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A note on Roman domination: changing and unchanging

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Abstract

A Roman dominating function (RD-function) on a graph $G = (V(G), E(G))$ is a labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The weight $f(V(G))$ of a RD-function f on G is the value $\Sigma_{v \in V(G)} f(v)$. The Roman domination number $\gamma_R(G)$ of G is the minimum weight of a RD-function on G. The six classes of graphs resulting from the changing or unchanging of the Roman domination number of a graph when a vertex is deleted, or an edge is deleted or added are considered. We consider relationships among the classes, which are illustrated in a Venn diagram.

1 Introduction and preliminaries

By a graph, we mean a finite, undirected graph with neither loops nor multiple edges. For basic notation and graph theory terminology not explicitly defined here, in general we follow Haynes et al. [7]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. In a graph G, for a subset $S \subseteq V(G)$ the *subgraph induced* by S is the graph $G[S]$ with vertex set S and edge set $\{xy \in E(G) \mid x, y \in S\}$. We write K_n for the *complete graph* of order n, $K_{m,n}$ for the *complete bipartite graph* with partite sets of order m and n, P_n for the *path* on n vertrices, and C_m for the *cycle* of length m. For vertices x and y in a connected graph G, the *distance* $dist(x, y)$ is the length of a shortest $x - y$ path in G. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of x is $deg(x, G) = |N_G(x)|$. The *minimum* and *maximum* degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a graph G, let $x \in X \subseteq V(G)$. A vertex $y \in V(G)$ is an X-*private neighbor* of x if $N_G[y] \cap X = \{x\}$. The set of all X-private neighbors of x is denoted by $pn_G[x, X]$. A *leaf* of a graph is a vertex of degree 1, while a *support vertex* is a vertex adjacent to a leaf. A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set.

The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a comprehensive introduction to the theory of domination in graphs, we refer the reader to Haynes et al. [7]. A *dominating set* for a graph G is a subset $D \subseteq V(G)$ of vertices such that every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the *domination number* of G and is denoted by $\gamma(G)$.

A variation of domination called Roman domination was introduced by ReVelle [11, 12]. Also see ReVelle and Rosing [13] for an integer programming formulation of the problem. The concept of Roman domination can be formulated in terms of graphs ([3]). A *Roman dominating function* (RD-*function*) on a graph G is a vertex labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. For a RD-function f, let $V_i^f = \{v \in V(G) : f(v) = i\}$ for i = 0, 1, 2. Since these 3 sets determine f, we can equivalently write $f = (V_0^f; V_1^f; V_2^f)$. The *weight* $f(V(G))$ of a RD-function f on G is the value $\Sigma_{v\in V(G)} f(v)$, which equals $|V_1^f| + 2|V_2^f|$. The *Roman domination number* $\gamma_R(G)$ of G is the minimum weight of a RD-function on G. A RD-function with minimum weight in a graph G will be referred to as a γ_R-*function* on G. If H is a subgraph of G and f a γ_R-function on G, then we denote the restriction of f on H by $f|H$.

It is often of interest to known how the value of a graph parameter μ is affected when a change is made in a graph. The addition of a set of edges, or the removal of a set of vertices/edges may increase or decrease μ, or leave μ unchanged. Thus, it is naturally to consider the following classes of graphs. We use acronyms to denote these classes (V represents vertex; E: edge; R: removal; A: addition). Let k be a positive integer.

(i) (k-VR^-_μ) $\mu(G - S) < \mu(G)$ for any set $S \subseteq V(G)$ with $|S| = k$,

(ii) (k-VR^+_μ) $\mu(G - S) > \mu(G)$ for any set $S \subseteq V(G)$ with $|S| = k$,

(iii) (k-$VR^=_\mu$) $\mu(G - S) = \mu(G)$ for any set $S \subseteq V(G)$ with $|S| = k$,

(iv) (k-$VR^{\neq}_\mu$) $\mu(G - S) \neq \mu(G)$ for any set $S \subseteq V(G)$ with $|S| = k$

