

Unimodality of the independence polynomials of non-regular caterpillars

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Abstract

The *independence polynomial* $I(G, x)$ of a graph G is the polynomial in variable x in which the coefficient a_n on x^n gives the number of independent subsets $S \subseteq V(G)$ of vertices of G such that $|S| = n$. $I(G, x)$ is *unimodal* if there is an index μ such that $a_0 \leq a_1 \leq \dots \leq a_{\mu-1} \leq a_{\mu} \geq a_{\mu+1} \geq \dots \geq a_{d-1} \geq a_d$. While the independence polynomials of many families of graphs with highly regular structure are known to be unimodal, little is known about less regularly-structured graphs. We analyze the independence polynomials of a large infinite family of trees without regular structure and show that these polynomials are unimodal through a combinatorial analysis of the polynomials' coefficients.

1 Introduction

Let $G = (V, E)$ be a graph with vertices V and edges E . An *independent set* in G is a subset of V in which no two distinct vertices are adjacent. (In other words, the shortest path between each pair of vertices in an independent set is at least length two.) The *independence number* of G , denoted $\alpha(G)$, is the cardinality of a largest independent set in V . The *independence polynomial*, $I(G; x)$, of G is defined by

$$I(G; x) = \sum_{n=0}^{\alpha(G)} a_n x^n,$$

where a_n is the number of independent subsets of V of cardinality n .

We say that a sequence $\{a_0, a_1, \dots, a_d\}$ is *unimodal* if there is some index μ such that $a_0 \leq a_1 \leq \dots \leq a_{\mu-1} \leq a_{\mu} \geq a_{\mu+1} \geq \dots \geq a_{d-1} \geq a_d$; in this case we say that μ is a *mode* of the sequence. A polynomial is said to be unimodal if its coefficient sequence is unimodal. Throughout this article, we will abuse the terminology and say that a graph is unimodal if its independence polynomial is unimodal. We will

further say that a sequence $\{a_0, a_1, \dots, a_d\}$ (or, analogously, polynomial or graph) is *strictly* unimodal if it has a unique mode μ and if

$$a_0 < a_1 < \dots < a_{\mu-1} < a_\mu > a_{\mu+1} > \dots > a_{d-1} > a_d.$$

The unimodality of various families of graphs has been the focus of a large amount of study. The survey [8] provides an overview of a number of early results concerning unimodality and related properties (*e.g.*, symmetry and logarithmic concavity). More recent studies include [2–7, 9–15]. Despite the considerable attention paid to unimodality, little is known about the unimodality of any but the most regular families of graphs. In particular, little progress has been made in answering the following simple question, first posed by Alavi, Malde, Schwenk, and Erdős in [1]: is the independence polynomial of an arbitrary tree unimodal?

A graph T is called a *tree* if it is connected and acyclic. Further, a tree T is called a *caterpillar* if the collection of all of T 's vertices of degree at least 2 forms a path P , which we call the *spine* of the caterpillar. Several authors have investigated the independence polynomials of “regular” caterpillars, in which the subtree of T pendant at each vertex of T 's spine is identical. Beginning with [15], Zhu proves that caterpillars in which every spine vertex has two pendant edges is unimodal. In fact, several studies of regular caterpillars encompass more general path-like graphs; see, for example, [2] and [4]. More recently, Galvin and Hilyard [7] investigate the behavior of $I(T, x)$ for a family of “semi-regular” caterpillar-like graphs, in which the subtrees pendant at the vertices in the graph's spine are not all identical to one another but rather alternate in a regular pattern.

Here we consider much more general caterpillars. Let $\vec{m} = (m_1, m_2, \dots)$ be a sequence of natural numbers and let $T(\vec{m}, n)$ be the caterpillar with n vertices on its spine, the k th of which has m_k pendant edges.

Theorem 1.1 *Let $T(\vec{m}, n)$ be defined as above for $\vec{m} = (m_1, m_2, \dots)$ such that*

1. $m_k \leq m_{k+1}$ for all $1 \leq k < n$,
2. $3 \leq m_1 < m_2$ and $m_3 < m_4$, and
3. $2(m_1 + m_3 + \dots + m_k) < 3(m_2 + m_4 + \dots + m_{k-1})$ for k odd and $2(m_2 + m_4 + \dots + m_k) < 3(m_1 + m_3 + \dots + m_{k-1})$ for k even.

