a,\\
&\quad 14+35a, \square^{8a-6}, -12-37a),\\
R_{5+8i} &= (\square^{8i}, -12-24a-i, -21-48a-8i, 8+24a-4i, 22+48a+8i, -1-8a+2i,\\
&\quad 27+64a+8i, -7-24a+4i, -28-64a-8i, 12+32a-i, \square^{8a-6-8i}),\\
R_{6+8i} &= (\square^{1+8i}, -13-33a-i, -19-40a-8i, 6+20a-4i, 20+40a+8i, -6a+2i,\\
&\quad 25+56a+8i, -5-20a+4i, -26-56a-8i, 12+39a-i, \square^{8a-7-8i}),\\
R_{7+8i} &= (\square^{2+8i}, -12-38a+i, -23-48a-8i, 6+8a+4i, 24+48a+8i, -1+4a-2i,\\
&\quad 29+64a+8i, -7-8a-4i, -30-64a-8i, 14+34a+i, \square^{8a-8-8i}),\\
R_{8+8j} &= (\square^{3+8j}, -11-29a+j, -21-40a-8j, 8+12a+4j, 22+40a+8j, -2+2a-2j,\\
&\quad 27+56a+8j, -9-12a-4j, -28-56a-8j, 14+27a+j, \square^{8a-9-8j}),\\
R_{9+8j} &= (\square^{4+8j}, -13-32a-j, -25-48a-8j, 6+24a-4j, 26+48a+8j, -8a+2j,\\
&\quad 31+64a+8j, -5-24a+4j, -32-64a-8j, 12+40a-j, \square^{8a-10-8j}),\\
R_{10+8j} &= (\square^{5+8j}, -13-25a-j, -23-40a-8j, 4+20a-4j, 24+40a+8j, 1-6a+2j,\\
&\quad 29+56a+8j, -3-20a+4j, -30-56a-8j, 11+31a-j, \square^{8a-11-8j}),\\
R_{11+8j} &= (\square^{6+8j}, -11-30a+j, -27-48a-8j, 8+8a+4j, 28+48a+8j, -2+4a-2j,\\
&\quad 33+64a+8j, -9-8a-4j, -34-64a-8j, 14+26a+j, \square^{8a-12-8j}),\\
R_{12+8h} &= (\square^{7+8h}, -11-37a+h, -25-40a-8h, 10+12a+4h, 26+40a+8h,\\
&\quad -3+2a-2h, 31+56a+8h, -11-12a-4h, -32-56a-8h, 15+35a+h,\\
&\quad \square^{8a-13-8h}),
\end{aligned}$$

$$\begin{aligned}
R_{8a-4} &= (\square^{8a-9}, -13-36a, -9-48a, 2+16a, 10+48a, -1-4a, 15+64a, -3-16a,\\
&\quad -16-64a, 15+40a, \square^{3}),\\
R_{8a} &= (11+24a, \square^{8a-6}, -10-24a, -13-48a, 4+16a, 14+48a, 1, 19+64a, -6-16a,\\
&\quad -20-64a),\\
R_{8a+1} &= (-24-72a, 13+39a, \square^{8a-6}, -12-33a, -17-56a, 10+20a, 18+56a, -2-6a,\\
&\quad 23+72a, -9-20a),\\
R_{8a+2} &= (-5-16a, -22-64a, 13+34a, \square^{8a-6}, -12-26a, -15-48a, 7+16a, 16+48a,\\
&\quad -3-8a, 21+64a),\\
R_{8a+3} &= (25+72a, -5-12a, -26-72a, 13+27a, \square^{8a-6}, -12-29a, -19-56a, 4+12a,\\
&\quad 20+56a, 2a),
\end{aligned}$$

where $i = 0, \ldots, a-1$, $j = 0, \ldots, a-2$ and $h = 0, \ldots, a-3$. Note that every row contains exactly 9 elements. Since

$$\begin{aligned}
\left\| \cup_{t=3}^{5} R_{8a-2t} \cup \cup_{i=1}^{3} R_{8a-i} \right\| &= \{2, 2a+1, 4a+2, 4a+3, 4a+4, 6a+3, 12a+2, 12a+3,\\
&\quad 16a, 16a+1\} \cup \{16a+9, \ldots, 16a+14\} \cup \{20a+11,\\
&\quad 20a+12, 25a+11, 26a+11, 28a+12, 28a+13, 30a+13,\\
&\quad 31a+13, 34a+11, 34a+12, 35a+13, 37a+13, 38a+13,\\
&\quad 38a+14\} \cup \{48a+3, \ldots, 48a+8\} \cup \{48a+11, 48a+12\}\\
&\quad \cup \{56a+13, \ldots, 56a+16\} \cup \{64a+9, \ldots, 64a+14\} \cup\\
&\quad \{64a+17, 64a+18\} \cup \{72a+19, \ldots, 72a+22\},\\
\cup_{j=0}^{a-2} \left\| \cup_{i=2}^{5} R_{2i+1+8j} \right\| &= \{2a+2, \ldots, 4a-1\} \cup \{6a+4, \ldots, 8a+1\} \cup \{8a+6, \ldots,\\
&\quad 12a+1\} \cup \{20a+13, \ldots, 24a+8\} \cup \{24a+12, \ldots, 25a+10\} \cup\\
&\quad \{26a+14, \ldots, 27a+12\} \cup \{29a+13, \ldots, 30a+11\} \cup\\
&\quad \{31a+14, \ldots, 33a+11\} \cup \{34a+14, \ldots, 35a+12\} \cup\\
&\quad \{37a+14, \ldots, 38a+12\} \cup \{39a+14, \ldots, 40a+12\} \cup\\
&\quad \{48a+21, \ldots, 56a+12\} \cup \{64a+27, \ldots, 72a+18\},\\
\cup_{j=0}^{a-3} \left\| \cup_{i=3}^{6} R_{2i+8j} \right\| &= \{3, \ldots, 2a-2\} \cup \{4a+5, \ldots, 6a\} \cup \{12a+8, \ldots, 16a-1\}\\
&\quad \cup \{16a+15, \ldots, 20a+6\} \cup \{25a+13, \ldots, 26a+10\} \cup\\
&\quad \{27a+14, \ldots, 28a+11\} \cup \{28a+14, \ldots, 29a+11\} \cup\\
&\quad \{30a+14, \ldots, 31a+11\} \cup \{33a+13, \ldots, 34a+10\} \cup\\
&\quad \{35a+15, \ldots, 36a+12\} \cup \{36a+14, \ldots, 37a+11\} \cup\\
&\quad \{38a+15, \ldots, 39a+12\} \cup \{40a+19, \ldots, 48a+2\} \cup\\
&\quad \{56a+25, \ldots, 64a+8\},\\
\left\| \cup_{j=1}^{4} R_j \cup R_{8a-4} \cup \cup_{t=8a}^{8a+3} R_t \right\| &= \{1, 2a-1, 2a, 4a, 4a+1, 6a+1, 6a+2\} \cup \{8a+2, \ldots,\\
&\quad 8a+5\} \cup \{12a+4, \ldots, 12a+7\} \cup \{16a+2, \ldots,\\
&\quad 16a+8\} \cup \{20a+7, \ldots, 20a+10\} \cup \{24a+9, 24a+10,\\
&\quad 24a+11, 25a+12, 26a+12, 26a+13, 27a+13,\\
&\quad 29a+12, 30a+12, 31a+12, 33a+12, 34a+13,\\
&\quad 35a+14, 36a+13, 37a+12, 39a+13\} \cup \{40a+13, \ldots,\\
&\quad 40a+18\} \cup \{48a+9, 48a+10\} \cup \{48a+13, \ldots,\\
&\quad 48a+20\} \cup \{56a+17, \ldots, 56a+24\} \cup \{64a+15,\\
&\quad 64a+16\} \cup \{64a+19, \ldots, 64a+26\} \cup \{72a+23, \ldots,\\
&\quad 72a+27\},
\end{aligned}$$

