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Disjoint cycles of order at least 5

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Abstract

We prove that if G is a graph of order at least $5k$ with $k \geq 2$ and the minimum degree of G is at least $3k$ then G contains k disjoint cycles of length at least 5. This supports the conjecture by Wang [*Australas. J. Combin.* 54 (2012), 59–84]: if G is a graph of order at least $(2d+1)k$ and the minimum degree of G is at least $(d+1)k$ with $k \geq 2$ then G contains k disjoint cycles of length at least $2d+1$.

1 Introduction

A set of graphs is said to be disjoint if no two of them have any common vertex. Corrádi and Hajnal [2] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. Erdős and Faudree [4] conjectured that if G is a graph of order $4k$ with minimum degree at least $2k$, then G contains k disjoint cycles of length 4. To solve this conjecture, partial results were obtained in [5] and [6]. We finally confirmed this conjecture in [7]. In [8], we proposed the following two conjectures:

Conjecture 1 [8] *Let d and k be two positive integers with $k \geq 2$. If G is a graph of order at least $(2d+1)k$ and the minimum degree of G is at least $(d+1)k$ then G contains k disjoint cycles of length at least $2d+1$.*

Conjecture 2 [8] *Let d and k be two positive integers with $k \geq 2$ and $d \geq 3$. Let G be a graph of order $n \geq 2dk$ with minimum degree at least dk. Then G contains k disjoint cycles of length at least $2d$, unless k is odd and $n = 2dk + r$ for some $1 \leq r \leq 2d-2$.*

The above two conjectures are related with El-Zahar's conjcture [3]. El-Zahar conjectured that if G is a graph of order $n = n_1+n_2+\cdots+n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ and the minimum degree of G is at least $\lceil n_1/2\rceil + \lceil n_2/2\rceil + \cdots + \lceil n_k/2\rceil$, then G contains k disjoint cycles of lengths $n_1, n_2, \ldots, n_k$, respectively. In Conjecture 1, if G has order $(2d+1)k$ then the conjecture reduces to the special case of El-Zahar's conjecture where $n_i = 2d+1$ for all $1 \leq i \leq k$. Similarly, if G has order $2dk$ in

Conjecture 2, then the conjecture reduces to the special case of El-Zahar's conjecture where $n_i = 2d$ for all $1 \le i \le k$.

With [7], we showed in [8] that if a graph G of order $n \ge 4k$ with $k \ge 2$ has minimum degree at least $2k$ then with three easily recognized exceptions, G contains k disjoint cycles of length at least 4. When $d = 1$, Conjecture 1 holds by Corrádi and Hajnal [2]. Comparing the proof of Conjecture 1 in the case $d = 1$ with our work in [7] and [8], the work in [7] and [8] is significantly more complicated and involved. It would be sound to make some progress on Conjecture 1 which includes Corrádi and Hajnal Theorem as a special case. As said so, our purpose in this paper is to show Conjecture 1 in the case $d = 2$.

Another motivation for us to consider Conjecture 1 in the case $d = 2$ is the result we proved in [9]:

Theorem 1 [9] *Let k and n be two integers with $k \ge 1$. If G is a graph of order $n = 5k$ and the minimum degree of G is at least $3k$, then G contains k disjoint cycles of length of 5.*

In this paper, we prove the following:

Theorem 2 *Let k and n be two integers with $k \ge 2$ and $n \ge 5$. If G is a graph of order $n \ge 5k$ and the minimum degree of G is at least $3k$, then G contains k disjoint cycles of length at least 5.*

This extends Theorem 1 and also further supports Conjecture 1.

1.1 Terminology and Notation

We use [1] for standard terminology and notation except as indicated. Let G be a graph. We use $|G|$ to denote the order of G, i.e., $|G| = |V(G)|$. Let H be a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G. Let $u \in V(G)$. We define $N(u, H)$ to be the set of neighbors of u contained in H, and let $e(u, H) = |N(u, H)|$. Clearly, $N(u, G) = N(u)$ and $e(u, G)$ is the degree of u in G. Let $v \in V(G)$. We define $I(uv, H) = N(u, H) \cap N(v, H)$ and let $i(uv, H) = |I(uv, H)|$.

