AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(1)** (2018), Pages 106–112

Tic-Tac-Toe on graphs

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Abstract

Tic-Tac-Toe is a two player pencil and paper game. Players alternate turns placing marks on a three by three grid. The first player to have three of their respective marks on a horizontal, vertical, or diagonal row wins the game. In this paper, we generalize this game to graphs. In our main result, we provide simple necessary and sufficient conditions for the first player to have a winning strategy on a graph. We prove that both players have a drawing strategy on all remaining graphs. We provide simple explicit strategies for both players. Finally, we give open problems related to this study.

1 Introduction

There has been much research in the area of games on graphs (see [8] for a survey of references). Several tabletop games such as Lights Out [7], Nim [5], pebbling (inspired by Mancala, see for example [9]), and peg solitaire [3] have been adapted for play on graphs. A classic game that would lend itself to such a treatment is Tic-Tac-Toe. For this reason, we are motivated to introduce the study of Tic-Tac-Toe on graphs in this paper.

Games similar to Tic-Tac-Toe (or Noughts and Crosses) have been played for at least two thousand years [6]. The most well-known variation of Tic-Tac-Toe is played on a three by three grid. Two players alternate turns placing marks on the grid. The first player to have three of their respective marks in a horizontal, vertical, or diagonal row wins the game. In the traditional game, perfect play from both players will result in a draw each time. However, generalizations of the game are usually more complicated and often unsolved. See Beck [2] for more information on variations of Tic-Tac-Toe.

In games of no chance, the goal is usually to determine the optimal strategy. A *strategy* is one of the options available to a player where the outcome depends not only on the player's actions, but the actions of others. A strategy is *winning* if the player following it will win, regardless of the actions of their opponents. Similarly, a *drawing strategy* is one in which the player following it can force a draw, no matter the actions of their opponents. The Fundamental Theorem of Combinatorial Game Theory (see for example [1, 10]) states that in games such as Tic-Tac-Toe either one player has a winning strategy or both players have a drawing strategy. Further, Nash's Strategy Stealing Argument (see for example [2, 10]) says that in positional games such as this, there is no disadvantage in going first. Therefore, if the second player has a winning strategy or a drawing strategy, then the first player could waste their opening move and steal the second player's strategy. Combining these two facts leads to the following observation.

Observation 1.1 *[1, 2, 10] If the second player has a drawing strategy in Tic-Tac-Toe, then both players have a drawing strategy.*

In this paper, we generalize Tic-Tac-Toe to graphs. A graph $G = (V, E)$ is a set of vertices, V, and a set of edges, E. We will assume that all graphs are finite, connected, undirected graphs with no loops or multiple edges. Our notation and terminology will be consistent with West [12]. The *star* with n arms will be denoted $K_{1,n}$. The *path* and *cycle* on n vertices will be denoted P_n and C_n, respectively. Most of our conditions will be in terms of the *degree* of a vertex $v \in V(G)$, that is the number of vertices adjacent to v. The *neighbors* of v are the vertices that share an edge with v. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

As with the traditional game, two players take turns placing their respective marks on the vertices of a graph G. Following the convention of [1], we will refer to the first player as "Alice" and the second player as "Bob." The first player to place their marks on vertices x, y, and z such that $xy \in E(G)$ and $yz \in E(G)$ wins. This will be referred to as *capturing* a P_3. Thus, each P_3 subgraph of G constitutes a *winning set*. The goal of this paper is to classify all graphs in which Alice has a winning strategy. We will show that both players have a drawing strategy on all other graphs.

2 The Basic Game

We note that if G has at most four vertices, then neither player can win, regardless of strategy. Thus, we need only consider the case where G has at least five vertices.

In this section, we will prove our main result which is given in Theorem 2.3. Our result will be a subgraph characterization, similar to Beineke's Theorem (for line graphs) and Kuratowski's Theorem (for planar graphs). We will show that Alice has a winning strategy on a graph G if and only if G contains one of the graphs in Figure 1 as a subgraph. For this reason, we start by determining the graphs with maximum degree three that have neither B_1 nor B_2 as a subgraph. This is given in the following proposition.

