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Some enumerations on non-decreasing Motzkin paths

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Abstract

We introduce non-decreasing Motzkin paths similar to non-decreasing Dyck paths. Some generating functions on several variables are constructed to count several aspects of non-decreasing Motzkin paths. In particular, we use those formal power series to count the number of non-decreasing Motzkin paths according to length, the area under a path, prefixes, and the number of paths with a fixed number of peaks.

1 Introduction

In 1997 Barcucci et al. [2] introduced the concept of non-decreasing Dyck paths. Several authors have been interested in this topic [2, 3, 6, 7, 10, 13, 19]. For example, Deutsch and Prodinger [10] gave a bijection between these paths and directed column-convex polyominoes. In 2002 Vella [19] found that the number of non-decreasing Dyck paths is equal to the number of $\{132, 3241\}$-avoiding permutations. A recent study about non-decreasing Dyck paths was given in [6, 7] where several statistics about them were studied.

Motzkin paths are paths in the first quadrant of the xy-plane having North-East steps, Horizontal steps, and South-East steps. These paths start at the origin, end on the x-axis, and do not cross the x-axis (see for example [17]). Here we generalize the concept of non-decreasing Dyck paths to Motzkin paths. Some aspects of non-decreasing Motzkin paths are counted using the symbolic method [11]. For example, the generating functions given using the symbolic method count: the number of paths, the number of paths with a fixed number of peaks, the area under the path, and the number of prefixes (using Riordan arrays). Additionally, we use coloring and horizontal steps to generalize our non-decreasing Motzkin paths. Varying the definition of non-decreasing Motzkin paths we find some connections between those paths and other combinatorial objects.

2 Preliminaries and Examples

A word in the letters X, Z and Y with as many X's as Y's and in which no initial segment has more Y's than X's is a Motzkin word. For example, $XZZXZYXYY$ and $XXZZYXYYXYZ$ are *Motzkin words*, but $ZXXYXYYYXZZ$ is not. Whoever is familiar with Dyck words may realize a Motzkin word is like a Dyck word but with the letter Z spread around. The *length* of a Motzkin word with exactly n $X's$ and m $Z's$ is $2n+m$.

A *path* P of length $2n+m$ is a $(2n+m+1)$-tuple of points in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. A *step* in a path $P = (p_0, p_1, \dots, p_m)$ is a pair of two consecutive points (p_i, p_{i+1}) for $i \in \{0, \dots, m-1\}$. A *North-East* step has the form $(p_i, p_{i+1}) = ((i,j), (i+1, j+1))$. Similarly, a *South-East* step has the form $(p_i, p_{i+1}) = ((i,j), (i+1, j-1))$ and a *Horizontal* step has the form $(p_i, p_{i+1}) = ((i,j), (i+1, j))$. The *altitude* of $p_i = (i,j)$, denoted by $alt(p_i)$, is the component j. We identify a path $P = (p_0, p_1, \dots, p_m)$ with its broken-line graph obtained by joining p_i to p_{i+1} with a line segment for $i \in \{0, \dots, m-1\}$.

Each Motzkin word L gives rise to a path (*Motzkin path*) P_L having only North-East steps, Horizontal steps and South-East steps. Indeed, X corresponds to a North-East step, Z corresponds to horizontal steps and Y corresponds to a South-East step. For instance, the Motzkin word $ZX^4ZY^4Z^2X^2Y^2X^2Y^2ZX^3Y^2X^3YZXY^4$ corresponds to the Motzkin path depicted in Figure 1. A Motzkin path of length $2n+m$ starts at the origin ($p_0 = (0,0)$), ends on the x-axis ($p_{2n+m} = (2n+m, 0)$), and does not cross the x-axis. It is easy to see that the correspondence $L \mapsto P_L$ is a bijection between the collection of all Motzkin words and the collection of all Motzkin paths.

A *pyramid* of *height* $h > 0$ is a sub-path of the form X^hY^h with h maximal. A *truncated pyramid* of *height* h is a sub-path of the form $X^hZ^mY^h$ where $h, m > 0$ with h maximal. The *valleys* and *peaks* of a Motzkin path P are the local minima and local maxima of P. (A *peak* is a sub-path of the form XY and a *valley* is a sub-path of the form YX.) A *left valley* is preceded by a South-East step and followed by a Horizontal step (is a subword of the form YZ). A *right valley* is preceded by a Horizontal step and followed by a North-East step (is a subword of the form ZX).

We say that a Motzkin path P is *non-decreasing* if the y-coordinates of all valleys, left valleys, and the right valleys of the path P form a non-decreasing sequence. That is, if $v_1, \dots, v_t$ are all valley points, left valley points, and right valley points of a Motzkin path P, then P is *non-decreasing* if

$$alt(v_1) \leq alt(v_2) \leq \cdots \leq alt(v_t).$$

In Figure 1 we show an example of a non-decreasing Motzkin path of length 36. Let us denote by $\mathcal{NM}$ the set of all non-decreasing Motzkin paths. In the following sections we study several statistics over the set $\mathcal{NM}$.