(v) (k-ER^-_μ) $\mu(G - R) < \mu(G)$ for any set $R \subseteq E(G)$ with $|R| = k$,

(vi) (k-ER^+_μ) $\mu(G - R) > \mu(G)$ for any set $R \subseteq E(G)$ with $|R| = k$,

(vii) (k-$ER^=_\mu$) $\mu(G - R) = \mu(G)$ for any set $R \subseteq E(G)$ with $|R| = k$,

(viii) (k-$ER^{\neq}_\mu$) $\mu(G - R) \neq \mu(G)$ for any set $R \subseteq E(G)$ with $|R| = k$,

(ix) (k-EA^-_μ) $\mu(G + U) < \mu(G)$ for any set $U \subseteq E(\overline{G})$ with $|U| = k$,

(x) (k-EA^+_μ) $\mu(G + U) > \mu(G)$ for any set $U \subseteq E(\overline{G})$ with $|U| = k$,

(xi) (k-$EA^=_\mu$) $\mu(G + U) = \mu(G)$ for any set $U \subseteq E(\overline{G})$ with $|U| = k$,

(xii) (k-$EA^{\neq}_\mu$) $\mu(G + U) \neq \mu(G)$ for any set $U \subseteq E(\overline{G})$ with $|U| = k$.

Two mathematical problems arise immediately: 1) to find a nontrivial characterization of every one of the above classes, and 2) to establish relationships among these twelve classes. Here we concentrate on the second problem in the case when $\mu \equiv \gamma_R$ and $k = 1$.

We end this section with some known results which will be useful in proving our main results.

Observation A ([3]) *Let $f = (V_0^f; V_1^f; V_2^f)$ be any γ_R-function on a graph G. Then $\Delta(G[V_1^f]) \leq 1$ and no edge of G joins V_1^f and V_2^f. If $|V_1^f|$ is a minimum then V_1^f is independent and if in addition G is isolate-free then $V_0^f \cup V_2^f$ is a vertex cover.*

In most cases, Observation A will be used in the sequel without specific reference.

Theorem B ([10]) *Let v be a vertex of a graph G. Then $\gamma_R(G-v) < \gamma_R(G)$ if and only if there is a γ_R-function f on G such that $v \in V_1^f$. If $\gamma_R(G-v) < \gamma_R(G)$ then $\gamma_R(G-v) = \gamma_R(G) - 1$. If $\gamma_R(G-v) > \gamma_R(G)$ then for every γ_R-function f on G, $f(v) = 2$.*

According to the effects of vertex removal on the Roman domination number of a graph G, let

- $V_R^+(G) = \{v \in V(G) \mid \gamma_R(G-v) > \gamma_R(G)\}$,
- $V_R^-(G) = \{v \in V(G) \mid \gamma_R(G-v) < \gamma_R(G)\}$,
- $V_R^=(G) = \{v \in V(G) \mid \gamma_R(G-v) = \gamma_R(G)\}$.

Clearly $V_R^-(G), V_R^=(G)$ and $V_R^+(G)$ are pairwise disjoint, and their union is $V(G)$.

Theorem C *Let G be a graph.*

(i) ([6]) *Let x and y be non-adjacent vertices of G. Then $\gamma_R(G) \geq \gamma_R(G+xy) \geq \gamma_R(G) - 1$. Moreover, $\gamma_R(G+xy) = \gamma_R(G) - 1$ if and only if there is a γ_R-function f on G such that $\{f(x), f(y)\} = \{1, 2\}$.*

(ii) ([10]) *If e is an edge of G, then $\gamma_R(G) \leq \gamma_R(G-e) \leq \gamma_R(G) + 1$.*