If X is a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G, we define $N(X, H) = \cup_u N(u, H)$ and $e(X, H) = \sum_u e(u, H)$ where u runs over all the vertices in X. Let each of $X_1, X_2, \dots, X_r$ be a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G. We use $[X_1, X_2, \dots, X_r]$ to denote the subgraph of G induced by the set of all the vertices that belong to at least one of $X_1, X_2, \dots, X_r$.

For each integer $k \ge 3$, a k-cycle is a cycle of length k and a $(\ge k)$-cycle is a cycle of length at least k. A feasible cycle is a (≥ 5)-cycle. For each integer $i \ge 3$, we use C_i to denote a cycle of length i and $C_{\ge i}$ to denote a cycle of length at least i. Use P_j to denote a path of order j for all integers $j \ge 1$. For a cycle or path L of G, a *chord* of L is an edge of $G - E(L)$ which joins two vertices of L, and we use $\tau(L)$ to denote the number of chords of L in G. For each $x \in V(L)$, use $\tau(x, L)$ to denote the number of chords of L that are incident with x. The length of L is denoted by $l(L)$.

If S is a set of subgraphs of G, we write $G \supseteq S$. For an integer $k \geq 1$ and a graph G', we use kG' to denote a set of k disjoint graphs isomorphic to G'. If G_1 and G_2 are two graphs, we use $G_1 \uplus G_2$ to denote a set of two disjoint graphs, one isomorphic to G_1 and the other isomorphic to G_2. For two graphs H_1 and H_2, the union of H_1 and H_2 is still denoted by $H_1 \cup H_2$ as usual, that is, $H_1 \cup H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Let each of Y and Z be a subgraph of G, or a subset of $V(G)$, or a sequence of distinct vertices of G. If Y and Z do not have any common vertices, we define $E(Y, Z)$ to be the set of all the edges of G between Y and Z. Clearly, $e(Y, Z) = |E(Y, Z)|$. If $C = x_1 x_2 \dots x_r x_1$ is a cycle, then the operations on the subscripts of the x_i's will be taken by modulo r in $\{1, 2, \dots, r\}$.

If we write a graph G as a sequence $x_1 x_2 \dots x_l$ of its vertices, it means that $V(G) = \{x_1, x_2, \dots, x_l\}$ and $E(G) = \{x_i x_{i+1} | 1 \leq i \leq l-1\}$. Note that the sequence may have repeated vertices. We use R_t^i to denote a graph of order t such that $R_t^i = x_1 x_2 \dots x_t x_{t-i+1}$ with $3 \leq i \leq t$. We use B to denote a graph of order 5 such that $B = x_3 x_1 x_2 x_3 x_4 x_5 x_3$. Let P be a path of G. We use $r(P)$ to denote the order of a largest cycle in $P + f$ where f runs over all the chords f of P that are incident with an endvertex of P. If P does not have such a chord, then $r(P) = 0$. Clearly, if R_t^i is a subgraph of G then $r(R_t^i) \geq i$.

Let C be a 5-cycle of G and $u \in V(C)$. Let $x \in V(G) - V(C)$. We write $x \to (C, u)$ if $[C - u + x] \supseteq C' \cong C_5$. If $x \to (C, u)$ for all $u \in V(C)$ then we write $x \to C$.

2 Lemmas

Let $G = (V, E)$ be a graph. We will use the following lemmas. Lemma 2.1 and Lemma 2.2 are two easy observations.

Lemma 2.1 *If P is a path of order 3 and u and v are two vertices in $G - V(P)$ such that $e(uv, P) \geq 5$, then $[P + u + v]$ contains a cycle of order 5.*

Lemma 2.2 *The following four statements hold:*

(*a*) *If L is a cycle of order $p \geq 6$ and $v \in V(G) - V(L)$ such that $e(v, L) \geq 3$, then either $[L + v]$ contains a feasible cycle C with $l(C) < p$, or $e(v, L) = 3$ and v is adjacent to three consecutive vertices of L, or $e(v, L) = 3$, $p = 6$ and v is adjacent to every other vertex of L.*

(*b*) *If P is a path of order $p \geq 5$ and $u \in V(G) - V(P)$ such that $e(u, P) \geq 4$, then for some endvertex z of P, $[P + u - z]$ contains a feasible cycle C with $l(C) \leq p$. Moreover, if $p \geq 6$, then $[P + u]$ contains a feasible cycle of length less than p.*