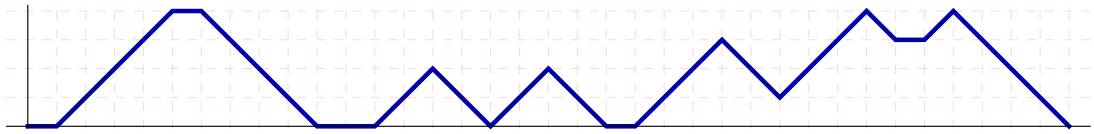

Figure 1: Non-decreasing Motzkin path of length 36.

3 Some Enumerations of Non-Decreasing Motzkin Paths

In this section we use the symbolic method, with the factorization given in Figure 2, to construct two generating functions. From those generating functions we obtain the number of non-decreasing Motzkin paths and the number of non-decreasing Motzkin paths with exactly s peaks. The *area* under a path P is the sum of the altitudes of P. In Section 3.1 we give a generating function that counts the area under a path P.

$\ell(P)$ the length of P;	$h(P)$ the number of horizontal steps of P;
$p(P)$ the number of peaks of P;	$r(P)$ the number of North-East steps of P;
$a(P)$ the area of P.	

Table 1: Parameters.

Let P be a non-decreasing Motzkin path. Using some parameters given in Table 1 we define the generating function:

$$F(x,y,z,q) := \sum_{P\in\mathcal{NM}} x^{\ell(P)}y^{h(P)}z^{r(P)}q^{p(P)}.$$

Theorem 3.1. *The generating function $F(x,y,z,q)$ is given by*

$$F(x,y,z,q) = \frac{x(1-xy)(xzq+y)(1-x^2z)}{1-2xy+x^2y^2-(2+q)x^2z+(2+q)x^3yz-x^4y^2z+x^4z^2-x^5yz^2}.$$

Proof. We use T' to mean a non-decreasing Motzkin path, Δ to mean a pyramid, and W to mean a truncated pyramid. From the definition of non-decreasing Motzkin paths we have that each non-empty non-decreasing Motzkin path P may be uniquely decomposed using one of the following forms: Z, XY, ZT', $XT'Y$, WT', or $\Delta T'$ (see Figure 2). This and the symbolic method (cf. [11]) imply that

$$\begin{aligned}F(x,y,z,q) &= xy + x^2zq + xyF(x,y,z,q) + x^2zF(x,y,z,q)\\ &\quad + \frac{x^3yz}{(1-xy)(1-x^2z)}F(x,y,z,q) + \frac{x^2zq}{1-x^2z}F(x,y,z,q).\end{aligned}$$

After some simplifications we obtain the desired result. $\square$

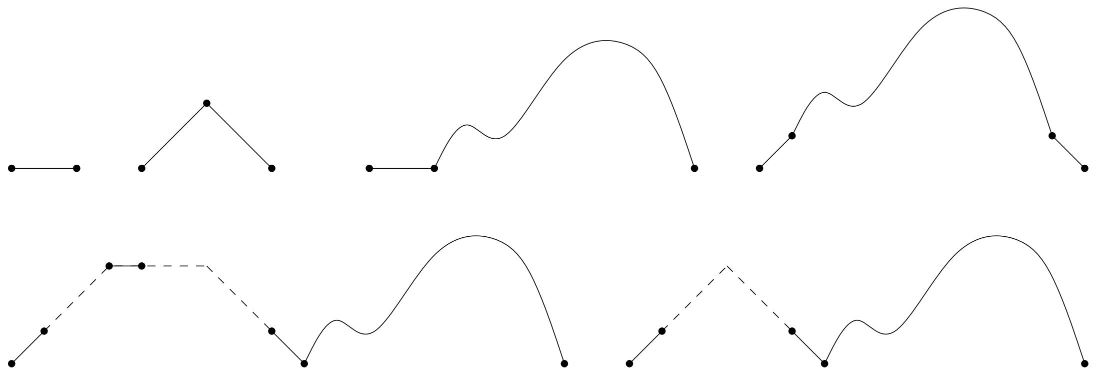

Figure 2: Factorizations of any non-decreasing Motzkin path.

We now give some special cases of our first main generating function. Theorem 3.1 with $y = z = q = 1$ gives rise to the generating function that counts the number of non-decreasing Motzkin paths with respect to the length:

$$\begin{aligned} F(x,1,1,1) &= \frac{x(x^2-1)^2}{1-2x-2x^2+3x^3-x^5} \\ &= x+2x^2+4x^3+9x^4+21x^5+49x^6+115x^7+269x^8+630x^9+\cdots \end{aligned}$$

Moreover, if a_n is the number of non-decreasing Motzkin paths of length n, then a_n has order five and satisfies

$$a_n = 2a_{n-1} + 2a_{n-2} - 3a_{n-3} + a_{n-5},$$

with initial conditions $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 9$, and $a_5 = 21$.

Theorem 3.1 with $y = 0$ and $q = z = 1$ gives rise to the generating function that counts the number of non-decreasing Dyck paths of length $2n$:

$$F(x,0,1,1) = \frac{x^2(1-x^2)}{1-3x^2+x^4} = \sum_{n=1}^{\infty} F_{2n-1}x^{2n}.$$

Therefore, the number of non-decreasing Dyck paths of length $2n$ is the $(2n-1)$-th Fibonacci number (see [2]).