H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$
\begin{array}{rcl}
\mathcal{S}(R_1) & = & (2+8a, 25+72a, 16+48a, -8-16a, 5+24a, -9-16a, -26-64a,\\
& & -18-48a, 0),\\
\mathcal{S}(R_2) & = & (16+40a, 15+34a, 42+106a, 35+86a, 13+30a, 25+61a, 13+36a,\\
& & -8-20a, 0),\\
\mathcal{S}(R_3) & = & (4+8a, 24+56a, 24+60a, 49+124a, 44+116a, 18+52a, 31+78a, 19+48a, 0),\\
\mathcal{S}(R_4) & = & (-17-40a, -11-28a, 7+12a, 6+14a, 29+70a, 22+58a, -2+2a, 12+37a, 0),\\
\mathcal{S}(R_{5+8i}) & = & (-12-24a-i, -33-72a-9i, -25-48a-13i, -3-5i, -4-8a-3i,\\
& & 23+56a+5i, 16+32a+9i, -12-32a+i, 0),\\
\mathcal{S}(R_{6+8i}) & = & (-13-33a-i, -32-73a-9i, -26-53a-13i, -6-13a-5i, -6-19a-3i,\\
& & 19+37a+5i, 14+17a+9i, -12-39a+i, 0),\\
\mathcal{S}(R_{7+8i}) & = & (-12-38a+i, -35-86a-7i, -29-78a-3i, -5-30a+5i, -6-26a+3i,\\
& & 23+38a+11i, 16+30a+7i, -14-34a-i, 0),\\
\mathcal{S}(R_{8+8j}) & = & (-11-29a+j, -32-69a-7j, -24-57a-3j, -2-17a+5j, -4-15a+3j,\\
& & 23+41a+11j, 14+29a+7j, -14-27a-j, 0),\\
\mathcal{S}(R_{9+8j}) & = & (-13-32a-j, -38-80a-9j, -32-56a-13j, -6-8a-5j, -6-16a-3j,\\
& & 25+48a+5j, 20+24a+9j, -12-40a+j, 0),\\
\mathcal{S}(R_{10+8j}) & = & (-13-25a-j, -36-65a-9j, -32-45a-13j, -8-5a-5j, -7-11a-3j,\\
& & 22+45a+5j, 19+25a+9j, -11-31a+j, 0),\\
\mathcal{S}(R_{11+8j}) & = & (-11-30a+j, -38-78a-7j, -30-70a-3j, -2-22a+5j, -4-18a+3j,\\
& & 29+46a+11j, 20+38a+7j, -14-26a-j, 0),\\
\mathcal{S}(R_{12+8h}) & = & (-11-37a+h, -36-77a-7h, -26-65a-3h, -25a+5h, -3-23a+3h,\\
& & 28+33a+11h, 17+21a+7h, -15-35a-h, 0),\\
\mathcal{S}(R_{8a-4}) & = & (-13-36a, -22-84a, -20-68a, -10-20a, -11-24a, 4+40a, 1+24a,\\
& & -15-40a, 0),\\
\mathcal{S}(R_{8a}) & = & (11+24a, 1, -12-48a, -8-32a, 6+16a, 7+16a, 26+80a, 20+64a, 0),\\
\mathcal{S}(R_{8a+1}) & = & (-24-72a, -11-33a, -23-66a, -40-122a, -30-102a, -12-46a,\\
& & -14-52a, 9+20a, 0),\\
\mathcal{S}(R_{8a+2}) & = & (-5-16a, -27-80a, -14-46a, -26-72a, -41-120a, -34-104a,\\
& & -18-56a, -21-64a, 0),\\
\mathcal{S}(R_{8a+3}) & = & (25+72a, 20+60a, -6-12a, 7+15a, -5-14a, -24-70a, -20-58a, -2a, 0).
\end{array}
$$

By a long direct calculation, the reader can check that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct both modulo $18(8a+3)+1$ and modulo $18(8a+3)+2$ and in particular each row sums to 0. From the definition of H we obtain the following expression of the columns:

$$
\begin{array}{rcl}
C_1 & = & (2+8a, 16+40a, 4+8a, -17-40a, -12-24a, \square^{8a-6}, 11+24a, -24-72a,\\
& & -5-16a, 25+72a)^T,\\
C_2 & = & (23+64a, -1-6a, 20+48a, 6+12a, -21-48a, -13-33a, \square^{8a-6}, 13+39a,\\
& & -22-64a, -5-12a)^T,\\
C_3 & = & (-9-24a, 27+72a, 4a, 18+40a, 8+24a, -19-40a, -12-38a, \square^{8a-6},\\
& & 13+34a, -26-72a)^T,
\end{array}
$$

$$\begin{array}{rcl}
C_4 & = & (-24-64a,-7-20a,25+64a,-1+2a,22+48a,6+20a,-23-48a,\\
 & & -11-29a,\square^{8a-6},13+27a)^T,\\
C_{5+8i} & = & (\square^{8i},13+40a-i,-22-56a-8i,-5-8a-4i,23+56a+8i,-1-8a+2i,\\
 & & 20+40a+8i,6+8a+4i,-21-40a-8i,-13-32a-i,\square^{8a-6-8i})^T,\\
C_{6+8i} & = & (\square^{1+8i},12+31a-i,-26-64a-8i,-7-12a-4i,27+64a+8i,-6a+2i,\\
 & & 24+48a+8i,8+12a+4i,-25-48a-8i,-13-25a-i,\square^{8a-7-8i})^T,\\
C_{7+8i} & = & (\square^{2+8i},13+26a+i,-24-56a-8i,-7-24a+4i,25+56a+8i,-1+4a-2i,\\
 & & 22+40a+8i,6+24a-4i,-23-40a-8i,-11-30a+i,\square^{8a-8-8i})^T,\\
C_{8+8j} & = & (\square^{3+8j},14+35a+j,-28-64a-8j,-5-20a+4j,29+64a+8j,-2+2a-2j,\\
 & & 26+48a+8j,4+20a-4j,-27-48a-8j,-11-37a+j,\square^{8a-9-8j})^T,\\
C_{9+8j} & = & (\square^{4+8j},12+32a-j,-26-56a-8j,-7-8a-4j,27+56a+8j,-8a+2j,\\
 & & 24+40a+8j,8+8a+4j,-25-40a-8j,-13-24a-j,\square^{8a-10-8j})^T,\\
C_{10+8j} & = & (\square^{5+8j},12+39a-j,-30-64a-8j,-9-12a-4j,31+64a+8j,1-6a+2j,\\
 & & 28+48a+8j,10+12a+4j,-29-48a-8j,-14-33a-j,\square^{8a-11-8j})^T,\\
C_{11+8j} & = & (\square^{6+8j},14+34a+j,-28-56a-8j,-5-24a+4j,29+56a+8j,-2+4a-2j,\\
 & & 26+40a+8j,4+24a-4j,-27-40a-8j,-11-38a+j,\square^{8a-12-8j})^T,\\
C_{12+8h} & = & (\square^{7+8h},14+27a+h,-32-64a-8h,-3-20a+4h,33+64a+8h,-3+2a-2h,\\
 & & 30+48a+8h,2+20a-4h,-31-48a-8h,-10-29a+h,\square^{8a-13-8h})^T,\\
C_{8a-4} & = & (\square^{8a-9},12+28a,-16-72a,-11-16a,17+72a,-1-4a,14+56a,10+16a,\\
 & & -15-56a,-10-24a,\square^{3})^T,\\
C_{8a} & = & (-14-40a,\square^{8a-6},15+40a,-20-72a,-9-16a,21+72a,1,18+56a,\\
 & & 7+16a,-19-56a)^T,\\
C_{8a+1} & = & (-17-48a,-12-25a,\square^{8a-6},13+31a,-18-64a,-3-12a,19+64a,-2-6a,\\
 & & 16+48a,4+12a)^T,\\
C_{8a+2} & = & (8+16a,-21-56a,-12-30a,\square^{8a-6},13+38a,-22-72a,-6-16a,23+72a,\\
 & & -3-8a,20+56a)^T,\\
C_{8a+3} & = & (18+48a,8+20a,-19-48a,-12-37a,\square^{8a-6},13+35a,-20-64a,-9-20a,\\
 & & 21+64a,2a)^T,
\end{array}$$

where $i=0,\dots,a-1$, $j=0,\dots,a-2$ and $h=0,\dots,a-3$. Each column contains 9 elements, hence condition (c) holds. The partial sums of the columns are:

$$\begin{array}{rcl}
\mathcal{S}(C_1) & = & (2+8a,18+48a,22+56a,5+16a,-7-8a,4+16a,-20-56a,-25-72a,0),\\
\mathcal{S}(C_2) & = & (23+64a,22+58a,42+106a,48+118a,27+70a,14+37a,27+76a,\\
 & & 5+12a,0),\\
\mathcal{S}(C_3) & = & (-9-24a,18+48a,18+52a,36+92a,44+116a,25+76a,13+38a,\\
 & & 26+72a,0),\\
\mathcal{S}(C_4) & = & (-24-64a,-31-84a,-6-20a,-7-18a,15+30a,21+50a,-2+2a,\\
 & & -13-27a,0),\\
\mathcal{S}(C_{5+8i}) & = & (13+40a-i,-9-16a-9i,-14-24a-13i,9+32a-5i,8+24a-3i,\\
 & & 28+64a+5i,34+72a+9i,13+32a+i,0),\\
\mathcal{S}(C_{6+8i}) & = & (12+31a-i,-14-33a-9i,-21-45a-13i,6+19a-5i,6+13a-3i,\\
 & & 30+61a+5i,38+73a+9i,13+25a+i,0),
\end{array}$$

$$
\begin{aligned}
\mathcal{S}(C_{7+8i}) &= (13+26a+i,-11-30a-7i,-18-54a-3i,7+2a+5i,6+6a+3i,\\
&\quad 28+46a+11i,34+70a+7i,11+30a-i,0),\\
\mathcal{S}(C_{8+8j}) &= (14+35a+j,-14-29a-7j,-19-49a-3j,10+15a+5j,8+17a+3j,\\
&\quad 34+65a+11j,38+85a+7j,11+37a-j,0),\\
\mathcal{S}(C_{9+8j}) &= (12+32a-j,-14-24a-9j,-21-32a-13j,6+24a-5j,6+16a-3j,\\
&\quad 30+56a+5j,38+64a+9j,13+24a+j,0),\\
\mathcal{S}(C_{10+8j}) &= (12+39a-j,-18-25a-9j,-27-37a-13j,4+27a-5j,5+21a-3j,\\
&\quad 33+69a+5j,43+81a+9j,14+33a+j,0),\\
\mathcal{S}(C_{11+8j}) &= (14+34a+j,-14-22a-7j,-19-46a-3j,10+10a+5j,8+14a+3j,\\
&\quad 34+54a+11j,38+78a+7j,11+38a-j,0),\\
\mathcal{S}(C_{12+8h}) &= (14+27a+h,-18-37a-7h,-21-57a-3h,12+7a+5h,9+9a+3h,\\
&\quad 39+57a+11h,41+77a+7h,10+29a-h,0),\\
\mathcal{S}(C_{8a-4}) &= (12+28a,-4-44a,-15-60a,2+12a,1+8a,15+64a,25+80a,10+24a,0),\\
\mathcal{S}(C_{8a}) &= (-14-40a,1,-19-72a,-28-88a,-7-16a,-6-16a,12+40a,19+56a,0),\\
\mathcal{S}(C_{8a+1}) &= (-17-48a,-29-73a,-16-42a,-34-106a,-37-118a,-18-54a,\\
&\quad -20-60a,-4-12a,0),\\
\mathcal{S}(C_{8a+2}) &= (8+16a,-13-40a,-25-70a,-12-32a,-34-104a,-40-120a,\\
&\quad -17-48a,-20-56a,0),\\
\mathcal{S}(C_{8a+3}) &= (18+48a,26+68a,7+20a,-5-17a,8+18a,-12-46a,-21-66a,-2a,0).
\end{aligned}
$$

Since each column sums to 0, H is a $\mathrm{H}(8a+3;9)$. Finally, one can check that the elements of each $\mathcal{S}(C_t)$ are pairwise distinct both modulo $18(8a+3)+1$ and modulo $18(8a+3)+2$. We conclude that H is an $\mathrm{SH}^*(8a+3;9)$.