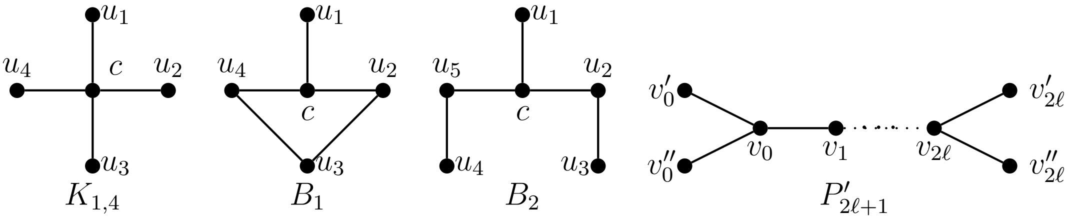

Figure 1: Examples of graphs in which Player One has a winning strategy

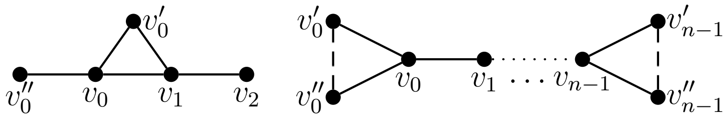

Figure 2: Graphs with maximum degree three that have neither B_1 nor B_2 as a subgraph

Proposition 2.1 *Suppose that G is a graph with at least five vertices and maximum degree three such that G has neither B_1 nor B_2 as a subgraph, where B_1 and B_2 are illustrated in Figure 1. Then either* (i) *G is isomorphic to the graph on the left of Figure 2 or* (ii) *G is obtained from the path on the vertices v_0, v_1,..., v_{n-1} (where $n \geq 2$) by appending two vertices (v_0' and v_0'') to v_0 and at most two vertices (v_{n-1}' and v_{n-1}'') to v_{n-1}. In this case, the edges $v_0'v_0''$ and $v_{n-1}'v_{n-1}''$ may or may not be in the graph. An example is illustrated on the right of Figure 2.*

Proof. Suppose that G is a graph with maximum degree three such that G has neither B_1 nor B_2 as a subgraph. Let v_0 be a vertex of degree three in G with neighbors v_0', v_0'', and v_1. Since G must have at least five vertices, we assume that v_1 is adjacent to a vertex $v_2 \notin \{v_0, v_0', v_0''\}$. We now have two cases to consider.

Case 1: Suppose that v_0' is adjacent to a vertex $u \notin \{v_0, v_0''\}$. If $v_0'v_2 \in E(G)$, then G has B_1 as a subgraph. Likewise, if v_0' is adjacent to a vertex $u \notin \{v_0'', v_0, v_1, v_2\}$, then G has B_2 as a subgraph. Thus, we can assume that $v_0'v_1 \in E(G)$. Reversing the roles of v_0' and v_0'' shows that v_0'' can only be adjacent to v_0' or v_1. In either case, the resulting graph has B_1 as a subgraph. Thus the degree of v_0'' must be one. Further, if the degree of v_2 is at least two, then G will have B_2 as a subgraph. Hence, G will be isomorphic to the graph illustrated on the left of Figure 2.

Case 2: Suppose that v_0' has no neighbor outside of the set $\{v_0, v_0''\}$. Note that if v_0'' has a neighbor outside of $\{v_0, v_0'\}$, then this reduces to Case 1. So we may assume that neither v_0' nor v_0'' has a neighbor outside of the set $\{v_0, v_0', v_0''\}$. Let $P = \{v_0, v_1, ..., v_n\}$ be a path containing v_0, where $n \geq 2$. If v_i is a vertex of degree three, where $1 \leq i \leq n-2$, then G has B_2 as a subgraph. Thus, we may assume that the degree of v_i is two for all i, where $1 \leq i \leq n-2$. Suppose that v_{n-1} is of degree three, with neighbors v_{n-2}, v_n, and v_{n-1}'. If v_n has a neighbor outside of $\{v_{n-1}, v_{n-1}'\}$, then G has B_2 as a subgraph. Thus, if v_{n-1} is of degree three, then this is isomorphic to the graph in (ii) with $v_n = v_{n-1}''$. If v_{n-1} is of degree two, then v_n may have at most two neighbors v'_n and v''_n other than v_{n-1}. Using a similar argument as above, v'_n and v''_n can have no neighbor outside of the set $\{v_n, v'_n, v''_n\}$. Thus, any acceptable graph can be obtained from the path on the vertices v_0, v_1,...,v_{n-1} by appending two vertices (v'_0 and v''_0) to v_0 and at most two vertices (v'_{n-1} and v''_{n-1}) to v_{n-1}. The edges $v'_0v''_0$ and $v'_{n-1}v''_{n-1}$ may or may not be in the graph. ■