We use $\mathcal{NM}_s$ to mean the set of non-decreasing Motzkin paths with exactly s peaks. Using some parameters given in Table 1 we define the generating function:

$$Q_s(x,y) := \sum_{P\in\mathcal{NM}_s} x^{\ell(P)}y^{h(P)}.$$

Setting $q = 0$ and $z = 1$ in Theorem 3.1 we obtain that $Q_0(x,y) = F(x,y,1,0)$. Therefore, the generating function for the set of all non-decreasing Motzkin paths of length n without peaks is

$$Q_0(x,y) = F(x,y,1,0) = \frac{xy(1-x^2)(1-xy)}{(1-xy-x^2)(1-x^2)(1-xy)-x^3y}.$$

Figure 3 shows all non-decreasing Motzkin paths of length 5 without peaks.

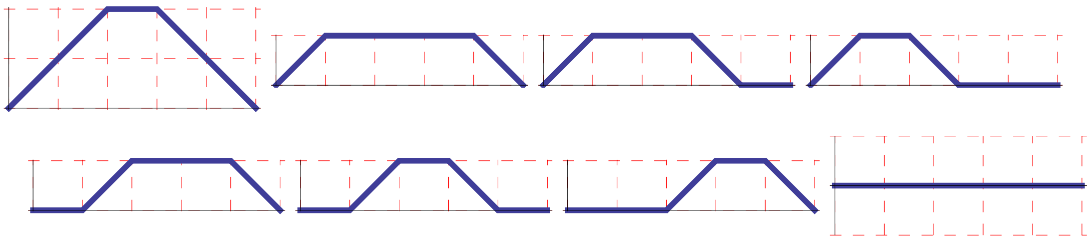

Figure 3: Non-decreasing Motzkin paths of length 5 without peaks.

Corollary 3.2 studies the generating function of the non-decreasing Motzkin paths of length n with exactly s peaks. The proof follows by taking $z = 1$ and $q = 0$ in Theorem 3.1. Taking $y = 0$ in $Q_s(x, y)$ we obtain a well-known result for Dyck paths. Corollary 3.3 studies the generating function of the non-decreasing Dyck paths of length $2n$ with exactly s peaks.

Corollary 3.2. *If $Q_s(x, y)$ is the generating function for the non-decreasing Motzkin paths with exactly s peaks, then the following hold:*

1. *For all $s \geq 2$*

$$Q_s(x, y) = \frac{x^2(1 - xy)}{(1 - xy - x^2)(1 - x^2)(1 - xy) - x^3y} Q_{s-1}(x, y).$$

2. *For all $s \geq 1$*

$$Q_s(x, y) = \frac{x^{2s}(1 - xy)^s(1 - x^2 + Q_0(x, y))}{((1 - xy - x^2)(1 - x^2)(1 - xy) - x^3y)^s}.$$

Proof. We prove part (1). From the factorization given in Figure 2 we obtain that

$$Q_s(x, y) = xyQ_s + x^2Q_s(x, y) + \frac{x^3y}{(1 - x^2)(1 - xy)} Q_s(x, y) + \frac{x^2}{1 - x^2} Q_{s-1}(x, y).$$

After some simplification it is easy to see that part (1) holds.

Proof of part (2). From

$$Q_1(x, y) = x^2 + xyQ_1(x, y) + x^2Q_1(x, y) + \frac{x^3y}{(1 - x^2)(1 - xy)} Q_1(x, y) + \frac{x^2}{1 - x^2} Q_0(x, y)$$

and iterating the equation in part (1) the result in part (2) follows. □

We now give some special cases of the generating function given in Corollary 3.2. Notice that this corollary with $y = 0$ gives [6, Proposition 3]. Now, taking $y = 1$ and $s = 1, 2, 3$ in Corollary 3.2 part (2) we obtain the following series:

$$Q_1(x, 1) = x^2 + 2x^3 + 4x^4 + 10x^5 + 22x^6 + 51x^7 + 115x^8 + 259x^9 + 579x^{10} + \cdots$$
$$Q_2(x, 1) = x^4 + 3x^5 + 9x^6 + 25x^7 + 65x^8 + 167x^9 + 416x^{10} + 1022x^{11} + \cdots$$
$$Q_3(x, 1) = x^6 + 4x^7 + 15x^8 + 48x^9 + 143x^{10} + 407x^{11} + 1114x^{12} + 2970x^{13} + \cdots$$

Figure 4 shows all non-decreasing Motzkin paths of length 7 with three peaks.

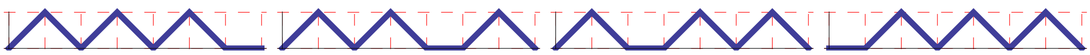

Figure 4: Non-decreasing Motzkin paths of length 7 with exactly three peaks.