Case 2. Let $n=8a+7$ and H be the $n\times n$ array whose rows R_t are defined as follows:

$$
\begin{aligned}
R_1 &= (6+8a,55+64a,-21-24a,-56-64a,33+40a,\square^{8a-2},-34-40a,\\
&\quad -41-48a,16+16a,42+48a),\\
R_2 &= (36+40a,-4-6a,63+72a,-17-20a,-50-56a,32+39a,\square^{8a-2},\\
&\quad -29-33a,-49-56a,18+20a),\\
R_3 &= (8+8a,44+48a,2+4a,57+64a,-9-8a,-58-64a,26+26a,\square^{8a-2},\\
&\quad -27-30a,-43-48a),\\
R_4 &= (-37-40a,12+12a,38+40a,2a,51+56a,-13-12a,-52-56a,\\
&\quad 27+27a,\square^{8a-2},-26-29a),\\
R_{5+8i} &= (\square^{8i},-24-24a-i,-45-48a-8i,20+24a-4i,46+48a+8i,-5-8a+2i,\\
&\quad 59+64a+8i,-19-24a+4i,-60-64a-8i,28+32a-i,\square^{8a-2-8i}),\\
R_{6+8i} &= (\square^{1+8i},-25-25a-i,-39-40a-8i,16+20a-4i,40+40a+8i,-3-6a+2i,\\
&\quad 53+56a+8i,-15-20a+4i,-54-56a-8i,27+31a-i,\square^{8a-3-8i}),\\
R_{7+8i} &= (\square^{2+8i},-31-38a+i,-47-48a-8i,10+8a+4i,48+48a+8i,1+4a-2i,\\
&\quad 61+64a+8i,-11-8a-4i,-62-64a-8i,31+34a+i,\square^{8a-4-8i}),\\
R_{8+8j} &= (\square^{3+8j},-30-37a+j,-41-40a-8j,14+12a+4j,42+40a+8j,-1+2a-2j,\\
&\quad 55+56a+8j,-15-12a-4j,-56-56a-8j,32+35a+j,\square^{8a-5-8j}),
\end{aligned}
$$

$$\begin{aligned}
R_{9+8i} &= (\square^{4+8i}, -29-32a-i, -49-48a-8i, 18+24a-4i, 50+48a+8i, -4-8a+2i,\\
&\quad 63+64a+8i, -17-24a+4i, -64-64a-8i, 32+40a-i, \square^{8a-6-8i}),\\
R_{10+8i} &= (\square^{5+8i}, -30-33a-i, -43-40a-8i, 14+20a-4i, 44+40a+8i, -2-6a+2i,\\
&\quad 57+56a+8i, -13-20a+4i, -58-56a-8i, 31+39a-i, \square^{8a-7-8i}),\\
R_{11+8i} &= (\square^{6+8i}, -26-30a+i, -51-48a-8i, 12+8a+4i, 52+48a+8i, 4a-2i,\\
&\quad 65+64a+8i, -13-8a-4i, -66-64a-8i, 27+26a+i, \square^{8a-8-8i}),\\
R_{12+8j} &= (\square^{7+8j}, -25-29a+j, -45-40a-8j, 16+12a+4j, 46+40a+8j, -2+2a-2j,\\
&\quad 59+56a+8j, -17-12a-4j, -60-56a-8j, 28+27a+j, \square^{8a-9-8j}),\\
R_{8a} &= (\square^{8a-5}, -31-36a, -33-48a, 10+16a, 34+48a, -3-4a, 47+64a, -11-16a,\\
&\quad -48-64a, 35+40a, \square^{3}),\\
R_{8a+4} &= (23+24a, \square^{8a-2}, -22-24a, -37-48a, 12+16a, 38+48a, 1, 51+64a,\\
&\quad -14-16a, -52-64a),\\
R_{8a+5} &= (-60-72a, 28+31a, \square^{8a-2}, -24-25a, -45-56a, 20+20a, 46+56a, -5-6a,\\
&\quad 59+72a, -19-20a),\\
R_{8a+6} &= (-13-16a, -54-64a, 30+34a, \square^{8a-2}, -25-26a, -39-48a, 15+16a,\\
&\quad 40+48a, -7-8a, 53+64a),\\
R_{8a+7} &= (61+72a, -11-12a, -62-72a, 31+35a, \square^{8a-2}, -31-37a, -47-56a, 10+12a,\\
&\quad 48+56a, 1+2a),
\end{aligned}$$

where $i = 0, \dots, a-1$ and $j = 0, \dots, a-2$. Each row contains 9 elements. Since

$$\begin{aligned}
\left\|\cup_{j=1}^{4} R_j \cup \cup_{t=8a+4}^{8a+7} R_t\right\| &= \{1, 2a, 2a+1, 4a+2, 6a+4, 6a+5\} \cup \{8a+6, \dots, 8a+9\}\\
&\quad \cup\{12a+10, \dots, 12a+13\} \cup \{16a+12, \dots, 16a+16\}\cup\\
&\quad \{20a+17, \dots, 20a+20\} \cup \{24a+21, 24a+22, 24a+23,\\
&\quad 25a+24, 26a+25, 26a+26, 27a+27, 29a+26, 30a+27,\\
&\quad 31a+28, 33a+29, 34a+30, 35a+31, 37a+31, 39a+32,\\
&\quad 40a+33, 40a+34, 40a+36, 40a+37, 40a+38\}\cup\\
&\quad \{48a+37, \dots, 48a+44\} \cup \{56a+45, \dots, 56a+52\}\cup\\
&\quad \{64a+51, \dots, 64a+58\} \cup \{72a+59, \dots, 72a+63\},\\
\cup_{j=0}^{a-1} \left\|\cup_{i=2}^{5} R_{2i+1+8j}\right\| &= \{2a+2, \dots, 4a+1\} \cup \{6a+6, \dots, 8a+5\}\cup\\
&\quad \{8a+10, \dots, 12a+9\} \cup \{20a+21, \dots, 24a+20\}\cup\\
&\quad \{24a+24, \dots, 25a+23\} \cup \{26a+27, \dots, 27a+26\}\cup\\
&\quad \{29a+27, \dots, 30a+26\} \cup \{31a+29, \dots, 33a+28\}\cup\\
&\quad \{34a+31, \dots, 35a+30\} \cup \{37a+32, \dots, 38a+31\}\cup\\
&\quad \{39a+33, \dots, 40a+32\} \cup \{48a+45, \dots, 56a+44\}\cup\\
&\quad \{64a+59, \dots, 72a+58\},\\
\cup_{j=0}^{a-2} \left\|\cup_{i=3}^{6} R_{2i+8j}\right\| &= \{2, \dots, 2a-1\} \cup \{4a+6, \dots, 6a+3\}\cup\\
&\quad \{12a+14, \dots, 16a+9\} \cup \{16a+21, \dots, 20a+16\}\cup\\
&\quad \{25a+25, \dots, 26a+23\} \cup \{27a+28, \dots, 29a+25\}\cup\\
&\quad \{30a+29, \dots, 31a+27\} \cup \{33a+30, \dots, 34a+28\}\cup\\
&\quad \{35a+32, \dots, 36a+30\} \cup \{36a+32, \dots, 37a+30\}\cup\\
&\quad \{38a+33, \dots, 39a+31\} \cup \{40a+39, \dots, 48a+30\}\cup\\
&\quad \{56a+53, \dots, 64a+44\},\\
\left\|\cup_{t=-1}^{1} R_{8a+2t}\right\| &= \{4a+3, 4a+4, 4a+5, 16a+10, 16a+11\} \cup \{16a+17 \dots, 16a+20\}\cup\\
&\quad \{26a+24, 30a+28, 34a+29, 36a+31, 38a+32, 40a+35\}\cup\\
&\quad \{48a+31, \dots, 48a+36\} \cup \{64a+45, \dots, 64a+50\},
\end{aligned}$$