Then $I(T(\vec{m}, n), x)$ is unimodal with mode $\mu_n \in \{\lfloor \frac{d_n}{2} \rfloor, \lceil \frac{d_n}{2} \rceil\}$, where

$$d_n = \deg(I(T(\vec{m}, n))) = \sum_{k=1}^n m_k.$$

Condition (3) is a technical one that follows if m_k does not grow too quickly. For instance, when \vec{m} is non-decreasing, Condition (3) holds whenever $m_k \leq m_1 + m_3 + \dots + m_{k-1}$ when k is even and $m_k \leq m_2 + m_4 + \dots + m_{k-1}$ when k is odd.

In the following section, we establish several important lemmas and in Section 3 we prove Theorem 1.1.

2 The relationship between $q(x)$ and $(1+x)^t q(x)$

Our proofs depend on careful analysis of products of the form $(1+x)^t q(x)$ where q is a polynomial whose coefficients are well understood. We will say that the strictly unimodal polynomial $q(x) = \sum_{i=0}^d b_i x^i$ with unique mode μ is *left-dominant* (abbreviated LD) if

$$b_\mu > b_{\mu-1} > b_{\mu+1} > \cdots > b_1 > b_{2\mu-1} > b_0 \geq b_{2\mu}$$

when $\mu \leq \frac{d}{2}$ and

$$b_\mu > b_{\mu-1} > b_{\mu+1} > \cdots > b_{2\mu-d+1} > b_{d-1} > b_{2\mu-d} \geq b_d$$

when $\mu > \frac{d}{2}$. Similarly, we will say that q is *right-dominant* (abbreviated RD) if

$$b_\mu > b_{\mu+1} > b_{\mu-1} > \cdots > b_{2\mu-1} > b_1 > b_{2\mu} \geq b_0$$

when $\mu \leq \frac{d}{2}$ and

$$b_\mu > b_{\mu+1} > b_{\mu-1} > \cdots > b_{d-1} > b_{2\mu-d+1} > b_d \geq b_{2\mu-d}$$

when $\mu > \frac{d}{2}$. We will say that q is *weakly LD* (or *weakly RD*) if the respective ordering permits equality to hold between terms strictly less than b_μ , rather than requiring strict inequality. Finally, assuming that q is both strictly unimodal and either LD or RD, we will say that q is *balanced* if $\mu = \lceil \frac{d}{2} \rceil$ when q is LD and $\mu = \lfloor \frac{d}{2} \rfloor$ when q is RD.

We now examine how strict unimodality and left- and right-dominance are affected when multiplying a polynomial q by powers of $1+x$:

Lemma 2.1 *Suppose that $q(x) = \sum_{i=0}^d b_i x^i$ is a balanced strictly unimodal polynomial with mode μ . Let $t \in \mathbb{N}$ such that $t \leq \mu$, and suppose $(1+x)^t q(x) = \sum_{i=0}^{d+t} \beta_i x^i$.*

1. *If t is even and q is weakly LD (respectively, weakly RD), then $(1+x)^t q(x)$ is a balanced weakly LD (respectively, weakly RD) strictly unimodal polynomial with unique mode $\mu + \frac{t}{2}$.*
2. *If t is odd and q is weakly LD (respectively, weakly RD), then $(1+x)^t q(x)$ is a balanced weakly RD (respectively, weakly LD) strictly unimodal polynomial with unique mode $\mu + \frac{t-1}{2}$ (respectively, $\mu + \frac{t+1}{2}$).*

Moreover, if q is LD or RD (and not merely weakly LD or weakly RD), then $(1+x)^t q(x)$ is LD or RD, accordingly as above.

Proof: Let us assume that t is even and q is weakly LD; the remaining three cases are proven analogously.