Corollary 3.3. *The generating function for the non-decreasing Dyck paths of length* $2n$ *with exactly* s *peaks satisfies*

1. *for all* $s \geq 2$
$$Q_s(x, 0) = \frac{x}{(1-x)^2} Q_{s-1}(x, 0).$$

2. *For all* $s \geq 1$
$$Q_s(x, 0) = \frac{x^s}{(1-x)^{2s-1}}.$$

Therefore, the number of non-decreasing Dyck paths of length $2n$ *with exactly* s *peaks is*
$$\binom{n+s-2}{2s-2}.$$

3.1 The Area

We recall that the area under a given non-decreasing Motzkin path P is the sum of the altitudes of all points of P. For instance, the path in Figure 1 has area 62. Using the parameters given in the Table 1 we define the generating function:

$$T(x, y, z, q) := \sum_{P \in \mathcal{NM}} x^{\ell(P)} y^{h(P)} z^{a(P)} q^{p(P)}.$$

Theorem 3.4. *The generating function for the area of the non-decreasing Motzkin paths is given by the following equation*

$$T(x, y, z, q) = \sum_{k=1}^{\infty} \frac{x^{2k} z^{k^2} q + x^{2k-1} y z^{k^2-k}}{\prod_{j=1}^{k-1} (1 - xyz^{k-1} - P_1(xz^j, y, z) - P_2(xz^j, z, q))},$$

where

$$P_1(x, y, z) = y \sum_{k=1}^{\infty} \frac{x^{2k+1} z^{k^2+k}}{1 - xyz^k}$$

and

$$P_2(x, z, q) = q \sum_{k=1}^{\infty} x^{2k} z^{k^2}.$$

Proof. The generating function of the area of a truncated pyramid $X^i Z^j Y^i$, is given by:

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} (x^{2k} z^{k^2})(x^\ell y^\ell z^{\ell k}) = y \sum_{k=1}^{\infty} \frac{x^{2k+1} z^{k^2+k}}{1 - xyz^k} = P_1(x, y, z).$$

Analogously, the generating function of the area of a pyramid $X^i Y^i$, $(i \geq 1)$ is given by $P_2(x, z, q)$.

From the decomposition given in Figure 2 we have

$$\begin{aligned} T(x, y, z, q) = xy + x^2 zq + xyT(x, y, z, q) + x^2 zT(xz, y, z, q) \\ + P_1(x, y, z)T(x, y, z, q) + P_2(x, z, q)T(x, y, z, q). \end{aligned}$$

Therefore the desired result follows. □

We now give a special case of Theorem 3.4. Taking $y = 0$ and seting $x^2 := x$ in $T(x, y, z, q)$ we obtain the following corollary.

Corollary 3.5 (Theorem 3.2 [2]). *The generating function for the area of the non-decreasing Dyck paths is*

$$T(x, z, q) = q \sum_{k=1}^{\infty} \frac{x^k z^{k^2}}{\prod_{j=1}^{k-1} (1 - P(xz^{2j}, z, q))},$$

where

$$P(x, z, q) = q \sum_{k=1}^{\infty} x^k z^{k^2}.$$

4 Prefixes of non-decreasing Dyck paths and Motzkin paths

A path *prefix* is any initial sub-path. Let P be a non-decreasing Motzkin path. A prefix of P is called *non-decreasing Motzkin*. Intuitively, a initial sub-path Q of P is a prefix, if Q contains more X's than Y's. The prefixes of the classical Dyck paths are also called Ballot paths. The height of a path is defined as the final height of the path, i.e., the stopping y-coordinate. The number of non-decreasing Motzkin prefixes of length n and height k is denoted by $w_{n,k}$. Figure 5 shows that $w_{4,2} = 9$. Let $\mathcal{M}$ be the infinite lower triangular matrix defined by $\mathcal{M} = [w_{n,k}]_{n,k \geq 0}$. The first few terms of this matrix are

$$\mathcal{M} = [w_{n,k}]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 9 & 12 & 9 & 4 & 1 & 0 & 0 & 0 \\ 21 & 30 & 25 & 14 & 5 & 1 & 0 & 0 \\ 49 & 74 & 69 & 44 & 20 & 6 & 1 & 0 \\ 115 & 182 & 185 & 133 & 70 & 27 & 7 & 1 \\ \vdots & & & \vdots & & & & \vdots \end{pmatrix}.$$

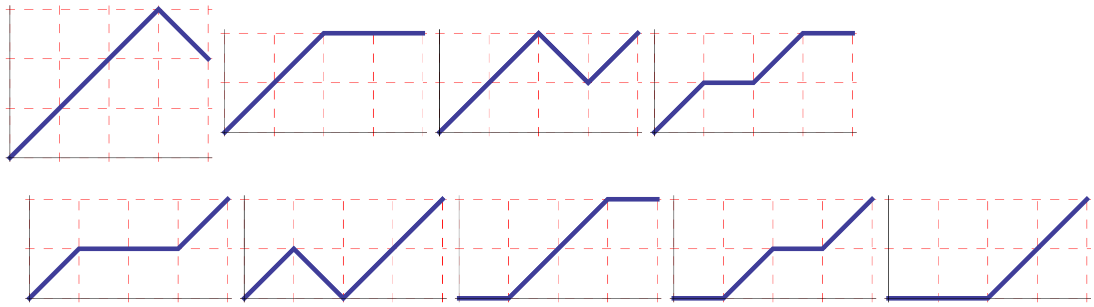

Figure 5: Non-decreasing Motzkin prefixes of length 4 and height 2.