H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums for each row. We have

$$\begin{aligned}
\mathcal{S}(R_1) &= (6+8a, 61+72a, 40+48a, -16-16a, 17+24a, -17-16a, -58-64a,\\
&\quad -42-48a, 0),\\
\mathcal{S}(R_2) &= (36+40a, 32+34a, 95+106a, 78+86a, 28+30a, 60+69a, 31+36a,\\
&\quad -18-20a, 0),\\
\mathcal{S}(R_3) &= (8+8a, 52+56a, 54+60a, 111+124a, 102+116a, 44+52a, 70+78a,\\
&\quad 43+48a, 0),\\
\mathcal{S}(R_4) &= (-37-40a, -25-28a, 13+12a, 13+14a, 64+70a, 51+58a, -1+2a,\\
&\quad 26+29a, 0),\\
\mathcal{S}(R_{5+8i}) &= (-24-24a-i, -69-72a-9i, -49-48a-13i, -3-5i,\\
&\quad -8-8a-3i, 51+56a+5i, 32+32a+9i, -28-32a+i, 0),\\
\mathcal{S}(R_{6+8i}) &= (-25-25a-i, -64-65a-9i, -48-45a-13i, -8-5a-5i,\\
&\quad -11-11a-3i, 42+45a+5i, 27+25a+9i, -27-31a+i, 0),\\
\mathcal{S}(R_{7+8i}) &= (-31-38a+i, -78-86a-7i, -68-78a-3i, -20-30a+5i,\\
&\quad -19-26a+3i, 42+38a+11i, 31+30a+7i, -31-34a-i, 0),\\
\mathcal{S}(R_{8+8j}) &= (-30-37a+j, -71-77a-7j, -57-65a-3j, -15-25a+5j,\\
&\quad -16-23a+3j, 39+33a+11j, 24+21a+7j, -32-35a-j, 0),\\
\mathcal{S}(R_{9+8i}) &= (-29-32a-i, -78-80a-9i, -60-56a-13i, -10-8a-5i,\\
&\quad -14-16a-3i, 49+48a+5i, 32+24a+9i, -32-40a+i, 0),\\
\mathcal{S}(R_{10+8i}) &= (-30-33a-i, -73-73a-9i, -59-53a-13i, -15-13a-5i,\\
&\quad -17-19a-3i, 40+37a+5i, 27+17a+9i, -31-39a+i, 0),\\
\mathcal{S}(R_{11+8i}) &= (-26-30a+i, -77-78a-7i, -65-70a-3i, -13-22a+5i,\\
&\quad -13-18a+3i, 52+46a+11i, 39+38a+7i, -27-26a-i, 0),\\
\mathcal{S}(R_{12+8j}) &= (-25-29a+j, -70-69a-7j, -54-57a-3j, -8-17a+5j,\\
&\quad -10-15a+3j, 49+41a+11j, 32+29a+7j, -28-27a-j, 0),\\
\mathcal{S}(R_{8a}) &= (-31-36a, -64-84a, -54-68a, -20-20a, -23-24a, 24+40a,\\
&\quad 13+24a, -35-40a, 0),\\
\mathcal{S}(R_{8a+4}) &= (23+24a, 1, -36-48a, -24-32a, 14+16a, 15+16a, 66+80a,\\
&\quad 52+64a, 0),\\
\mathcal{S}(R_{8a+5}) &= (-60-72a, -32-41a, -56-66a, -101-122a, -81-102a,\\
&\quad -35-46a, -40-52a, 19+20a, 0),\\
\mathcal{S}(R_{8a+6}) &= (-13-16a, -67-80a, -37-46a, -62-72a, -101-120a,\\
&\quad -86-104a, -46-56a, -53-64a, 0),\\
\mathcal{S}(R_{8a+7}) &= (61+72a, 50+60a, -12-12a, 19+23a, -12-14a, -59-70a,\\
&\quad -49-58a, -1-2a, 0).
\end{aligned}$$

By a long direct calculation, the reader can check that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct both modulo $18(8a+7)+1$ and modulo $18(8a+7)+2$ and in particular each row sums to 0. From the definition of H we obtain the following expression of the columns:

$$\begin{aligned}
C_1 &= (6+8a, 36+40a, 8+8a, -37-40a, -24-24a, \square^{8a-2}, 23+24a, -60-72a,\\
&\quad -13-16a, 61+72a)^T,
\end{aligned}$$

$$
\begin{aligned}
C_2 &= (55+64a, -4-6a, 44+48a, 12+12a, -45-48a, -25-25a, \square^{8a-2}, 28+31a,\\
&\quad -54-64a, -11-12a)^T,\\
C_3 &= (-21-24a, 63+72a, 2+4a, 38+40a, 20+24a, -39-40a, -31-38a, \square^{8a-2},\\
&\quad 30+34a, -62-72a)^T,\\
C_4 &= (-56-64a, -17-20a, 57+64a, 2a, 46+48a, 16+20a, -47-48a, -30-37a,\\
&\quad \square^{8a-2}, 31+35a)^T,\\
C_{5+8i} &= (\square^{8i}, 33+40a-i, -50-56a-8i, -9-8a-4i, 51+56a+8i, -5-8a+2i,\\
&\quad 40+40a+8i, 10+8a+4i, -41-40a-8i, -29-32a-i, \square^{8a-2-8i})^T,\\
C_{6+8i} &= (\square^{8i+1}, 32+39a-i, -58-64a-8i, -13-12a-4i, 59+64a+8i, -3-6a+2i,\\
&\quad 48+48a+8i, 14+12a+4i, -49-48a-8i, -30-33a-i, \square^{8a-3-8i})^T,\\
C_{7+8i} &= (\square^{8i+2}, 26+26a+i, -52-56a-8i, -19-24a+4i, 53+56a+8i, 1+4a-2i,\\
&\quad 42+40a+8i, 18+24a-4i, -43-40a-8i, -26-30a+i, \square^{8a-4-8i})^T,\\
C_{8+8j} &= (\square^{8j+3}, 27+27a+j, -60-64a-8j, -15-20a+4j, 61+64a+8j, -1+2a-2j,\\
&\quad 50+48a+8j, 14+20a-4j, -51-48a-8j, -25-29a+j, \square^{8a-5-8j})^T,\\
C_{9+8i} &= (\square^{8i+4}, 28+32a-i, -54-56a-8i, -11-8a-4i, 55+56a+8i, -4-8a+2i,\\
&\quad 44+40a+8i, 12+8a+4i, -45-40a-8i, -25-24a-i, \square^{8a-6-8i})^T,\\
C_{10+8i} &= (\square^{8i+5}, 27+31a-i, -62-64a-8i, -15-12a-4i, 63+64a+8i, -2-6a+2i,\\
&\quad 52+48a+8i, 16+12a+4i, -53-48a-8i, -26-25a-i, \square^{8a-7-8i})^T,\\
C_{11+8i} &= (\square^{8i+6}, 31+34a+i, -56-56a-8i, -17-24a+4i, 57+56a+8i, 4a-2i,\\
&\quad 46+40a+8i, 16+24a-4i, -47-40a-8i, -30-38a+i, \square^{8a-8-8i})^T,\\
C_{12+8j} &= (\square^{8j+7}, 32+35a+j, -64-64a-8j, -13-20a+4j, 65+64a+8j, -2+2a-2j,\\
&\quad 54+48a+8j, 12+20a-4j, -55-48a-8j, -29-37a+j, \square^{8a-9-8j})^T,\\
C_{8a} &= (\square^{8a-5}, 26+28a, -52-72a, -19-16a, 53+72a, -3-4a, 42+56a, 18+16a,\\
&\quad -43-56a, -22-24a, \square^3)^T,\\
C_{8a+4} &= (-34-40a, \square^{8a-2}, 35+40a, -56-72a, -17-16a, 57+72a, 1, 46+56a,\\
&\quad 15+16a, -47-56a)^T,\\
C_{8a+5} &= (-41-48a, -29-33a, \square^{8a-2}, 33+39a, -50-64a, -9-12a, 51+64a, -5-6a,\\
&\quad 40+48a, 10+12a)^T,\\
C_{8a+6} &= (16+16a, -49-56a, -27-30a, \square^{8a-2}, 32+38a, -58-72a, -14-16a, 59+72a,\\
&\quad -7-8a, 48+56a)^T,\\
C_{8a+7} &= (42+48a, 18+20a, -43-48a, -26-29a, \square^{8a-2}, 26+27a, -52-64a, -19-20a,\\
&\quad 53+64a, 1+2a)^T,
\end{aligned}
$$

where $i = 0, \ldots, a-1$ and $j = 0, \ldots, a-2$. Clearly, each column has exactly 9 filled cells. We have