Since

$$\beta_k = \sum_{i=0}^k \binom{t}{i} b_{k-i}$$

when $0 \leq k \leq t$ and

$$\beta_{d+k} = \sum_{i=k}^t \binom{t}{i} b_{d+k-i}$$

when $d \leq k \leq d + t$, the strict unimodality of q shows that $\beta_0 \leq \beta_1 \leq \dots \leq \beta_t$ and $\beta_d \geq \beta_{d+1} \geq \dots \geq \beta_{d+t}$. When $t \leq k \leq d$,

$$\beta_k = \sum_{i=0}^t \binom{t}{i} b_{k-i},$$

and straightforward computation thus yields

$$\beta_{\mu+t/2} - \beta_{\mu+t/2-1} = \left[\sum_{i=0}^{t/2-1} \left(\binom{t}{i+1} - \binom{t}{i} \right) (b_{\mu+t/2-i-1} - b_{\mu-t/2+i}) \right] + b_{\mu+t/2} - b_{\mu-t/2-1}.$$

Because q is LD, every parenthesized term in this expression is non-negative. Moreover, the term corresponding to $i = \frac{t}{2} - 1$ involves the strictly positive difference $b_{\mu} - b_{\mu-1}$. Therefore the sum must be strictly positive, and $\beta_{\mu+t/2-1} < \beta_{\mu+t/2}$. Similarly,

$$\beta_{\mu+t/2} - \beta_{\mu+t/2+1} = \left[\sum_{i=0}^{t/2-1} \left(\binom{t}{i+1} - \binom{t}{i} \right) (b_{\mu-t/2+i+1} - b_{\mu+t/2-i}) \right] + b_{\mu-t/2} - b_{\mu+t/2+1},$$

in which, again, every parenthesized term is non-negative and the term corresponding to $i = \frac{t}{2} - 1$ is strictly positive, showing that $\beta_{\mu+t/2+1} < \beta_{\mu+t/2}$ and establishing that $\mu + \frac{t}{2}$ is the unique mode of $(1+x)^t q(x)$.

Completely analogous computations show that $\beta_k < \beta_{k+1}$ whenever $k \in [t, \mu+t/2-2]$ and $\beta_k > \beta_{k+1}$ whenever $k \in [\mu+t/2+1, d-1]$. Together with the above inequalities, these inequalities establish the strict unimodality of $(1+x)^t q(x)$.

To establish weak left dominance, we must consider the differences $\beta_{\mu+t/2-s} - \beta_{\mu+t/2+s}$ for $s \leq \mu + \frac{t}{2}$. Let us first assume that $\frac{t}{2} \leq s \leq \mu - \frac{t}{2}$. Expanding in a similar manner as above, we obtain

$$\beta_{\mu+t/2-s} - \beta_{\mu+t/2+s} = \sum_{i=0}^t \binom{t}{i} (b_{\mu-(s+t/2-i)} - b_{\mu+(s+t/2-i)}).$$

Since q is weakly LD, every parenthesized term in this expression is non-negative, establishing $\beta_{\mu+t/2+s} \leq \beta_{\mu+t/2-s}$. If $s < t/2$, a different arrangement of the terms in the difference gives

$$\begin{aligned} \beta_{\mu+t/2-s} - \beta_{\mu+t/2+s} &= \sum_{i=s}^{t/2-1} \left(\binom{t}{i+s} - \binom{t}{i-s} \right) (b_{\mu-(t/2-i)} - b_{\mu+(t/2-i)}) \\ &\quad + \sum_{i=0}^{2s-1} \binom{t}{i} (b_{\mu-(t/2+s+i)} - b_{\mu+(t/2+s+i)}), \end{aligned}$$

where once more every parenthesized term is non-negative, establishing $\beta_{\mu+t/2+s} \leq \beta_{\mu+t/2-s}$ in this case as well. Finally, consider s such that $\mu - \frac{t}{2} + 1 \leq s \leq \mu + \frac{t}{2}$. Because $(1+x)^t q(x)$ is now known to be balanced, establishing weak left dominance for these values s is equivalent to showing $\beta_{d+t-j} \leq \beta_j$ for $0 \leq j \leq t-1$. However,

$$\beta_j - \beta_{d+t-j} = \sum_{i=0}^j \binom{t}{i} (b_{j-i} - b_{d-(j-i)}),$$

in which, yet again, every parenthesized term is non-negative because q is itself balanced and weakly LD.