Theorem 4.1. *Let $T_k(x)$ be the generating function for the non-decreasing Motzkin prefixes defined by*

$$T_k(x) := \sum_{n=0}^{\infty} w_{n,k} x^n.$$

Then

$$\begin{aligned} T_k(x) &= (1 + F(x,1,1,1)) \left(x \frac{(1-x)^2(1+x)}{1-2x-x^2+2x^3-x^4} \right)^k \\ &= \frac{1-x-2x^2+x^3}{1-2x-2x^2+3x^3-x^5} \left(x \frac{(1-x)^2(1+x)}{1-2x-x^2+2x^3-x^4} \right)^k. \end{aligned}$$

Proof. We use U to mean a Motzkin path without valleys or valleys of height zero, T to mean a non-decreasing Motzkin path (possible empty). Each non-decreasing Motzkin prefix P of height k may be uniquely decomposed as $UXUX \cdots UXT$ (see Figure 6).

Let $\mathcal{U}$ be the family of Motzkin paths without valleys or valleys of height zero. Then $T_k(x) = (xU(x,1))^k(1 + F(x,1,1,1))$, where

$$U(x,y) := \sum_{P \in \mathcal{U}} x^{\ell(P)} y^{h(P)}.$$

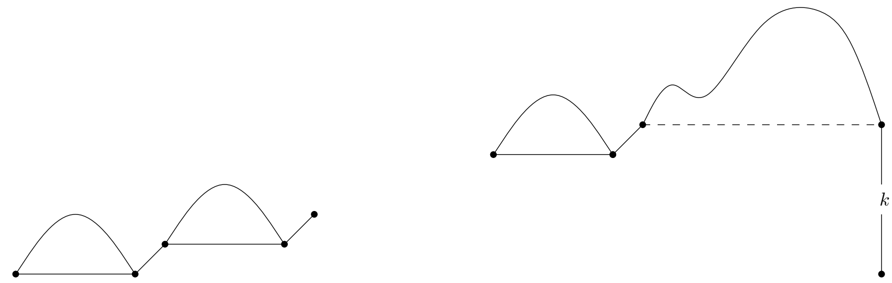

Figure 6: Factorization of any non-decreasing Motzkin prefix of height k.

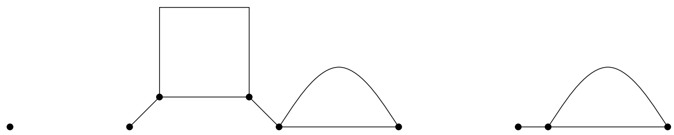

Figure 7: Factorizations of any Motzkin path without valleys or valleys of height zero.

Any path of $\mathcal{U}$ may be uniquely decomposed as either λ (empty path), $X\widetilde{U}YU$ or ZU, where $\widetilde{U}$ is a Motzkin path without valleys, see Figure 7.

Let $\widetilde{U}(x,y)$ be the generating function defined by

$$\widetilde{U}(x,y) := \sum_{P\in\widetilde{\mathcal{U}}} x^{\ell(P)}y^{h(P)},$$

where $\widetilde{\mathcal{U}}$ is the family of Motzkin paths without valleys. It is not difficult to show that (see Figure 8)

$$\widetilde{U}(x,y) = 1 + x^2\widetilde{U}(x,y) + \frac{xy}{1-xy}.$$

Therefore,

$$\widetilde{U}(x,y) = \frac{1}{(1-xy)(1-x^2)}.$$

So,

$$U(x,y) = 1 + x^2\widetilde{U}(x,y)U(x,y) + xyU(x,y).$$

Then

$$U(x,y) = \frac{(1-xy)(1-x^2)}{(1-xy)^2(1-x^2)-x^2}.$$

This completes the proof. □

We recall that an infinite lower triangular matrix is called a Riordan array [14] if its kth column satisfies the generating function $g(z)\,(f(z))^k$ for $k \geq 0$, where $g(z)$ and $f(z)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$ (where $f'(x)$ is the formal derivative of $f(x)$). The matrix corresponding to the pair $f(z), g(z)$

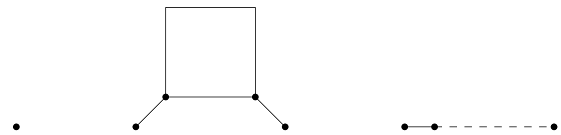

Figure 8: Factorizations of any Motzkin path without valleys.

is denoted by $(g(z), f(z))$. If we multiply (g, f) by a column vector $(c_0, c_1, \dots)^T$ with the generating function $h(z)$, then the resulting column vector has generating function $gh(f)$. This property is known as the fundamental theorem of Riordan arrays or summation property.