$$
\begin{aligned}
\mathcal{S}(C_1) &= (6+8a, 42+48a, 50+56a, 13+16a, -11-8a, 12+16a, -48-56a,\\
&\quad -61-72a, 0),\\
\mathcal{S}(C_2) &= (55+64a, 51+58a, 95+106a, 107+118a, 62+70a, 37+45a, 65+76a,\\
&\quad 11+12a, 0),\\
\mathcal{S}(C_3) &= (-21-24a, 42+48a, 44+52a, 82+92a, 102+116a, 63+76a, 32+38a,\\
&\quad 62+72a, 0),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}(C_4) &= (-56-64a, -73-84a, -16-20a, -16-18a, 30+30a, 46+50a, -1+2a,\\
&\quad -31-35a, 0),\\
\mathcal{S}(C_{5+8i}) &= (33+40a-i, -17-16a-9i, -26-24a-13i, 25+32a-5i, 20+24a-3i,\\
&\quad 60+64a+5i, 70+72a+9i, 29+32a+i, 0),\\
\mathcal{S}(C_{6+8i}) &= (32+39a-i, -26-25a-9i, -39-37a-13i, 20+27a-5i, 17+21a-3i,\\
&\quad 65+69a+5i, 79+81a+9i, 30+33a+i, 0),\\
\mathcal{S}(C_{7+8i}) &= (26+26a+i, -26-30a-7i, -45-54a-3i, 8+2a+5i, 9+6a+3i,\\
&\quad 51+46a+11i, 69+70a+7i, 26+30a-i, 0),\\
\mathcal{S}(C_{8+8j}) &= (27+27a+j, -33-37a-7j, -48-57a-3j, 13+7a+5j, 12+9a+3j,\\
&\quad 62+57a+11j, 76+77a+7j, 25+29a-j, 0),\\
\mathcal{S}(C_{9+8i}) &= (28+32a-i, -26-24a-9i, -37-32a-13i, 18+24a-5i, 14+16a-3i,\\
&\quad 58+56a+5i, 70+64a+9i, 25+24a+i, 0),\\
\mathcal{S}(C_{10+8i}) &= (27+31a-i, -35-33a-9i, -50-45a-13i, 13+19a-5i, 11+13a-3i,\\
&\quad 63+61a+5i, 79+73a+9i, 26+25a+i, 0),\\
\mathcal{S}(C_{11+8i}) &= (31+34a+i, -25-22a-7i, -42-46a-3i, 15+10a+5i, 15+14a+3i,\\
&\quad 61+54a+11i, 77+78a+7i, 30+38a-i, 0),\\
\mathcal{S}(C_{12+8j}) &= (32+35a+j, -32-29a-7j, -45-49a-3j, 20+15a+5j, 18+17a+3j,\\
&\quad 72+65a+11j, 84+85a+7j, 29+37a-j, 0),\\
\mathcal{S}(C_{8a}) &= (26+28a, -26-44a, -45-60a, 8+12a, 5+8a, 47+64a, 65+80a, 22+24a, 0),\\
\mathcal{S}(C_{8a+4}) &= (-34-40a, 1, -55-72a, -72-88a, -15-16a, -14-16a, 32+40a,\\
&\quad 47+56a, 0),\\
\mathcal{S}(C_{8a+5}) &= (-41-48a, -70-81a, -37-42a, -87-106a, -96-118a, -45-54a,\\
&\quad -50-60a, -10-12a, 0),\\
\mathcal{S}(C_{8a+6}) &= (16+16a, -33-40a, -60-70a, -28-32a, -86-104a, -100-120a,\\
&\quad -41-48a, -48-56a, 0),\\
\mathcal{S}(C_{8a+7}) &= (42+48a, 60+68a, 17+20a, -9-9a, 17+18a, -35-46a, -54-66a,\\
&\quad -1-2a, 0).
\end{aligned}
$$

Since each column sums to 0, H is an $\mathrm{H}(8a+7;9)$. Also in this case, the elements of each $\mathcal{S}(C_t)$ are pairwise distinct both modulo $18(8a+7)+1$ and modulo $18(8a+7)+2$. We conclude that H is an $\mathrm{SH}^*(8a+7;9)$. □

Proposition 4.11 *Let $n \geq 10$ be even. Then, there exists an* $\mathrm{SH}^*(n;10)$.

PROOF: An $\mathrm{SH}^*(10;10)$ can be found in [12]. So let $n \geq 12$ be even and let H be the $n \times n$ partially filled array whose rows R_t are the following:

$$
\begin{aligned}
R_1 &= (16-10n, 10n-9, 12-10n, 6-10n, 8-10n, 1-10n, 10n-4, \square,\\
&\quad 10n, \square^{n-12}, 10n-17, \square, 10n-13),\\
R_{2+2i} &= (\square^{2i+1}, -21-20i, \square, -25-20i, 22+20i, -29-20i, 26+20i, 30+20i,\\
&\quad 33+20i, 37+20i, -35-20i, \square, -38-20i, \square^{n-13-2i}),\\
R_3 &= (\square, 7, -4, 11, -8, -14, -12, -19, 16, \square, 20, \square^{n-12}, 3),\\
R_{5+2i} &= (\square^{2i+1}, 23+20i, \square, 27+20i, -24-20i, 31+20i, -28-20i, -34-20i,\\
&\quad -32-20i, -39-20i, 36+20i, \square, 40+20i, \square^{n-13-2i}),
\end{aligned}
$$

$$\begin{array}{rcl}
R_{n-10} & = & (82-10n, \square^{n-12}, 99-10n, \square, 95-10n, -98+10n, 91-10n, -94+10n,\\
& & -90+10n, -87+10n, -83+10n, 85-10n, \square),\\
R_{n-8} & = & (65-10n, \square, 62-10n, \square^{n-12}, 79-10n, \square, 75-10n, -78+10n, 71-10n,\\
& & -74+10n, -70+10n, -67+10n, -63+10n),\\
R_{n-7} & = & (-80+10n, \square^{n-12}, -97+10n, \square, -93+10n, 96-10n, -89+10n,\\
& & 92-10n, 86-10n, 88-10n, 81-10n, -84+10n, \square),\\
R_{n-6} & = & (-47+10n, -43+10n, 45-10n, \square, 42-10n, \square^{n-12}, 59-10n, \square,\\
& & 55-10n, -58+10n, 51-10n, -54+10n, -50+10n),\\
R_{n-5} & = & (-64+10n, \square, -60+10n, \square^{n-12}, -77+10n, \square, -73+10n, 76-10n,\\
& & -69+10n, 72-10n, 66-10n, 68-10n, 61-10n),\\
R_{n-4} & = & (-34+10n, -30+10n, -27+10n, -23+10n, 25-10n, \square, 22-10n,\\
& & \square^{n-12}, 39-10n, \square, 35-10n, -38+10n, 31-10n),\\
R_{n-3} & = & (48-10n, 41-10n, -44+10n, \square, -40+10n, \square^{n-12}, -57+10n, \square,\\
& & -53+10n, 56-10n, -49+10n, 52-10n, 46-10n),\\
R_{n-2} & = & (-18+10n, 11-10n, -14+10n, -10+10n, -7+10n, -3+10n,\\
& & 5-10n, \square, 2-10n, \square^{n-12}, 19-10n, \square, 15-10n),\\
R_{n-1} & = & (32-10n, 26-10n, 28-10n, 21-10n, -24+10n, \square, -20+10n, \square^{n-12},\\
& & -37+10n, \square, -33+10n, 36-10n, -29+10n),\\
R_n & = & (\square, -5, 2, -9, 6, 10, 13, 17, -15, \square, -18, \square^{n-12}, -1),
\end{array}$$

where $i = 0, \ldots, \frac{n-14}{2}$. Note that every row contains exactly 10 filled cells. Also, it is easy to see that $\|R_3 \cup R_n\| = \{1, \ldots, 20\}$, $\|R_1 \cup R_{n-2}\| = \{10n-19, \ldots, 10n\}$ and $\|R_{2h} \cup R_{2h+3}\| = \{1+20h, \ldots, 20+20h\}$ for all $h = 1, \ldots, \frac{n-4}{2}$. Hence, H satisfies conditions (a) and (b) of Definition 1.1. Now, we list the partial sums of each row. We have

$$\begin{array}{rcl}
\mathcal{S}(R_1) & = & (16-10n, 7, 19-10n, 25-20n, 33-30n, 34-40n, 30-30n, 30-20n,\\
& & 13+10n, 0),\\
\mathcal{S}(R_{2+2i}) & = & (-21-20i, -46-40i, -24-20i, -53-40i, -27-20i, 3, 36+20i,\\
& & 73+40i, 38+20i, 0),\\
\mathcal{S}(R_3) & = & (7, 3, 14, 6, -8, -20, -39, -23, -3, 0),\\
\mathcal{S}(R_{5+2i}) & = & (23+20i, 50+40i, 26+20i, 57+40i, 29+20i, -5, -37-20i, -76-40i,\\
& & -40-20i, 0),\\
\mathcal{S}(R_{n-10}) & = & (82-10n, 181-20n, 276-30n, 178-20n, 269-30n, 175-20n,\\
& & 85-10n, -2, -85+10n, 0),\\
\mathcal{S}(R_{n-8}) & = & (65-10n, 127-20n, 206-30n, 281-40n, 203-30n, 274-40n,\\
& & 200-30n, 130-20n, 63-10n, 0),\\
\mathcal{S}(R_{n-7}) & = & (-80+10n, -177+20n, -270+30n, -174+20n, -263+30n,\\
& & -171+20n, -85+10n, 3, 84-10n, 0),\\
\mathcal{S}(R_{n-6}) & = & (-47+10n, -90+20n, -45+10n, -3, 56-10n, 111-20n, 53-10n,\\
& & 104-20n, 50-10n, 0),
\end{array}$$

$$\begin{aligned}
\mathcal{S}(R_{n-5}) &= (-64+10n, -124+20n, -201+30n, -274+40n, -198+30n,\\
&\quad -267+40n, -195+30n, -129+20n, -61+10n, 0),\\
\mathcal{S}(R_{n-4}) &= (-34+10n, -64+20n, -91+30n, -114+40n, -89+30n, -67+20n,\\
&\quad -28+10n, 7, -31+10n, 0),\\
\mathcal{S}(R_{n-3}) &= (48-10n, 89-20n, 45-10n, 5, -52+10n, -105+20n, -49+10n,\\
&\quad -98+20n, -46+10n, 0),\\
\mathcal{S}(R_{n-2}) &= (-18+10n, -7, -21+10n, -31+20n, -38+30n, -41+40n, -36+30n,\\
&\quad -34+20n, -15+10n, 0),\\
\mathcal{S}(R_{n-1}) &= (32-10n, 58-20n, 86-30n, 107-40n, 83-30n, 63-20n, 26-10n,\\
&\quad -7, 29-10n, 0),\\
\mathcal{S}(R_n) &= (-5, -3, -12, -6, 4, 17, 34, 19, 1, 0).
\end{aligned}$$

Clearly, each row sums to 0. By a long direct check one can see that the elements of each $\mathcal{S}(R_t)$ are pairwise distinct modulo $20n+1$ for any even $n \geq 12$. Also, they are pairwise distinct modulo $20n+2$ if $n \equiv 0 \pmod 6$, while, if $n \equiv 2 \pmod 6$ and $n \geq 32$, the partial sums $26+20i$ and $-76-40i$ of $\mathcal{S}(R_{5+2i})$ are equivalent modulo $20n+2$ when $i = \frac{n-5}{3}$ and if $n \equiv 4 \pmod 6$ and $n \geq 34$ the partial sums $-46-40i$ and $36+20i$ of $\mathcal{S}(R_{2+2i})$ are equivalent modulo $20n+2$ when $i = \frac{n-4}{3}$. Moreover, the partial sums s and s' of $\mathcal{S}(R_t)$ listed in Table 1 are congruent modulo $20n+2$.

n	t	s	s'	n	t	s	s'
14	$n-3$	$89-20n$	$-49+10n$	22	$n-8$	$281-40n$	$63-10n$
16	$n-6$	$-47+10n$	$111-20n$	26	$n-7$	$-174+20n$	$84-10n$
20	$n-5$	$-198+30n$	0	28	$n-10$	$276-30n$	-2

Table 1: Partial sums of $\mathcal{S}(R_t)$ for small values of n.