2 Six classes

We will write $\mathcal{R}_{CVR}$, $\mathcal{R}_{UVR}$, $\mathcal{R}_{CER}$, $\mathcal{R}_{UER}$, $\mathcal{R}_{CEA}$, and $\mathcal{R}_{UEA}$ instead of 1-$VR^-_{\gamma_R}$, 1-$VR^=_{\gamma_R}$, 1-$ER^+_{\gamma_R}$, 1-$ER^=_{\gamma_R}$, 1-$EA^-_{\gamma_R}$, and 1-$EA^=_{\gamma_R}$, respectively. The first four classes of graphs were introduced in [10] by Jafari Rad and Volkmann. On the other hand, the graphs in $\mathcal{R}_{CEA}$ and $\mathcal{R}_{UEA}$ were investigated by Hansberg et al. [6], and Chellali and Jafari Rad [9], respectively. Let us note that Theorems B and C imply that (a) the class 1-$VR^+_{\gamma_R}$ is empty, (b) the class 1-$EA^+_{\gamma_R}$ consists of all complete graphs, (c) the class 1-$ER^-_{\gamma_R}$ consists of all edgeless graphs, (d) 1-$VR^{\neq}_{\gamma_R} \equiv \mathcal{R}_{CVR}$, (e) 1-$ER^{\neq}_{\gamma_R} \equiv \mathcal{R}_{CER}$, and (f) 1-$EA^{\neq}_{\gamma_R} \equiv \mathcal{R}_{CEA}$. That is why we concentrate, in what follows, on the establishing relationships among the following six classes: $\mathcal{R}_{CVR}$, $\mathcal{R}_{UVR}$, $\mathcal{R}_{CER}$, $\mathcal{R}_{UER}$, $\mathcal{R}_{CEA}$, and $\mathcal{R}_{UEA}$. For further results on these classes see [2, 4, 5, 14]. Our main goal is to show that these six classes are related as in the Venn diagram of Fig. 1.

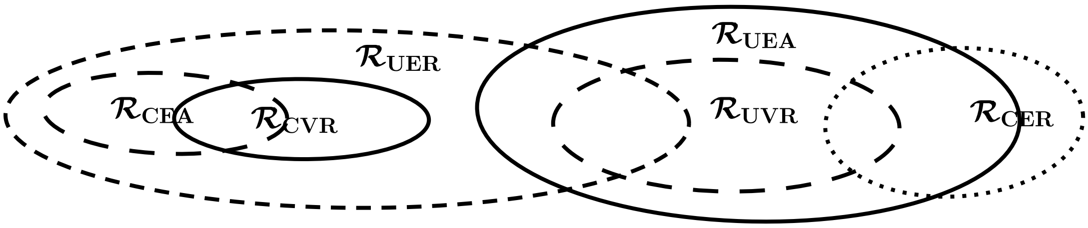

Figure 1: Classes of changing and unchanging graphs.

Theorem 1 *Let a graph G be in $\mathcal{R}_{CEA}$. Then all the following hold.*

(i) ([2]) *$V(G) = V^-(G) \cup V^=(G)$ and either $V^=(G)$ is empty or $G[V^=(G)]$ is a complete graph.*

(ii) A vertex $x \in V^=(G)$ if and only if there are γ_R-functions f_x and g_x on G with $\{f_x(x), g_x(x)\} = \{0, 2\}$.

(iii) If $V^=(G)$ is not empty and $G[V^=(G)]$ is not a connected component of G, then each vertex in $V^=(G)$ has a neighbor in $V^-(G)$.

(iv) G is in $\mathcal{R}_{UER}$.

Proof. For complete graphs the results are obvious. So, let G be noncomplete.

(ii) By Theorem B, $V^=(G) = A \cup B \cup C$, where $A = \{x \in V(G) \mid f(x) = 0 \text{ for each } \gamma_R - \text{function } f \text{ on } G\}$, $B = \{x \in V(G) \mid \gamma_R(G - x) = \gamma_R(G) \text{ and } f(x)$ $\coprod = 2$ for each γ_R − function f on $G\}$, and $C = \{x \in V(G) \mid$ there are γ_R − functions f_x and g_x with $\{f_x(x), g_x(x)\} = \{0, 2\}\}$.