The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h\left(f(z)\right), l\left(f(z)\right)).$$

We recall that the set of all Riordan matrices is a group under the operator "$*$" [14]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is

$$(g(z), f(z))^{-1} = \left(1/\left(g \circ \overline{f}\right)(z), \overline{f}(z)\right), \tag{1}$$

where $\overline{f}(z)$ is the compositional inverse of $f(z)$.

The following theorem is straightforward from the definition of Riordan matrix.

Theorem 4.2. *The matrix $\mathcal{M}$ is a Riordan matrix given by*

$$\mathcal{M} = \left(\frac{1 - x - 2x^2 + x^3}{1 - 2x - 2x^2 + 3x^3 - x^5}, \frac{x(1-x)^2(1+x)}{1 - 2x - x^2 + 2x^3 - x^4}\right).$$

From the summation property for the Riordan matrices we obtain the generating function for the total number of non-decreasing Motzkin prefixes:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} w_{n,k}x^n = \frac{1 - 3x - x^2 + 8x^3 - 3x^4 - 4x^5 + 4x^6 - x^7}{1 - 5x + 4x^2 + 12x^3 - 17x^4 - 3x^5 + 16x^6 - 6x^7 - 3x^8 + 2x^9}.$$

4.1 Non-decreasing Dyck prefixes paths

The number of non-decreasing Dyck prefixes of length n and height k is denoted by $d_{n,k}$. Let us define the matrix $\mathcal{D}_1 = [d_{n,k}]_{n,k \geq 0}$. The first few terms of this matrix are:

$$\mathcal{D}_1 = [d_{n,k}]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 13 & 0 & 14 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\ 13 & 0 & 26 & 0 & 20 & 0 & 7 & 0 & 1 & 0 & 0 \\ 0 & 34 & 0 & 45 & 0 & 27 & 0 & 8 & 0 & 1 & 0 \\ 34 & 0 & 73 & 0 & 71 & 0 & 35 & 0 & 9 & 0 & 1 \\ \vdots & & & & & \vdots & & & & & \vdots \end{pmatrix}.$$

Taking $y = 0$ in the proof of Theorem 4.1 we obtain the following corollary.

Corollary 4.3. *The Riordan matrix $\mathcal{D}_1$ is given by*

$$\mathcal{D}_1 = \left(\frac{1-2x^2}{1-3x^2+x^4}, x\frac{(1-x^2)}{1-2x^2} \right).$$

Notice that the first column of $\mathcal{D}_1$ are the odd-indexed Fibonacci numbers. From the summation property for the Riordan matrices we have the Corollary 4.4.

Corollary 4.4. *The bivariate generating function is*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d_{n,k}x^n y^k = \frac{(1-2x^2)^2}{(1-3x^2+x^4)(1-2x^2-xy+x^3y)}.$$

In particular the generating function of the total non-decreasing Dyck prefixes is

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} d_{n,k}x^n = \frac{(1-2x^2)^2}{(1-3x^2+x^4)(1-x-2x^2+x^3)}.$$

Note that the total number of Dyck prefixes of length n is $\binom{n}{\lfloor n/2 \rfloor}$ (cf. [12]).

Corollary 4.5 gives an explicit expression for the entries $d_{n,k}$. This corollary follows from the definition of Riordan matrix and Corollary 4.3.

Corollary 4.5. *For $n, k \geq 0$ we have*

$$d_{n,k} = \sum_{\ell=0}^{\frac{n-k}{2}} F_{2\ell-1} \sum_{j=0}^{\frac{n-k}{2}-\ell} \binom{k+j-1}{j} \binom{k}{\frac{n-k}{2}-\ell-j} 2^j (-1)^{\frac{n-k}{2}-j-\ell},$$

if $n+k$ is even; and 0 otherwise.

Let $\mathcal{D}_2 = [d_{2n-k,k}]_{n,k\geq 0} = [\widetilde{d}_{n,k}]_{n,k\geq 0}$. The first few terms of the matrix $\mathcal{D}_2$ are

$$\mathcal{D}_2 = [\widetilde{d}_{n,k}]_{n,k\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 13 & 13 & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\ 34 & 34 & 26 & 14 & 5 & 1 & 0 & 0 & 0 \\ 89 & 89 & 73 & 45 & 20 & 6 & 1 & 0 & 0 \\ 233 & 233 & 201 & 137 & 71 & 27 & 7 & 1 & 0 \\ 610 & 610 & 546 & 402 & 234 & 105 & 35 & 8 & 1 \end{pmatrix}.$$

Corollary 4.6 is a straightforward application of the summation property.