Very similar arguments show that $\beta_{\mu+t/2-(s+1)} \leq \beta_{\mu+t/2+s}$ for all s . Together these inequalities show that $(1+x)^t q(x)$ is weakly LD, as desired. Moreover, we note that were q to be LD and not merely weakly LD, all of the above expressions would involve strictly positive terms and not merely non-negative ones, showing that $(1+x)^t q(x)$ would be LD as well.

As mentioned above, the proofs in case q is (weakly) RD, or in which t is odd, are analogous. □

We note that the proof above can easily be modified to show that the lemma remains true when we replace $(1+x)^t$ with any even-degree polynomial $p(x) = \sum_{i=0}^t a_i x^i$ that is both unimodal and *symmetric* (for which $a_j = a_{t-j}$ for any j , $0 \leq j \leq t$). Moreover, we may also show that if q itself is symmetric and unimodal, then the polynomial $(1+x)^t q(x)$ is likewise symmetric and unimodal.

Similar techniques yield estimates for the differences $\beta_{k+1} - \beta_k$ and $\beta_k - \beta_{k+1}$ in terms of the differences $b_j - b_{j+1}$:

Lemma 2.2 *Suppose that $q(x) = \sum_{i=0}^d b_i x^i$ is a balanced strictly unimodal polynomial with mode μ , and further that q is either weakly LD or weakly RD. Let $t \in \mathbb{N}$ be such that $t \leq \mu$, and let $(1+x)^t q(x) = \sum_{i=0}^{d+t} \beta_i x^i$ and $\nu = \mu((1+x)^t q(x)) \in \{\mu + \lfloor \frac{t}{2} \rfloor, \mu + \lceil \frac{t}{2} \rceil\}$.*

1. Let $k \in [\mu + 1, \nu - 1]$. Then $\beta_{k+1} - \beta_k \geq \left(\binom{t}{k-j+1} - \binom{t}{k-j} \right) (b_j - b_{j+1})$ for $j \in [k - \lfloor \frac{t}{2} \rfloor + 1, k]$ and $\beta_{k+1} - \beta_k \geq b_{k+1} - b_{k+2}$.
2. Let $k \in [\nu, d - 1]$. Then $\beta_k - \beta_{k+1} \geq \left(\binom{t}{k-j} - \binom{t}{k-j-1} \right) (b_j - b_{j+1})$ for $j \in [k - \lfloor \frac{t}{2} \rfloor, k - 1]$ and $\beta_k - \beta_{k+1} \geq b_k - b_{k+1}$.

Proof: Let us consider Case (1) when q is weakly LD. Then

$$\beta_{k+1} - \beta_k = \left[\sum_{i=0}^{t/2-1} \left(\binom{t}{i+1} - \binom{t}{i} \right) (b_{k-i} - b_{k-t+i+1}) \right] + b_{k+1} - b_{k-t}.$$

Isolating the i th term in the sum gives

$$\beta_{k+1} - \beta_k \geq \left(\binom{t}{i+1} - \binom{t}{i} \right) (b_{k-i} - b_{k-t+i+1}) \geq \left(\binom{t}{i+1} - \binom{t}{i} \right) (b_{k-i} - b_{k-i+1}),$$

and letting $j = k - i$ gives the desired inequality. The inequality $\beta_{k+1} - \beta_k \geq b_{k+1} - b_{k+2}$ follows from isolating the lone term outside of the sum. Proving Case (1) for q weakly RD is analogous.

Case (2) is proven in similar fashion, using the fact that

$$\beta_k - \beta_{k+1} = \left[\sum_{i=0}^{t/2-1} \left(\binom{t}{i+1} - \binom{t}{i} \right) (b_{k-t+i+1} - b_{k-i}) \right] + b_{k-t} - b_{k+1}.$$

□

We note that in general, the bounds given in Lemma 2.2 will offer very coarse estimates, given both the number of terms ignored in the proof above and the enormity of the coefficients on those terms. However, these bounds are sufficiently tight for our purposes, as we shall see in the next section.