Theorem C implies that A is empty. Suppose B is not empty, and $u \in B$. By (i) we have $B \subseteq V^=(G) \subseteq N[u]$. Now Observation A and Theorem B lead to $N[u] = V^=(G)$ and $B \subsetneq V^=(G)$. Since $A = \emptyset$, there is $v \in C$. But then there exists a γ_R-function f on G with $f(v) = 2$. Define a RD-function f' on G as follows: $f'(u) = 0$ and $f'(x) = f(x)$ for all $x \in V(G - x)$. Since f' has a weight less than $\gamma_R(G)$, we arrive to a contradiction. Thus $V^=(G) = C$, as required.

(iii) Assume to the contrary, that $N[v] = V^=(G)$ for some $v \in V^=(G)$. Clearly, there are $u \in V^-(G)$ and $w \in V^=(G)$ which are adjacent. Since $uv \notin E(G)$ and G is in $\mathcal{R}_{CEA}$, there is a γ_R-function f'' on G with $f''(u) = 1$ and $f''(v) = 2$. But then $f''(w) = 0$ and $f''' = ((V_0^{f''}(G) - \{w\}) \cup \{u, v\}; V_1^{f''} - \{u\}; (V_2^{f''} - \{v\}) \cup \{w\})$ is a RD-function on G with weight less than $\gamma_R(G)$, a contradiction.

(iv) Assume $G \in \mathcal{R}_{CEA} - \mathcal{R}_{UER}$. Then there is an edge $x_1x_2 \in E(G)$ with $\gamma_R(G_{12}) > \gamma_R(G)$, where $G_{12} = G - x_1x_2$. Now by Theorem C, applied to G_{12} and x_1x_2, there is a γ_R-function f on G_{12} with $\{f(x_1), f(x_2)\} = \{1, 2\}$, say without loss of generality, $f(x_1) = 2$. Note also that $f_{12} = (V_0^f(G) \cup \{x_2\}; V_1^f(G) - \{x_2\}; V_2^f(G))$ is a γ_R-function on G. Since G is in $\mathcal{R}_{CEA}$, we already know that $V(G) = V^=(G) \cup V^-(G)$.

If there is a γ_R-function f' on G with $f'(x_i) = 1$, then f' is a RD-function on G_{12}, a contradiction. Thus, $x_1, x_2 \in V^=(G) = C$.

Suppose that $x_1 \in V^+(G_{12}) \cup V^=(G_{12})$. Then $\gamma_R(G-x_1) = \gamma_R(G_{12}-x_1) \geq \gamma_R(G_{12}) > \gamma_R(G)$. This immediately implies $x_1 \in V^+(G)$, a contradiction.

So, in what follows let $x_1 \in V^-(G_{12})$. If $G[V^=(G)]$ is a component of G, then $\gamma_R(G_{12}) = \gamma_R(G)$, a contradiction. Hence each vertex in $V^=(G)$ is adjacent to a vertex in $V^-(G)$ (by (iii)). Assume first that $y \in V^-(G)$ is adjacent to both x_1 and x_2. Then there is a γ_R-function g on G with $g(y) = 1$. This implies $g(x_1) = g(x_2) = 0$ (recall that $x_1, x_2 \in V^=(G)$). But then g is a RD-function on G_{12} with weight less than $\gamma_R(G_{12})$, a contradiction. Thus, all common neighbors of x_1 and x_2 are in $V^=(G)$. Suppose $x_3 \in V^=(G)$ and $u \in N(x_1) \cap V^-(G)$. If $ux_3 \notin E(G)$ then there is a γ_R-function f_1 on G with $f_1(x_3) = 2$ and $f_1(u) = 1$. Since f_1 is a RD-function on G_{12}, we arrive to a contradiction. Therefore $N[x_1] = N[x_3]$, which implies $f(x_3) = 0$. But then $f_2 = (V_0^f - \{x_3\} \cup \{x_1, x_2\}; V_1^f - \{x_2\}; V_2^f - \{x_1\} \cup \{x_3\})$ is a RD-function on G_{12} of weight less than $\gamma_R(G_{12})$, a contradiction.