Corollary 4.6. *The Riordan matrix $\mathcal{D}_2$ is given by*

$$\mathcal{D}_2 = \left(\frac{1-2x}{1-3x+x^2}, x\frac{(1-x)}{1-2x}\right).$$

Moreover,

$$\sum_{k=0}^{n} \widetilde{d}_{n,k} = \frac{1}{5}\left((5+3n)F_{2n-1} + (3-n)F_{2n}\right)$$

and

$$\sum_{k=0}^{n} (-1)^k \widetilde{d}_{n,k} = \frac{1}{2}\left(F_{2n+3} - F_n\right), \quad (n \geq 2).$$

5 Two Additional Generalizations

The classic concept of Motzkin paths has been generalized to (u,ℓ,d)-colored Motzkin paths [20] and also to k-generalized Motzkin paths [5, 15, 16]. It is natural to ask if a generalization of the non-decreasing Motzkin paths to those mentioned concepts also have interesting results. The first generalization is called (u,ℓ,d)-*colored non-decreasing Motzkin paths*. In this case each step is coloured. Let u be the number of colors of each up-step $(1,1)$, let ℓ be the number of colors for each horizontal-step $(1,0)$ and let d be the number of colors of each each down-step $(1,-1)$. Let $\mathcal{NM}^{(u,\ell,d)}$ be the set of all (u,ℓ,d)-colored non-decreasing Motzkin paths. For brevity we are not going to give proofs of the following results. From the factorizations in Figure 2 and the symbolic method it is easy to see that the results hold.

We denote by $F^{(u,\ell,d)}(x,y,z,q)$ the generating function with the parameters given in Table 1:

$$F^{(u,\ell,d)}(x,y,z,q) := \sum_{P\in\mathcal{NM}^{(u,\ell,d)}} x^{\ell(P)} y^{h(P)} z^{r(P)} q^{p(P)}.$$

Theorem 5.1. *The generating function $F^{(u,\ell,d)}(x,y,z,q)$ is given by*

$$\frac{x(1-\ell xy)(\ell y + dquxz)(1-dux^2z)}{1 - 2\ell xy + \ell^2x^2y^2 - (2+q)(dux^2z + d\ell ux^3yz) - d\ell^2ux^4y^2z + d^2u^2x^4z^2 - d^2\ell u^2x^5yz^2}$$

In particular, the generating function for the (u, ℓ, d)-colored non-decreasing Motzkin paths is

$$F^{(u,\ell,d)}(x,1,1,1) = \frac{x(1-\ell x)(\ell + dux)(1-dux^2)}{1-2\ell x+(\ell^2-3du)x^2+3d\ell ux^3-(d\ell^2u-d^2u^2)x^4-d^2\ell u^2x^5}.$$

Moreover, the generating function for the (u, d)-colored non-decreasing Dyck paths is

$$F^{(u,d)}(x,0,1,1) = \frac{dux^2(1-dux^2)}{1-3dux^2+d^2u^2x^4} = \sum_{n=1}^{\infty}(ud)^n F_{2n-1}x^{2n}.$$

The second generalization is the k-generalized non-decreasing Motzkin paths. In this case the horizontal step is $(k, 0)$ for $k \in \mathbb{Z}_{>0}$. Let $\mathcal{NM}^{(k)}$ be the set of all k-generalized non-decreasing Motzkin paths. We denote by $F_k(x, y, z, q)$ the generating function:

$$F_k(x,y,z,q) := \sum_{P\in\mathcal{NM}^{(k)}} x^{\ell(P)}y^{h(P)}z^{r(P)}q^{p(P)}.$$

Theorem 5.2. *The generating function $F_k(x, y, z, q)$ is given by*

$$\frac{x(1-x^ky)(xzq+x^{k-1}y)(1-x^2z)}{1-2x^ky+x^{2k}y^2-(2+q)x^2z+(2+q)x^{2+k}yz-x^{2+2k}y^2z+x^4z^2-x^{4+k}yz^2}.$$

In particular, the generating function for the k-generalized non-decreasing Motzkin paths with respect to the length is

$$F_k(x,1,1,1) = \frac{x(1-x^k)(x+x^{k-1})(1-x^2)}{1-3x^2+x^4-2x^k+3x^{2+k}-x^{4+k}+x^{2k}-x^{2+2k}}.$$

6 Weak Non-Decreasing Motzkin Paths

A *weak valley* is a subpath of the form YX, YZ, ZX or ZZ. (This concept is called valley by Brennan and Mavhungu [4].) We say that a Motzkin path is *weak non-decreasing* if the weak valleys form a non-decreasing sequence. Let $\mathcal{WM}$ be the set of all weak non-decreasing Motzkin paths.

Theorem 6.1 gives an expression for the generating function in (2). The proof of Theorem 6.1 follows from the factorization given in Figure 2, where the last term (the leftmost on second line) factors as X^iZY^iT $(i \geq 1)$ with T a weak non-decreasing Motzkin path.

$$W(x,y,z,q) := \sum_{P\in\mathcal{WM}} x^{\ell(P)}y^{h(P)}z^{r(P)}q^{p(P)}. \tag{2}$$

Theorem 6.1. *The generating function $W(x, y, z, q)$ is given by*

$$W(x,y,z,q) = \frac{x(y+qxz)(1-x^2z)}{1-xy-(2+q)x^2z+x^4z^2}.$$

We now give some special cases of the generating function given in Theorem 6.1. Taking $y = z = q = 1$ in Theorem 6.1 we obtain the generating function for the weak non-decreasing Motzkin paths with respect to the length:

$$\begin{aligned} W(x,1,1,1) &= \frac{x - x^3}{1 - 2x - x^2 + x^3} \\ &= x + 2x^2 + 4x^3 + 9x^4 + 20x^5 + 45x^6 + 101x^7 + 227x^8 + 510x^9 + \cdots \end{aligned}$$

If b_n is the number of weak non-decreasing Motzkin paths of length n, then it is clear that the sequence b_n satisfies the recurrence relation

$$b_n = 2b_{n-1} + b_{n-2} - b_{n-3},$$

with the initial values $b_1 = 1, b_2 = 2$, and $b_3 = 4$.