From the definition of H we obtain the following expression of its columns:

$$\begin{aligned}
C_1 &= (16-10n, \square^{n-12}, 82-10n, \square, 65-10n, -80+10n, -47+10n,\\
&\quad -64+10n, -34+10n, 48-10n, -18+10n, 32-10n, \square)^T,\\
C_2 &= (-9+10n, -21, 7, \square, 23, \square^{n-12}, -43+10n, \square, -30+10n, 41-10n,\\
&\quad 11-10n, 26-10n, -5)^T,\\
C_3 &= (12-10n, \square, -4, \square^{n-12}, 62-10n, \square, 45-10n, -60+10n, -27+10n,\\
&\quad -44+10n, -14+10n, 28-10n, 2)^T,\\
C_4 &= (6-10n, -25, 11, -41, 27, \square, 43, \square^{n-12}, -23+10n, \square, -10+10n, 21-10n, -9)^T,\\
C_5 &= (8-10n, 22, -8, \square, -24, \square^{n-12}, 42-10n, \square, 25-10n, -40+10n, -7+10n,\\
&\quad -24+10n, 6)^T,\\
C_6 &= (1-10n, -29, -14, -45, 31, -61, 47, \square, 63, \square^{n-12}, -3+10n, \square, 10)^T,\\
C_7 &= (-4+10n, 26, -12, 42, -28, \square, -44, \square^{n-12}, 22-10n, \square, 5-10n, -20+10n, 13)^T,\\
C_8 &= (\square, 30, -19, -49, -34, -65, 51, -81, 67, \square, 83, \square^{n-12}, 17)^T,\\
C_9 &= (10n, 33, 16, 46, -32, 62, -48, \square, -64, \square^{n-12}, 2-10n, \square, -15)^T,\\
C_{10+2i} &= (\square^{2i+1}, 37+20i, \square, 50+20i, -39-20i, -69-20i, -54-20i, -85-20i,\\
&\quad 71+20i, -101-20i, 87+20i, \square, 103+20i, \square^{n-13-2i})^T,
\end{aligned}$$

$$
\begin{aligned}
C_{11} &= (\square, -35, 20, 53, 36, 66, -52, 82, -68, \square, -84, \square^{n-12}, -18)^T,\\
C_{13+2i} &= (\square^{2i+1}, -38-20i, \square, -55-20i, 40+20i, 73+20i, 56+20i, 86+20i,\\
&\quad -72-20i, 102+20i, -88-20i, \square, -104-20i, \square^{n-13-2i})^T,\\
C_{n-2} &= (-17+10n, \square^{n-12}, -83+10n, \square, -70+10n, 81-10n, 51-10n, 66-10n,\\
&\quad 35-10n, -49+10n, 19-10n, -33+10n, \square)^T,\\
C_n &= (-13+10n, \square, 3, \square^{n-12}, -63+10n, \square, -50+10n, 61-10n, 31-10n,\\
&\quad 46-10n, 15-10n, -29+10n, -1)^T,
\end{aligned}
$$

where $i = 0, \ldots, \frac{n-14}{2}$. We observe that each column contains exactly 10 filled cells, then condition (c) is satisfied. One can check that the partial sums of the columns are the following:

$$
\begin{aligned}
\mathcal{S}(C_1) &= (16-10n, 98-20n, 163-30n, 83-20n, 36-10n, -28, -62+10n, -14,\\
&\quad -32+10n, 0),\\
\mathcal{S}(C_2) &= (-9+10n, -30+10n, -23+10n, 10n, -43+20n, -73+30n, -32+20n,\\
&\quad -21+10n, 5, 0),\\
\mathcal{S}(C_3) &= (12-10n, 8-10n, 70-20n, 115-30n, 55-20n, 28-10n, -16, -30+10n,\\
&\quad -2, 0),\\
\mathcal{S}(C_4) &= (6-10n, -19-10n, -8-10n, -49-10n, -22-10n, 21-10n, -2,\\
&\quad -12+10n, 9, 0),\\
\mathcal{S}(C_5) &= (8-10n, 30-10n, 22-10n, -2-10n, 40-20n, 65-30n, 25-20n,\\
&\quad 18-10n, -6, 0),\\
\mathcal{S}(C_6) &= (1-10n, -28-10n, -42-10n, -87-10n, -56-10n, -117-10n,\\
&\quad -70-10n, -7-10n, -10, 0),\\
\mathcal{S}(C_7) &= (-4+10n, 22+10n, 10+10n, 52+10n, 24+10n, -20+10n, 2, 7-10n,\\
&\quad -13, 0),\\
\mathcal{S}(C_8) &= (30, 11, -38, -72, -137, -86, -167, -100, -17, 0),\\
\mathcal{S}(C_9) &= (10n, 33+10n, 49+10n, 95+10n, 63+10n, 125+10n, 77+10n,\\
&\quad 13+10n, 15, 0),\\
\mathcal{S}(C_{10+2i}) &= (37+20i, 87+40i, 48+20i, -21, -75-20i, -160-40i, -89-20i,\\
&\quad -190-40i, -103-20i, 0),\\
\mathcal{S}(C_{11}) &= (-35, -15, 38, 74, 140, 88, 170, 102, 18, 0),\\
\mathcal{S}(C_{13+2i}) &= (-38-20i, -93-40i, -53-20i, 20, 76+20i, 162+40i, 90+20i,\\
&\quad 192+40i, 104+20i, 0),\\
\mathcal{S}(C_{n-2}) &= (-17+10n, -100+20n, -170+30n, -89+20n, -38+10n, 28,\\
&\quad 63-10n, 14, 33-10n, 0),\\
\mathcal{S}(C_n) &= (-13+10n, -10+10n, -73+20n, -123+30n, -62+20n, -31+10n,\\
&\quad 15, 30-10n, 1, 0).
\end{aligned}
$$

Note that each column sums to 0, so condition (d) is satisfied, hence H is an $\mathrm{H}(n;10)$. Also, again by a direct check, one can see that the elements of each $\mathcal{S}(C_t)$ are pairwise distinct modulo $20n+1$ for any even $n \geq 12$ and that they are pairwise distinct modulo $20n+2$ for all $n \equiv 0, 4 \pmod 6$. While, if $n \equiv 2 \pmod 6$ and $n \geq 26$, the partial sums $87+40i$ and $-75-20i$ of $\mathcal{S}(C_{10+2i})$ are equivalent modulo $20n+2$ when $i=\frac{n-8}{3}$. Also, if $n=14$ the partial sums $-123+30n$ and 15 of $\mathcal{S}(C_n)$ are congruent modulo $20n+2$. Finally, if $n=20$ the partial sums $-170+30n$ and 28 of $\mathcal{S}(C_{n-2})$ are congruent modulo $20n+2$. We conclude that H is an $\mathrm{SH}(n;10)$ for any even $n \geq 12$ and that it satisfies also condition $(*)$ when $n \equiv 0 \pmod 6$.

Suppose $n \equiv 4 \pmod 6$. If $n=16,22,28$, an $\mathrm{SH}^*(n;10)$ can be found in [12]. So let $n=6a+4 \geq 34$ and note that, since $n \equiv 4 \pmod 6$, H is an $\mathrm{SH}(n;10)$ whose columns are simple also modulo $20n+2$. Recall that the only row which is not simple modulo $20n+2$ is R_{4a+2}. Interchanging the columns C_{4a+9} (i.e., C_{13+2i} with $i=2a-2$) and C_{4a+11} (i.e., C_{13+2i} with $i=2a-1$), that are the following:

$$\begin{aligned}
C_{4a+9} &= (\square^{4a-3}, 2-40a, \square, -15-40a, 40a, 33+40a, 16+40a, 46+40a, -32-40a,\\
&\quad 62+40a, -48-40a, \square, -64-40a, \square, \square, \square^{2a-7})^T,\\
C_{4a+11} &= (\square^{4a-3}, \square, \square, -18-40a, \square, -35-40a, 20+40a, 53+40a, 36+40a, 66+40a,\\
&\quad -52-40a, 82+40a, -68-40a, \square, -84-40a, \square^{2a-7})^T,
\end{aligned}$$

we obtain an $\mathrm{SH}(n;10)$, say H', satisfying also condition $(*)$. In fact, H' is again an $\mathrm{H}(n;10)$ whose columns are simple modulo $20n+1$ and $20n+2$ and so we are left to check its rows. Looking at the position of the empty cells of the previous two columns, it is easy to see that interchanging them, only the rows $R_{(4a-3)+1}$, $R_{(4a-3)+3}$, $R_{(4a-3)+4}, \ldots, R_{(4a-3)+12}$, $R_{(4a-3)+14}$ change. Hence, H' is the required array if these 12 rows are simple modulo $20n+1$ and $20n+2$. If $n=34,40,46$ we have checked by computer that H' works (the reader can find these arrays in [12]). For $n \geq 52$, we write explicitly the new rows, that will be denoted by R'_t, whose interchanged elements are in bold:

$$\begin{aligned}
R'_{4a-2} &= (\square^{4a-3}, 19-40a, \square, 15-40a, -18+40a, 11-40a, -14+40a, -10+40a,\\
&\quad -7+40a, -3+40a, 5-40a, \square, \boldsymbol{\square}, \square, \mathbf{2-40}\boldsymbol{a}, \square^{2a-7}),\\
R'_{4a} &= (\square^{4a-1}, -1-40a, \square, -5-40a, 2+40a, -9-40a, 6+40a, 10+40a,\\
&\quad 13+40a, 17+40a, \mathbf{-18-40}\boldsymbol{a}, \square, \mathbf{-15-40}\boldsymbol{a}, \square^{2a-7}),\\
R'_{4a+1} &= (\square^{4a-3}, -17+40a, \square, -13+40a, 16-40a, -9+40a, 12-40a, 6-40a,\\
&\quad 8-40a, 1-40a, -4+40a, \square, \boldsymbol{\square}, \square, \mathbf{40}\boldsymbol{a}, \square^{2a-7}),\\
R'_{4a+2} &= (\square^{4a+1}, -21-40a, \square, -25-40a, 22+40a, -29-40a, 26+40a, 30+40a,\\
&\quad \mathbf{-35-40}\boldsymbol{a}, 37+40a, \mathbf{33+40}\boldsymbol{a}, \square, -38-40a, \square^{2a-9}),\\
R'_{4a+3} &= (\square^{4a-1}, 3+40a, \square, 7+40a, -4-40a, 11+40a, -8-40a, -14-40a,\\
&\quad -12-40a, -19-40a, \mathbf{20+40}\boldsymbol{a}, \square, \mathbf{16+40}\boldsymbol{a}, \square^{2a-7}),\\
R'_{4a+4} &= (\square^{4a+3}, -41-40a, \square, -45-40a, 42+40a, -49-40a, \mathbf{53+40}\boldsymbol{a}, 50+40a,\\
&\quad \mathbf{46+40}\boldsymbol{a}, 57+40a, -55-40a, \square, -58-40a, \square^{2a-11}),\\
R'_{4a+5} &= (\square^{4a+1}, 23+40a, \square, 27+40a, -24-40a, 31+40a, -28-40a, -34-40a,\\
&\quad \mathbf{36+40}\boldsymbol{a}, -39-40a, \mathbf{-32-40}\boldsymbol{a}, \square, 40+40a, \square^{2a-9}),\\
R'_{4a+6} &= (\square^{4a+5}, -61-40a, \square, -65-40a, \mathbf{66+40}\boldsymbol{a}, -69-40a, \mathbf{62+40}\boldsymbol{a}, 70+40a,\\
&\quad 73+40a, 77+40a, -75-40a, \square, -78-40a, \square^{2a-13}),\\
R'_{4a+7} &= (\square^{4a+3}, 43+40a, \square, 47+40a, -44-40a, 51+40a, \mathbf{-52-40}\boldsymbol{a}, -54-40a,\\
&\quad \mathbf{-48-40}\boldsymbol{a}, -59-40a, 56+40a, \square, 60+40a, \square^{2a-11}),\\
R'_{4a+8} &= (\square^{4a+7}, -81-40a, \mathbf{82+40}\boldsymbol{a}, -85-40a, \boldsymbol{\square}, -89-40a, 86+40a, 90+40a,\\
&\quad 93+40a, 97+40a, -95-40a, \square, -98-40a, \square^{2a-15}),
\end{aligned}$$

$$\begin{aligned} R'_{4a+9} &= (\square^{4a+5}, 63+40a, \square, 67+40a, \mathbf{-68-40a}, 71+40a, \mathbf{-64-40a}, -74-40a,\\ &\quad -72-40a, -79-40a, 76+40a, \square, 80+40a, \square^{2a-13}),\\ R'_{4a+11} &= (\square^{4a+7}, 83+40a, \mathbf{-84-40a}, 87+40a, \boldsymbol{\square}, 91+40a, -88-40a, -94-40a,\\ &\quad -92-40a, -99-40a, 96+40a, \square, 100+40a, \square^{2a-15}). \end{aligned}$$

Notice that $\mathcal{S}(R'_t)$ is obtained from $\mathcal{S}(R_t)$ replacing A_t by B_t, according to Table 2. Now, one has only to check that the elements of B_t and those of $\mathcal{S}(R_t) \setminus A_t$ are pairwise distinct modulo $20n+1$ and $20n+2$. So H' is an $\mathrm{SH}^*(n;10)$.

t	A_t	B_t
$4a-2$	$\emptyset$	$\emptyset$
$4a$	$(18+40a)$	$(15+40a)$
$4a+1$	$\emptyset$	$\emptyset$
$4a+2$	$(36+40a, 73+80a)$	$(5, -32-40a)$
$4a+3$	$(-20-40a)$	$(-16-40a)$
$4a+4$	$(3, -47-40a)$	$(10, -40-40a)$
$4a+5$	$(-76-80a, -37-40a)$	$(-8, 31+40a)$
$4a+6$	$(-133-80a, -64-40a)$	$(-129-80a, -60-40a)$
$4a+7$	$(-5, 49+40a)$	$(-9, 45+40a)$
$4a+8$	$(-166-80a)$	(1)
$4a+9$	$(66+40a, 137+80a)$	$(62+40a, 133+80a)$
$4a+11$	$(170+80a)$	(-1)

Table 2: Subsequences A_t, B_t for $n \equiv 4 \pmod 6$.

Suppose now $n \equiv 2 \pmod 6$, say $n = 6a+2$. In this case H is an $\mathrm{SH}(n;10)$ such that neither its rows nor its columns are simple modulo $20n+2$. More precisely, the row R_{4a+3} and the column C_{4a+6} are not simple modulo $20n+2$. We construct a globally simple $\mathrm{H}(n;10)$ which satisfies also condition $(*)$ interchanging two rows and two columns of H.

Firstly, we interchange the columns C_{4a+7} and C_{4a+9} of H, obtaining a new array, say H'. Since

$$\begin{aligned} C_{4a+7} &= (\square^{4a-5}, 22-40a, \square, 5-40a, -20+40a, 13+40a, -4+40a, 26+40a,\\ &\quad -12-40a, 42+40a, -28-40a, \square, -44-40a, \square, \square, \square^{2a-7})^T,\\ C_{4a+9} &= (\square^{4a-5}, \square, \square, 2-40a, \square, -15-40a, 40a, 33+40a, 16+40a, 46+40a,\\ &\quad -32-40a, 62+40a, -48-40a, \square, -64-40a, \square^{2a-7})^T, \end{aligned}$$

looking at the position of the empty cells, it is easy to see that interchanging them only the rows $R_{(4a-5)+1}$, $R_{(4a-5)+3}$, $R_{(4a-5)+4}, \ldots, R_{(4a-5)+12}$, $R_{(4a-5)+14}$ change. We write explicitly the new rows, denoted by R'_t. As before the interchanged elements are in bold (for simplicity we assume $n \geq 50$, the cases $n = 14, 20, 26, 32, 38, 44$ can be found in [12]):

$$\begin{aligned} R'_{4a-4} &= (\square^{4a-5}, 39-40a, \square, 35-40a, -38+40a, 31-40a, -34+40a, -30+40a,\\ &\quad -27+40a, -23+40a, 25-40a, \square, \boldsymbol{\square}, \square, \mathbf{22-40a}, \square^{2a-7}), \end{aligned}$$

$$
\begin{aligned}
R'_{4a-2} &= (\square^{4a-3}, 19-40a, \square, 15-40a, -18+40a, 11-40a, -14+40a, -10+40a,\\
&\quad -7+40a, -3+40a, \mathbf{2-40a}, \square, \mathbf{5-40a}, \square^{2a-7}),\\
R'_{4a-1} &= (\square^{4a-5}, -37+40a, \square, -33+40a, 36-40a, -29+40a, 32-40a, 26-40a,\\
&\quad 28-40a, 21-40a, -24+40a, \square, \boldsymbol{\square}, \square, \mathbf{-20+40a}, \square^{2a-7}),\\
R'_{4a} &= (\square^{4a-1}, -1-40a, \square, -5-40a, 2+40a, -9-40a, 6+40a, 10+40a,\\
&\quad \mathbf{-15-40a}, 17+40a, \mathbf{13+40a}, \square, -18-40a, \square^{2a-9}),\\
R'_{4a+1} &= (\square^{4a-3}, -17+40a, \square, -13+40a, 16-40a, -9+40a, 12-40a, 6-40a,\\
&\quad 8-40a, 1-40a, \mathbf{40a}, \square, \mathbf{-4+40a}, \square^{2a-7}),\\
R'_{4a+2} &= (\square^{4a+1}, -21-40a, \square, -25-40a, 22+40a, -29-40a, \mathbf{33+40a}, 30+40a,\\
&\quad \mathbf{26+40a}, 37+40a, -35-40a, \square, -38-40a, \square^{2a-11}),\\
R'_{4a+3} &= (\square^{4a-1}, 3+40a, \square, 7+40a, -4-40a, 11+40a, -8-40a, -14-40a,\\
&\quad \mathbf{16+40a}, -19-40a, \mathbf{-12-40a}, \square, 20+40a, \square^{2a-9}),\\
R'_{4a+4} &= (\square^{4a+3}, -41-40a, \square, -45-40a, \mathbf{46+40a}, -49-40a, \mathbf{42+40a}, 50+40a,\\
&\quad 53+40a, 57+40a, -55-40a, \square, -58-40a, \square^{2a-13}),\\
R'_{4a+5} &= (\square^{4a+1}, 23+40a, \square, 27+40a, -24-40a, 31+40a, \mathbf{-32-40a}, -34-40a,\\
&\quad \mathbf{-28-40a}, -39-40a, 36+40a, \square, 40+40a, \square^{2a-11}),\\
R'_{4a+6} &= (\square^{4a+5}, -61-40a, \mathbf{62+40a}, -65-40a, \boldsymbol{\square}, -69-40a, 66+40a, 70+40a,\\
&\quad 73+40a, 77+40a, -75-40a, \square, -78-40a, \square^{2a-15}),\\
R'_{4a+7} &= (\square^{4a+3}, 43+40a, \square, 47+40a, \mathbf{-48-40a}, 51+40a, \mathbf{-44-40a}, -54-40a,\\
&\quad -52-40a, -59-40a, 56+40a, \square, 60+40a, \square^{2a-13}),\\
R'_{4a+9} &= (\square^{4a+5}, 63+40a, \mathbf{-64-40a}, 67+40a, \boldsymbol{\square}, 71+40a, -68-40a, -74-40a,\\
&\quad -72-40a, -79-40a, 76+40a, \square, 80+40a, \square^{2a-15}).
\end{aligned}
$$