Thus, $V^=(G) = \{x_1, x_2\}$ and $N(x_1) \cap N(x_2) = \emptyset$. Let $N(x_1) - \{x_2\} = \{y_1, y_2, .., y_r\}$ and $N(x_2) - \{x_1\} = \{z_1, z_2, .., z_s\}$. If there are nonadjacent y_i and y_j, then there is a γ_R-function g on G with $\{g(y_i), g(y_j)\} = \{1, 2\}$. Hence $g(x_1) = 0$ which implies that g is a RD-function on G_{12}, a contradiction. Thus $N[x_i] - \{x_j\}$ induces a complete graph for $\{i, j\} = \{1, 2\}$.

Assume now that $y_i z_j \notin E(G)$. Then, without loss of generality, there is a γ_R-function l on G with $l(y_i) = 2$ and $l(z_j) = 1$. Since $x_2 \in V^=(G)$, $l(x_2) = 0$. If $l(x_1) \neq 2$, then l is a RD-function on G_{12}, a contradiction. Thus $l(x_1) = 2$. But then $l_1 = (V_0^l(G) - \{x_2\}; V_1^l(G) \cup \{x_1, x_2\}; V_2^l(G) - \{x_1\})$ is a γ_R-function on G and $l_1(x_1) = l_1(x_2) = l_1(y_j) = 1$, a contradiction. So, $(N(x_1) \cup N(x_2)) - \{x_1, x_2\}$ induce a complete graph. Now, let h be any γ_R-function on G with $h(x_1) = 2$ and $h(z_1) = 1$. But then $h' = (V_0^h(G), (V_1^h(G) - \{z_1\}) \cup \{x_1\}; (V_2^h(G) - \{x_1\} \cup \{z_1\})$ is a γ_R-function on G with $h'(x_1) = 1$, a contradiction. □

Theorem 2 *For an edge $e = uv$ of a graph G is fulfilled $\gamma_R(G - e) = \gamma_R(G)$ if and only if there is a γ_R-function f_e on G such that at least one of the following holds:*

(i) $f_e(u) = f_e(v)$,

(ii) *at least one of u and v is in $V_1^{f_e}$,*

(iii) $f_e(u) = 2, f_e(v) = 0$ *and* $v \notin pn[u, V_2^{f_e}]$,

(iv) $f_e(u) = 0, f_e(v) = 2$ *and* $u \notin pn[v, V_2^{f_e}]$.

Proof. $\Leftarrow$: Let f_e be a γ_R-function on G and at least one of (i)–(iv) is true. Then obviously f_e is a RD-function on $G - e$, which implies $\gamma_R(G - e) \leq \gamma_R(G)$. The result now follows by Theorem C(ii).

$\Rightarrow$: Assume $\gamma_R(G - e) = \gamma_R(G)$. Hence each γ_R-function on $G - e$ is a γ_R-function on G. Suppose that for each γ_R-function f_e on G none of (i)–(iv) is valid and let

g be a γ_R-function on $G - e$. Then g is a γ_R-function on G and at least one of $(g(u) = 2, g(v) = 0$ and $v \in pn[u, V_2^g])$ and $(g(u) = 0, g(v) = 2$ and $u \in pn[v, V_2^g])$ is fulfilled. But clearly this is impossible. Thus, for each γ_R-function on G at least one of (i)–(iv) is valid, as required. □

Corollary 3 *Let G be a graph with edges. Then for each edge e incident to a vertex in $V^-(G)$, $\gamma_R(G - e) = \gamma_R(G)$. If $V^-(G)$ contains a vertex cover of G, then G is in $\mathcal{R}_{UER}$. In particular, if G is in $\mathcal{R}_{CVR}$, then G is in $\mathcal{R}_{UER}$.*

Proof. Let $x \in V^-(G)$. By Theorems B and 2, for each edge $e \in E(G)$ incident to x, $\gamma_R(G - e) = \gamma_R(G)$. Hence if $V^-(G)$ has as a subset some vertex cover of G, then G is in $\mathcal{R}_{UER}$. From this it immediately follows $\mathcal{R}_{CVR} \subseteq \mathcal{R}_{UER}$. □

Lemma D *[1] Let G be a graph of order $n \geq 3$. A graph G is in $\mathcal{R}_{UEA}$ if and only if for every γ_R-function $f = (V_0, V_1, V_2)$, $V_1 = \emptyset$.*