The graph in Proposition 2.1 (i) is called the *bull graph* in West [12]. We now introduce notation for an important subset of those graphs described in Proposition 2.1 (ii). Let P'_n denote the graph obtained from the path on n vertices by appending two pendant vertices to each of the two end vertices of the path. The notation for the vertices of P'_n will be consistent with Proposition 2.1. The graph $P'_{2\ell+1}$ is illustrated in Figure 1.

In order to prove our main result it is also useful to define the concept of a *fork*. A fork is a subgraph and a placement of marks on that subgraph such that one player, say Alice, can win on her next turn, provided that Bob cannot win first. As in the traditional game, recognizing forks and potential forks is central to the strategy of Tic-Tac-Toe on graphs. We will characterize forks based on the minimum subgraph in which they can appear. For this reason, we will assume that Bob has no marks on these subgraphs. This characterization is given in the following proposition.

Proposition 2.2 *There are precisely four possible forks:*

(i) The $K_{1,3}$-fork - *Alice takes the center and one arm of a* $K_{1,3}$.

(ii) The P_4-fork - *Alice takes the two center vertices of a* P_4.

(iii) The C_4-fork - *Alice takes the non-adjacent vertices of a* C_4.

(iv) The P_5-fork - *Alice takes the first, third, and fifth vertices of a* P_5.

In each case, Bob has no vertices in the respective subgraph.

Proof. In order for Alice to have a fork, she must have two vertices in each of two winning sets and Bob can have no vertices in these same sets. Assuming that Bob blocks when necessary, these two winning sets must share at least one common vertex. This common vertex must belong to Alice. For this reason, we assume that the only winning sets are $\{c, u_1, u_2\}$ and $\{c, u_3, u_4\}$.

If c is the only shared vertex, then we get the P_5-fork. In this case, Alice must take the first, third, and fifth vertices of the P_5. For the rest of the proof, we assume that the two sets share two common vertices. Without loss of generality, suppose that $u_1 = u_4$.

If c and u_1 are nonadjacent, then the two copies of P_3 are c, u_2, u_1 and c, u_3, u_1. This gives us the C_4-fork. Notice that Alice must take c and u_1 on this subgraph.

Thus we may assume that c and u_1 share an edge. If c has two neighbors in the set $\{u_2, u_3\}$, then this gives us the $K_{1,3}$-fork. If c has only one neighbor in the set

$\{u_2, u_3\}$, then this gives us the P_4-fork. In both cases, Alice must take c and u_1 on these subgraphs. ■

We are now prepared to prove our main result.

Theorem 2.3 *Alice has a winning strategy on a graph G if and only if G has one of the following as a subgraph: $K_{1,4}$, B_1, B_2, or $P'_{2\ell+1}$ for some $\ell \geq 1$ (see Figure 1). Both players have a drawing strategy on all other graphs.*

Proof. We begin by giving Alice's winning strategy on the above graphs. Suppose that G has a $K_{1,4}$ or B_1 as a subgraph. Alice takes c followed by one element from each of the sets $\{u_1, u_3\}$ and $\{u_2, u_4\}$. Since c together with one element from each of those pairs constitutes a winning set, Alice has a winning strategy.

Suppose that G has B_2 as a subgraph. For her opening move, Alice takes vertex c. Bob can take at most one element of $\{u_2, u_5\}$. Hence, Alice responds by taking the remaining element of that set. At this point, she has captured either the center of a P_4 or the center and one arm of a $K_{1,3}$. In either case, Alice can win on her next move by Proposition 2.2.

Suppose that G is the graph $P'_{2\ell+1}$, where $\ell \geq 1$. On Alice's ith turn ($i = 1, ..., \ell$), she takes vertex v_{2i-1}. Bob must respond by taking v_{2i-2} either to prevent both the P_4-fork and $K_{1,3}$-fork (on his first turn) or to block (on his remaining turns). On Alice's $(\ell + 1)$st turn, she takes vertex $v_{2\ell}$. This gives her the center and one arm of a $K_{1,3}$. Hence by Proposition 2.2, she can win on her next turn regardless of Bob's actions.