Note that $\mathcal{S}(R'_t)$ is obtained from $\mathcal{S}(R_t)$ replacing A_t by B_t, according to Table 3. As before, one has only to check that the elements of B_t and those of $\mathcal{S}(R_t) \setminus A_t$ are pairwise distinct modulo $20n+1$ and $20n+2$.

t	A_t	B_t
$4a-4$	$\emptyset$	$\emptyset$
$4a-2$	$(-2+40a)$	$(-5+40a)$
$4a-1$	$\emptyset$	$\emptyset$
$4a$	$(16+40a, 33+80a)$	$(5, -12-40a)$
$4a+1$	$(-40a)$	$(4-40a\}$
$4a+2$	$(3, -27-40a)$	$(10, -20-40a)$
$4a+3$	$(-36-80a, -17-40a)$	$(-8, 11+40a)$
$4a+4$	$(-93-80a, -44-40a)$	$(-89-80a, -40-40a)$
$4a+5$	$(-5, 29+40a)$	$(-9, 25+40a)$
$4a+6$	$(-126-80a)$	(1)
$4a+7$	$(46+40a, 97+80a)$	$(42+40a, 93+80a)$
$4a+9$	$(130+80a)$	(-1)

Table 3: Subsequences A_t, B_t for $n \equiv 2 \pmod 6$.

Next, we interchange the rows R'_{4a+3} and R'_{4a+4} of H' obtaining a new array H''. It is easy to see that after this operation only the columns $C'_{(4a-1)+1}$, $C'_{(4a-1)+3}$,

$C'_{(4a-1)+4}, \ldots, C'_{(4a-1)+14}$, $C'_{(4a-1)+16}$ of H' have been modified. Since we are swapping two consecutive rows, in each of these columns two consecutive elements are interchanged. If one of them is an empty cell, the partial sums of this column remain the same. So, we omit to write explicitly these columns and it remains only to consider the following ones:

$$\begin{aligned}
C''_{4a+4} &= (\square^{4a-5}, -23+40a, \square, -10+40a, 21-40a, -9-40a, 6-40a, -25-40a, \\
&\quad \boldsymbol{-41-40a}, \boldsymbol{11+40a}, 27+40a, \square, 43+40a, \square^{2a-5})^T, \\
C''_{4a+6} &= (\square^{4a-3}, -3+40a, \square, 10+40a, 1-40a, -29-40a, \boldsymbol{-45-40a}, \boldsymbol{-14-40a}, \\
&\quad 31+40a, -61-40a, 47+40a, \square, 63+40a, \square^{2a-7})^T, \\
C''_{4a+7} &= (\square^{4a-3}, 2-40a, \square, -15-40a, 40a, 33+40a, \boldsymbol{46+40a}, \boldsymbol{16+40a}, \\
&\quad -32-40a, 62+40a, -48-40a, \square, -64-40a, \square^{2a-7})^T, \\
C''_{4a+8} &= (\square^{4a-1}, 17+40a, \square, 30+40a, \boldsymbol{-49-40a}, \boldsymbol{-19-40a}, -34-40a, -65-40a, \\
&\quad 51+40a, -81-40a, 67+40a, \square, 83+40a, \square^{2a-9})^T, \\
C''_{4a+9} &= (\square^{4a-5}, 22-40a, \square, 5-40a, -20+40a, 13+40a, -4+40a, 26+40a, \\
&\quad \boldsymbol{42+40a}, \boldsymbol{-12-40a}, -28-40a, \square, -44-40a, \square^{2a-5})^T, \\
C''_{4a+11} &= (\square^{4a-1}, -18-40a, \square, -35-40a, \boldsymbol{53+40a}, \boldsymbol{20+40a}, 36+40a, 66+40a, \\
&\quad -52-40a, 82+40a, -68-40a, \square, -84-40a, \square^{2a-9})^T.
\end{aligned}$$

Notice that $\mathcal{S}(C''_t)$ is obtained from $\mathcal{S}(C'_t)$ replacing D_t by E_t, according to Table 4. Again by a direct check, one can verify that the partial sums of these 6 columns are pairwise distinct both modulo $20n+1$ and $20n+2$. Hence, H'' is an $\mathrm{SH}^*(n;10)$ for any $n \equiv 2 \pmod 4$. This concludes the proof. □

t	D_t	E_t	t	D_t	E_t
$4a+4$	$(-29-40a)$	$(-81-120a)$	$4a+8$	$(28+40a)$	$(-2+40a)$
$4a+6$	$(-35-40a)$	$(-66-40a)$	$4a+9$	$(30+40a)$	$(84+120a)$
$4a+7$	$(36+40a)$	$(66+40a)$	$4a+11$	$(-33-40a)$	$(-40a)$

Table 4: Subsequences D_t, E_t for $n \equiv 2 \pmod 6$.

Example 4.12 Let $n = 12$. By the construction given in the proof of Proposition 4.11, we obtain the following $\mathrm{SH}^*(12;10)$:

−104	111	−108	−114	−112	−119	116		120	103		107
−38	−21		−25	22	−29	26	30	33	37	−35	
	7	−4	11	−8	−14	−12	−19	16		20	3
−55		−58	−41		−45	42	−49	46	50	53	57
40	23		27	−24	31	−28	−34	−32	−39	36	
73	77	−75		−78	−61		−65	62	−69	66	70
56		60	43		47	−44	51	−48	−54	−52	−59
86	90	93	97	−95		−98	−81		−85	82	−89
−72	−79	76		80	63		67	−64	71	−68	−74
102	−109	106	110	113	117	−115		−118	−101		−105
−88	−94	−92	−99	96		100	83		87	−84	91
	−5	2	−9	6	10	13	17	−15		−18	−1

5 Conclusions

Theorem 5.1 *Let* $3 \leq k \leq 10$*. Then there exists an* $\mathrm{SH}^*(n;k)$ *if and only if* $n \geq k$ *and* $nk \equiv 0, 3 \pmod 4$*.*

PROOF: If $k = 3, 4, 5$ the result follows from Theorem 1.3 and from the considerations on partial sums given at the beginning of Section 4. So, assume $6 \leq k \leq 10$. We recall that, by Theorem 1.3, an $\mathrm{H}(n;k)$ exists only when $n \geq k$ and $nk \equiv 0, 3 \pmod 4$. For these cases, we give in Table 5 the proposition number where we constructed an $\mathrm{SH}^*(n;k)$ (the first column n_4 gives the congruence class of n modulo 4). □

$n_4 \diagdown k$	6	7	8	9	10
0	4.1	4.3	4.6	4.8	4.11
1	$\nexists$	4.4	4.7	$\nexists$	$\nexists$
2	4.1	$\nexists$	4.6	$\nexists$	4.11
3	$\nexists$	$\nexists$	4.7	4.10	$\nexists$

Table 5: Proposition number for the each case.

Now, Theorem 1.9 easily follows from Theorem 5.1 applying Proposition 2.6. The cases $1 \leq n < k$ have been obtained with the help of a computer starting from the constructions given in [6, 7], see [12].

PROOF OF THEOREM 1.11: The $\mathrm{H}(n;3)$ for $n \equiv 1 \pmod 4$ of [4] is cyclically 3-diagonal. Let $A = (a_{i,j})$ be the $\mathrm{H}(n;5)$ described in [17] for $n \equiv 3 \pmod 4$ and define $h_{2i,2j} = a_{i,j}$. Then $H = (h_{i,j})$ is a cyclically 5-diagonal $\mathrm{SH}^*(n;5)$. The $\mathrm{SH}^*(n;7)$ for $n \equiv 1 \pmod 4$ and the $\mathrm{SH}^*(n;9)$ for $11 < n \equiv 3 \pmod 4$ obtained in Section 4 are cyclically 7-diagonal and cyclically 9-diagonal, respectively. By Propositions 3.4 and 3.6 in each of these cases there are simple compatible orderings of the rows and columns. Then, the result follows from Theorem 3.1.

For the exceptional $\mathrm{SH}^*(11;9)$ described in [12], take as ω_r the natural ordering of the rows from left to right and as ω_c the natural ordering of the columns from top to bottom for the first 9 columns, from bottom to top for the last two columns. Then ω_r, ω_c are compatible orderings. Again, we apply Theorem 3.1. □

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