In order to establish a Venn diagram representing the classes $\mathcal{R}_{CVR}$, $\mathcal{R}_{UVR}$, $\mathcal{R}_{CER}$, $\mathcal{R}_{UER}$, $\mathcal{R}_{CEA}$, and $\mathcal{R}_{UEA}$, we do not consider the cases that are vacuously true. For example (a) the complete graphs are in both $\mathcal{R}_{CEA}$ and $\mathcal{R}_{UEA}$, and (b) the edgeless graphs are in both $\mathcal{R}_{CER}$ and $\mathcal{R}_{UER}$. Therefore we exclude edgeless graphs and complete graphs.

To continue, we need to relabel the Venn diagram of Fig. 1 in 11 regions $R_1 - R_{11}$ as shown in Fig. 2.

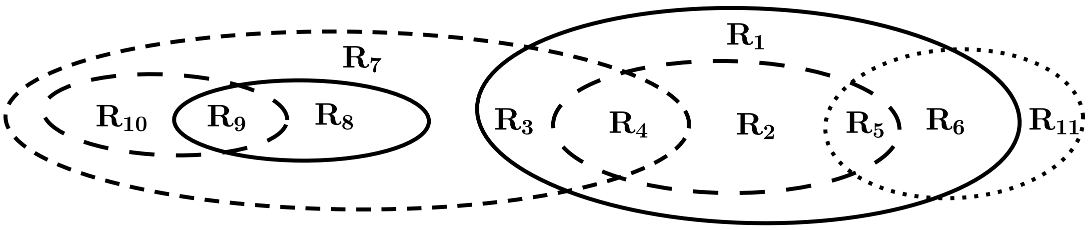

Figure 2: Regions of Venn diagram: general case

Theorem 4 *Classes $\mathcal{R}_{CVR}$, $\mathcal{R}_{CEA}$, $\mathcal{R}_{CER}$, $\mathcal{R}_{UVR}$, $\mathcal{R}_{UER}$ and $\mathcal{R}_{UEA}$ are related as shown in the Venn diagram of Fig. 1.*

Proof. By Theorem 1 and Corollary 3 we have $\mathcal{R}_{CEA} \cup \mathcal{R}_{CVR} \subseteq \mathcal{R}_{UER}$. It is obvious that all $\mathcal{R}_{UER} \cap \mathcal{R}_{CER}$, $\mathcal{R}_{UVR} \cap \mathcal{R}_{CVR}$, and $\mathcal{R}_{UEA} \cap \mathcal{R}_{CEA}$ are empty. If a graph G is in $\mathcal{R}_{UVR}$, then clearly $V(G) = V^=(G)$. Lemma D now implies $\mathcal{R}_{UVR} \subseteq \mathcal{R}_{UEA}$. If $G \in \mathcal{R}_{CVR}$ then $V^-(G) \neq \emptyset$ and by Lemma D, $\mathcal{R}_{CVR}$ and $\mathcal{R}_{UEA}$ are disjoint.

The next obvious claim shows that none of regions $R_1 - R_{11}$ is empty. The *double star* $S_{m,n}$, where $m, n \geq 2$, is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with an edge joining their centers.

Claim 4.1

(i) Any double star $S_{p,q}$ with $p, q \geq 3$, is in R_1.

(ii) The graph G obtained from $S_{2,2}$ by subdividing once the edge joining the support vertises of $S_{2,2}$, is in R_2.

(iii) The graph G obtained from K_4 by adding a new vertex v, joining it to three vertices of the K_4, and then subdividing once each of the edges incident to v, is in R_3.

(iv) C_6 is in R_4.

(v) $K_{1,2}$ is in R_5.

(vi) $K_{1,n}$, $n \geq 3$ is in R_6.

(vii) The double star $S_{2,2}$ is in R_7.

(viii) C_7 is in R_8.

(ix) C_4 is in R_9.

(x) The graph obtained from 2 disjoint copies of P_5 by joining their central vertices is in R_{10}.

(xi) $K_1 \cup K_{1,2}$ is in R_{11}.