Taking $y = 0$ and $q = 1 = z$ in Theorem 6.1, we recover the number of non-decreasing Dyck paths of length $2n$ (F_{2n-1} the $(2n-1)$-th Fibonacci numbers).

The sequence b_n coincides with the sequence A052534 of the OEIS [18]. Therefore, it is possible to obtain different combinatorial interpretations for this sequence.

For example, consider the Motzkin paths whose height is at most 2 and the horizontal steps all occur at level zero or one. This family of Motzkin paths is represented using the transition diagram of Figure 9.

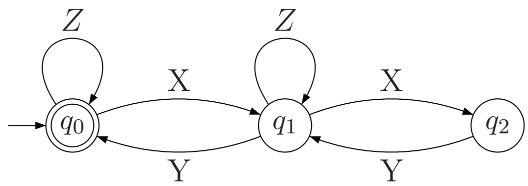

Figure 9: Transition diagram for the restricted Motzkin paths.

The technique on how to find the ordinary generating function corresponding to an automaton (or in general for a context-free language) is known as the *Chomsky-Schützenberger methodology*, (see for example [9, 11, 8]).

The finite automaton in Figure 9 gives rise to a system of equations for the associated generating functions:

$$\begin{cases} L_0(x) &= xL_0(x) + xL_1(x) + 1 \\ L_1(x) &= xL_0(x) + xL_1(x) + xL_2(x) \\ L_2(x) &= xL_1(x) \end{cases}$$

The augmented matrix associated to this system of equations is

$$\left[\begin{array}{ccc|c} 1-x & -x & 0 & 1 \\ -x & 1-x & -x & 0 \\ 0 & -x & 1 & 0 \end{array}\right].$$

Using Gaussian elimination, we obtain the generating function

$$L_0(x) = \frac{1 - x - x^2}{1 - 2x - x^2 + x^3}.$$

Notice that $L_0(x) = W(x, 1, 1, 1) + 1$. So, we have the following relation.

Corollary 6.2. *The number of weak non-decreasing Motzkin paths of length n is equal to the number of Motzkin paths of length n whose height is at most 2 and the horizontal steps occur at level zero or one.*

It will be interesting to have a bijective proof of this relation.

There are several interesting relations related to Corollary 6.2. The first is given by the red/black partitions (introduced by De Andrade et al. [1]). Other relations can be found in A052534. A *red-black partition* is a partition where the parts may be of two colors, red or black. Each red part from one up to the largest part appears at least once and at most twice. Each black part is counted twice and the largest black part is at most twice the largest red part. Let c_n be the number of red/black partitions into at most n parts. For example, $c_3 = 9$, where the partitions are

$$\emptyset;\ 1_r;\ 1_r + 1_r;\quad 2_r + 1_r;\ 1_r + 1_b;\ 1_r + 2_b;\ 2_r + 1_r + 1_r;\ 2_r + 2_r + 1_r;\ 3_r + 2_r + 1_r.$$

We observe that from [1, Theorem 10.1] the following holds.

Corollary 6.3. *The number of weak non-decreasing Motzkin paths of length $n + 1$ is equal to the number of red/black partitions into at most n parts.*

Let $\mathcal{WM}_s$ be the set of weak non-decreasing Motzkin paths with exactly s peaks (subwords XY). We introduce the following generating function:

$$J_s(x, y) := \sum_{P \in \mathcal{WM}_s} x^{\ell(P)} y^{h(P)}.$$

From Theorem 6.1 we have that the generating function for the weak non-decreasing Motzkin paths without peaks is

$$J_0(x, y) = W(x, 1, 1, 0) = \frac{x(1 - x^2)}{1 - x - 2x^2 + x^4}. \tag{3}$$

The number of weak non-decreasing Motzkin paths without peaks coincides with the sequence A052535. This is actually the number of compositions of n with parts in the set $\{2, 1, 3, 5, 7, 9, \dots\}$.

Corollary 6.4. *The generating function for the weak non-decreasing Motzkin paths with exactly s peaks satisfies*

1. *for all $s \geq 2$*

$$T_s(x, y) = \frac{x^2}{1 - 2x^2 + x^4 - xy} T_{s-1}(x, y).$$

2. For all $s \geq 1$

$$T_s(x,y) = \frac{x^{2s}(1 - x^2 + T_0(x,y))}{(1 - 2x^2 + x^4 - xy)^s}.$$

Notice that for all $s \geq 1$

$$T_s(x,1) = \frac{x^{2s}(1 - x^2)^3}{(1 - x - 2x^2 + x^4)^{s+1}}.$$

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