3 Applying the lemmas: unimodality of non-regular caterpillars

Recall that if G is a graph and $S \subseteq V(G)$, then $G - S$ is defined to be the graph resulting from G by removing all vertices in S and all edges incident to at least one vertex in S . If $S = \{v\}$ comprises a single vertex, we may write $G - v$ for $G - \{v\}$. If $v \in V(G)$, then the *closed neighborhood* of v , $N[v]$, is defined by $N[v] = \{u \in V(G) \mid uv \in E(V)\} \cup \{v\}$.

We first note a standard lemma that we will use frequently, often without explicit mention. Its proof is well-known and straightforward.

Lemma 3.1 *Suppose that G is a graph and $v \in V(G)$. Then $I(G, x) = I(G - v, x) + xI(G - N[v], x)$.*

Let us now recall the sequences $\vec{m} = (m_1, m_2, \dots)$ and corresponding caterpillars $T(\vec{m}, n)$ defined in the introduction. We let $p_{\vec{m}, n}(x) = I(T(\vec{m}, n), x)$, and if $k(\vec{m}, n)$ denotes the greatest power of $1 + x$ evenly dividing $p_{\vec{m}, n}$, we define $q_{\vec{m}, n}$ by $p_{\vec{m}, n} = (1 + x)^{k(\vec{m}, n)} q_{\vec{m}, n}$. When \vec{m} is understood, we may abbreviate $T(\vec{m}, n)$ to T_n , etc.

The following facts are proven by direct application of Lemma 3.1 (to the spine vertex v with m_n pendant edges) and straightforward inductions; compare the methods of [2–4], for instance:

Proposition 3.2 *Let \vec{m} be given a non-decreasing sequence of natural numbers. Then*

$$p_n = \begin{cases} (1+x)^{m_1} + x & \text{if } n = 1, \\ (1+x)^{m_1+m_2} + x((1+x)^{m_1} + (1+x)^{m_2}) & \text{if } n = 2, \text{ and} \\ (1+x)^{m_n} p_{n-1}(x) + x(1+x)^{m_{n-1}} p_{n-2}(x) & \text{if } n \geq 3. \end{cases}$$

For all n , $\deg(p_n) = \sum_{i=1}^n m_i$. Moreover, $k_1 = 0$ and

$$k_n = \begin{cases} m_1 + m_3 + \dots + m_{n-1} & \text{if } n \text{ is even and} \\ m_2 + m_4 + \dots + m_{n-1} & \text{if } n \text{ is odd, } n \geq 3, \end{cases}$$

so that $k_{n+1} \geq k_n$ for all n , and

$$q_n = \begin{cases} (1+x)^{m_1} + x & \text{if } n = 1, \\ (1+x)^{m_2} + x(1+x)^{m_2-m_1} + x & \text{if } n = 2, \text{ and} \\ (1+x)^{k_{n+1}-k_n} q_{n-1}(x) + x q_{n-2}(x) & \text{if } n \geq 3. \end{cases}$$

Thus, for all n ,

$$\deg(q_n) = \begin{cases} m_1 + m_3 + \dots + m_n & \text{if } n \text{ is odd and} \\ m_2 + m_4 + \dots + m_n & \text{if } n \text{ is even.} \end{cases}$$

We now apply Lemmas 2.1 and 2.2 to show that under the right hypotheses $q_n(x)$ is balanced, strictly unimodal, and either LD or RD. Once this is done, one more application of Lemma 2.1 will establish the same properties for $p_n(x) = I(T(\vec{m}, n), x)$, thereby proving our main result, Theorem 1.1.

Proposition 3.3 *Let $\vec{m} = (m_1, m_2, \dots)$ be a sequence of natural numbers such that*

1. $m_k \leq m_{k+1}$ for all $1 \leq k < n$,
2. $3 \leq m_1 < m_2$ and $m_3 < m_4$, and
3. $2(m_1 + m_3 + \dots + m_k) < 3(m_2 + m_4 + \dots + m_{k-1})$ for k odd and $2(m_2 + m_4 + \dots + m_k) < 3(m_1 + m_3 + \dots + m_{k-1})$ for k even.

Then the polynomial $q_n(x)$ defined as above is balanced, strictly unimodal, and either LD or RD.