□

Lemma E *[10] Let a graph G have at least one edge. Then G is in $\mathcal{R}_{CER}$ if and only if $\Delta(G) \geq 2$ and G is a forest in which each component is an isolated vertex or a star of order at least* 3.

Remark 5 *Using Lemma E it is easy to see that the following assertions hold.*

(i) A graph G is in R_5 if and only if $G = nK_{1,2}$, $n \geq 1$.

(ii) A graph G is in R_6 if and only if each component of G is a star of order at least 4.

(iii) A graph G is in R_{11} if and only if $\delta(G) = 0$ and each component of G is an isolated vertex or a star of order at least 3.

By Theorem 4, Claim 4.1 and Remark 5 we immediately obtain:

Corollary 6 *For connected graphs:*
(a) the subset R_{11} is empty, and (b) all $R_1, R_2, \ldots, R_{10}$ are nonempty.

Now our aim is to determine where trees of order at least 3 fit into the subsets of the Venn diagram.

Corollary 7 *For trees of order $n \geq 3$, (a) all regions R_3, R_4, R_8, R_9 and R_{11} of the Venn diagram (see Fig. 2) are empty, and (b) all regions R_1, R_2, R_5, R_6, R_7 and R_{10} are nonempty.*

Proof. Let T be a tree. By Corollary 6, R_{11} is empty. Clearly $K_{1,2}$ is in R_5 and $K_{1,r}$, $r \geq 2$, is in R_6. Since a tree T is in $\mathcal{R}_{CVR}$ if and only if $T = K_2$ (see [6]), R_8 and R_9 are empty. Assume T is in $\mathcal{R}_{UEA} \cap \mathcal{R}_{UER}$. By Lemma D, $V^-(T)$ is empty. Let x be a leaf of T and $\{y\} = N(x)$. As T is in $\mathcal{R}_{UER}$, $\gamma_R(T) = \gamma_R(T - xy) = \gamma_R(T - x) + 1$, a contradiction. Thus both R_3 and R_4 are empty.

The rest follows immediately by Theorem 4. □

Thus, we have shown that for trees of order $n \geq 3$, the regions of the Venn diagram can be reduced to the six shown in Fig. 3.

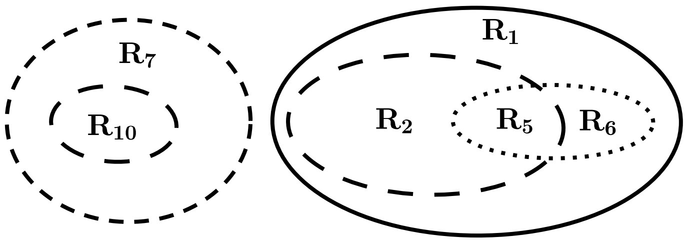

Figure 3: Regions of Venn diagram: trees

A constructive characterization of the trees belonging to $\mathcal{R}_{UEA}$ is given by Chellali and Jafari Rad [1], and for the trees belonging to $\mathcal{R}_{UVR}$, by the present author in [14]. By Remark 5, all trees in $\mathcal{R}_{CER}$ are $K_{1,r}$, $r \geq 2$; hence $K_{1,2}$ is the unique element of R_5, and R_6 consists of all stars $K_{1,r}$, $r \geq 3$.

Let U_i be the graph obtained by disjoint copies of P_5 and P_{3+i} by joining the central vertex of P_5 with a central vertex of P_{3+i}, $i = 1, 2$. Hansberg et al. [6] show that U_1 and U_2 are the only trees which are in $\mathcal{R}_{CEA}$ (i.e. R_{10}).

So, the following problem naturally arises.

Problem 1 *Find a constructive characterization for trees in* $\mathcal{R}_{UER}$.

We close with:

Problem 2 *Let* μ *be a domination-related parameter. 1) Give a characterization of every of the twelve classes of graphs stated in the introduction. 2) Establish relationships among these twelve classes.*

This problem has been well-studied in the case when $\mu = \gamma$. See the excellent article [8] of Haynes and Henning and the references therein.

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