We now give Bob's drawing strategy on the remaining graphs. Note that the maximum degree of such a graph is three since Alice has a winning strategy if the graph has a $K_{1,4}$ subgraph. Clearly, both players have a drawing strategy on a graph with maximum degree one. Let G be a graph with maximum degree two. In such a graph, there are at most two paths between any two vertices u and v. We say that such a path is a *P1-path* if Alice has taken u and v and Bob has not taken any vertex along this path.

Suppose that on such a graph, Alice takes vertex v_i on her ith turn. On his first turn, Bob responds by taking any neighbor of v_1. On his ith turn ($i \geq 2$), Bob takes any neighbor of v_i that is on a P1-path. Since the maximum degree of G is two, this neighbor (if it exists) is unique. Hence, Bob will prevent both the P_4-fork and the P_5-fork. If no such neighbor exists, then Bob can take any available vertex. By adopting this strategy, Bob will take at least one vertex from every winning set of G. Hence, he has a drawing strategy.

Suppose that G is a graph with maximum degree three that has neither B_1 nor B_2 as a subgraph. These graphs are described in Proposition 2.1. For Bob's drawing strategy on the bull graph from Proposition 2.1 (i), he takes a vertex of degree three on his first move and then blocks as necessary.

For the graphs described in Proposition 2.1 (ii), let G_ℓ be the graph obtained from $P'_{2\ell}$ by adding the edges $v'_0 v''_0$ and $v'_{2\ell-1} v''_{2\ell-1}$. We now give Bob's drawing strategy on G_ℓ. Whenever Alice takes an element from one of the pairs $\{v'_0, v''_0\}$, $\{v_0, v_1\}$,...,$\{v_{2\ell-2}, v_{2\ell-1}\}$, $\{v'_{2\ell-1}, v''_{2\ell-1}\}$, Bob responds by taking the remaining element from that set. Any P_3 subgraph of G_ℓ must contain two elements from some pair. Hence, this strategy will guarantee that Bob will have at least one vertex from every winning set. Ergo, he has a drawing strategy. Since all remaining graphs are subgraphs of G_ℓ, Bob has a drawing strategy on these graphs as well. ■

3 Open Problems

We end this paper by giving a number of open problems as possible avenues for future research. Since Alice has a winning strategy on most graphs, a natural question is how to neutralize her advantage. For this reason, we propose three open problems that suggest different approaches to reducing Alice's advantage.

Problem 3.1 *The* pie rule *(also known as the swap rule or Nash's rule from Hex) is a common method for mitigating the advantage of going first [4]. If the pie rule is implemented, then after the first move is made, Bob has one of two options. If he lets the move stand, then play proceeds as normal. Otherwise, Bob "takes" that move. In which case, Alice then plays as if she were the second player. What is the set of graphs in which each player has a winning strategy when the pie rule is implemented?*

Problem 3.2 *Suppose that we allow play to continue after one player has captured a P_3 (this is known as* full play convention *in [2, 10]).* (i) *What is the set of graphs in which Alice cannot prevent Bob from capturing a P_3?* (ii) *What is the set of graphs in which Alice can prevent Bob from capturing a P_3 only at the expense of capturing her own?* (iii) *What is the set of graphs in which Alice can both capture a P_3 and prevent Bob from capturing a P_3?*

Problem 3.3 *In the (a, b)-game (see for example [11]), players alternate turns as usual. On each of Alice's turns, she places a marks. Similarly, Bob places b marks on each of his turns. For each pair (a, b), determine necessary and sufficient conditions on a graph for each player to have a winning strategy in the (a, b)-game.*

Other open problems would center around the possibility of using other subgraphs as our winning sets. For example, suppose that we were to consider a variation in which the winning sets were *induced* P_3-subgraphs. In such a variation, the game would continue upon the capture of a C_3. What is the set of graphs in which Alice has a winning strategy in this variation? In addition, if we were to generalize Connect-Four to graphs, then we would likely assume that both players were trying to capture a P_4. Likewise, if we assumed that our grid in Tic-Tac-Toe were wrapped around something akin to a torus, then our winning sets would be C_3-subgraphs. We could generalize this further by assigning each player a family of graphs (which need not be the same for both players). The first player to capture any graph in their respective family wins.

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(Received 13 Dec 2017; revised 11 May 2018)