Proof: Straightforward computation shows that both q_1 and q_2 are balanced, with q_1 weakly LD and q_2 either LD or RD. The polynomial q_1 is always strictly unimodal, and q_2 is unimodal in general. It will be strictly unimodal except when m_1 is even, $m_1 \geq 6$, and $m_2 = m_1 + 1$; in this case, q_2 has consecutive modes at $\frac{m_2-1}{2}$ and $\frac{m_2+1}{2}$. In any case, we may take these polynomials to be the base cases for an induction. Assume that we have shown our result for all $k \leq n - 1$ for some $n \geq 3$.

Let $\mu = \mu(q_{n-2})$ and $\mu' = \mu(q_{n-1})$, and suppose

$$q_{n-2} = \sum a_i x^i, q_{n-1} = \sum b_i x^i, x q_{n-2} = \sum \alpha_i x^i, \text{ and } (1+x)^t q_{n-1} = \sum \beta_i x^i,$$

where

$$t = k_{n+1} - k_n = \begin{cases} (m_2 - m_1) + \cdots + (m_n - m_{n-1}) & \text{if } n \text{ is even and} \\ m_1 + (m_3 - m_2) + \cdots + (m_n - m_{n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, $\alpha_i = a_{i-1}$ for all $i \geq 1$, $\mu(xq_{n-2}) = \mu + 1$, and Lemma 2.1 implies that $(1 + x)^t q_{n-1}$ is balanced and unimodal and either LD or RD. Moreover, if we let $\mu'' = \mu((1 + x)^t q_{n-1})$, then $\mu'' = \mu' + \lfloor \frac{t}{2} \rfloor$ if q_{n-1} is LD and $\mu'' = \mu' + \lceil \frac{t}{2} \rceil$ if q_{n-1} is RD. (Note that Lemma 2.1 applies because our third hypothesis on the values m_i ensures that $t \leq \mu'$.) Finally, because $q_n = (1 + x)^t q_{n-1} + xq_{n-2}$, $q_n = \sum(\alpha_i + \beta_i)x^i = \sum(a_{i-1} + \beta_i)x^i$.

Clearly $\alpha_k + \beta_k < \alpha_{k+1} + \beta_{k+1}$ for $k \leq \mu$ and $\alpha_k + \beta_k > \alpha_{k+1} + \beta_{k+1}$ for $k \geq \mu''$. Thus, to prove strict unimodality of q_n we only need to consider $k \in [\mu + 1, \mu'' - 1]$, and for such k it suffices to show that $a_{k-1} - a_k < \beta_{k+1} - \beta_k$. First note that, by the definition of b_j and β_k , Part (1) of Lemma 2.2 implies $\beta_{k+1} - \beta_k \geq (t - 1)(b_k - b_{k+1})$. Moreover, because $q_{n-1} = (1 + x)^{k_n - k_{n-1}} q_{n-2} + xq_{n-3}$, Part (2) of the same lemma implies $b_k - b_{k+1} \geq (t' - 1)(a_{k-1} - a_k)$, where $t' = k_n - k_{n-1}$. (In fact, this last estimate ignores the non-negative contribution from the term xq_{n-3} , inclusion of which would only serve to increase the left-hand side of the last inequality.) Putting these together, we obtain $\beta_{k+1} - \beta_k \geq (t - 1)(t' - 1)(a_{k-1} - a_k)$. Moreover, our first two hypotheses on the values m_i imply that $t, t' \geq 2$ whenever $n \geq 3$. Thus $(t - 1)(t' - 1) \geq 1$, and the inequality above implies that $\beta_{k+1} - \beta_k \geq a_{k-1} - a_k$, as desired.

Thus q_n is strictly unimodal and

$$\mu(q_n) = \mu((1 + x)^t q_{n-1}) = \begin{cases} \mu(q_{n-1}) + \lfloor \frac{t}{2} \rfloor & \text{if } q_n \text{ is RD and} \\ \mu(q_{n-1}) + \lceil \frac{t}{2} \rceil & \text{if } q_n \text{ is LD,} \end{cases}$$

so that q_n is balanced as well. Moreover, Lemma 2.2 also implies that the addition of the terms α_k does not affect the LD or RD nature of $(1 + x)^t q_{n-1}$. That is, $q_n = (1 + x)^t q_{n-1} + xq_{n-2}$ remains LD or RD, accordingly. \square

As noted before the statement of the proposition, the fact that $p_n(x) = I(T(\vec{m}, n), x)$ is balanced, strictly unimodal (even in the case of p_2 when $m_1 \geq 6$ is even and $m_2 = m_1 + 1$), and either LD or RD follows from one more application of Lemma 2.1, because $p_n(x) = (1 + x)^{k_n} q_n(x)$ for all n . This completes our proof of Theorem 1.1.

We close by offering an explanation for our requirement that \vec{m} be non-decreasing. Observe that for $n \geq 3$, $k_n = \min\{k_{n-1} + m_n, k_{n-2} + m_{n-1}\}$, so that if \vec{m} were not non-decreasing, it could be that, for some n , $k_{n-1} + m_n < k_{n-2} + m_{n-1}$, giving us

$$q_n = q_{n-1} + x(1 + x)^{k_{n-2} + m_{n-1} - k_n} q_{n-2},$$

in which case our fundamental result, Lemma 2.1, would not apply. Therefore without significant further analysis of the recursive construction of an arbitrary caterpillar's independence polynomial, we cannot accommodate sequences \vec{m} in which $m_k > m_{k+1}$ for some k .

References

- [1] Y. ALAVI, P. MALDE, A. SCHWENK and P. ERDŐS, The vertex independence sequence of a graph is not constrained, *Congr. Numer.* **58** (1987), 15–23.
- [2] P. BAHLS, On the independence polynomials of path-like graphs, *Australas. J. Combin.* **53** (2012), 3–18.
- [3] P. BAHLS, E. BAILEY and M. OLSEN, New families of graphs whose independence polynomials have only real roots, *Australas. J. Combin.* **60** (2014), 128–135.
- [4] P. BAHLS and N. SALAZAR, Symmetry and unimodality of independence polynomials of generalized paths, *Australas. J. Combin.* **47** (2010), 165–176.
- [5] A. BHATTACHARYYA and J. KAHN, A bipartite graph with non-unimodal independent set sequence, *Electron. J. Combin.* **20** (2013), #P11.
- [6] M. CHUDNOVSKY and P. SEYMOUR, The roots of the stable set polynomial of a claw-free graph, *J. Combin. Theory Ser. B.* **97** (2007), 350–357.
- [7] D. GALVIN and J. HILYARD, The independent set sequence of some families of trees, (2017), preprint.
- [8] V. E. LEVIT and E. MANDRESCU, The independence polynomial of a graph—a survey, *Proc. 1st Int. Conf. Algebraic Informatics, Thessaloniki, 2005* (Eds. S. Bozapalidis, A. Kalampakas and G. Rahonis), Thessaloniki, Greece: Aristotle University (2005), 233–254.
- [9] V. E. LEVIT and E. MANDRESCU, Partial unimodality for independence polynomials of König-Egerváry graphs, *Congr. Numer.* **179** (2006), 109–119.
- [10] V. E. LEVIT and E. MANDRESCU, Independence polynomials and the unimodality conjecture for very well-covered, quasi-regularizable, and perfect graphs, *Graph Theory in Paris, Trends Math.*, Basel, Switzerland: Birkhäuser (2007), 243–254.
- [11] V. E. LEVIT and E. MANDRESCU, A family of graphs whose independence polynomials are both palindromic and unimodal, *Carpathian J. Math.* **23** (2007), 108–116.
- [12] E. MANDRESCU and A. SPIVAK, Maximal trees with log-concave independence polynomials, *Notes on Number Theory and Discrete Math.* **22** (2016), 44–53.
- [13] Y. WANG and B.-X. ZHU, On the unimodality of independence polynomials of some graphs, *European J. Combin.* **32** (2011), 10–20.
- [14] B.-X. ZHU, Clique cover products and unimodality of independence polynomials, *Discrete Appl. Math.* **206** (2016), 172–180.
- [15] Z.-F. ZHU, The unimodality of independence polynomials of some graphs, *Australas. J. Combin.* **38** (2007), 27–33.