(*c*) *If P is a path of order p and $u_1 u_2$ is an edge of $G - V(P)$ such that $e(u_1 u_2, P) \geq 4$, then $[P + u_1 + u_2]$ contains a feasible cycle, or $e(u_1 u_2, P) = 4$ and P has an edge xy such that $N(u_1 u_2, P) = \{x, y\}$, or $e(u_1 u_2, P) = 4$ and P has a subpath xyz such that $N(u_i, P) = \{x, y, z\}$ and $N(u_j, P) = \{y\}$ for some $\{i, j\} = \{1, 2\}$. Moreover, if $e(u_1 u_2, P) \geq 5$ then $[P + u_1 + u_2]$ contains a feasible cycle of order at most $p + 1$.*

Lemma 2.3 *Let P and Q be two disjoint paths of G. Suppose that $e(P,Q) \geq 5$ and $[P,Q]$ does not contain a feasible cycle of order at most $|P|+|Q|-1$. Then $e(P,Q)=5$ and one of the following two statements holds:*

(a) $|P|=3$, $|Q|=3$ and $[P,Q] \cong K_{3,3}$;

(b) P has a subpath uvw and Q has a subpath xyz such that $N(v,Q)=\{x,y,z\}$ and $N(y,P)=\{u,v,w\}$.

Proof. On the contrary, say the lemma fails. Let $|P|+|Q|$ be minimal such that the lemma fails for P and Q. By Lemma 2.2, we see that $|P| \geq 3$ and $|Q| \geq 3$. If $|P|=|Q|=3$, it is easy to check that one of (a) and (b) holds. So assume that $|P|+|Q| \geq 7$. Say $P = x_1 \dots x_s$ and $Q = y_1 \dots y_t$. By the minimality of $|P|+|Q|$, we see that $e(x_i,Q) \geq 1$ for $i \in \{1,s\}$ and $e(y_j,P) \geq 1$ for $j \in \{1,t\}$. If $\{x_1y_1, x_sy_t\} \subseteq E$ or $\{x_1y_t, x_sy_1\} \subseteq E$, then we readily see that $[P,Q]$ contains a feasible cycle of order at most $|P|+|Q|-1$. Therefore neither of these two situations will occur. This implies that $N(x_1,Q)=N(x_s,Q)=\{y_k\}$ for some $y_k \in V(Q)$ and $N(y_1,P)=N(y_t,P)=\{x_h\}$ for some $x_h \in V(P)$. Thus $s=t=3$ for otherwise $[P,Q]$ contains a feasible cycle of order at most $|P|+|Q|-1$. Then one of (a) and (b) holds. ∎

Lemma 2.4 *Let C be a 5-cycle of G. Let x and y be two vertices in $G-V(C)$. If $e(xy,C) \geq 7$, then there exists $z \in V(C)$ such that either $yz \in E$ and $[C-z+x]$ contains a 5-cycle C' with $\tau(C') \geq \tau(C)-1$, or $xz \in E$ and $[C-z+y]$ contains a 5-cycle C'' with $\tau(C'') \geq \tau(C)-1$.*

Proof. Say without loss of generality that $e(x,C) \geq 4$. For each $u \in V(C)$ with $x \to (C,u)$, we see that $[C-u+x] \supseteq C' \cong C_5$ and $\tau(C') \geq \tau(C)-1$. As $e(xy,C) \geq 7$, $yu \in E$ for such a vertex $u \in V(C)$ with $x \to (C,u)$ and so the lemma holds. ∎

Lemma 2.5 *Let p and q be two integers with $q \geq p \geq 5$ and $q \geq 6$. Let C and L be two disjoint cycles with $l(C)=p$ and $l(L)=q$. If $e(L,C) \geq 3q+1$, then $[C,L]$ contains two disjoint feasible cycles C' and L' such that either $l(C')<p$ or $l(C')=p$ and $l(L')<q$.*

Proof. Say $C = a_1a_2\dots a_pa_1$ and $L = x_1x_2\dots x_qx_1$. On the contrary, say the lemma fails. We first claim that $e(a_i,L) \leq 5$ and $e(x_j,C) \leq 5$ for all $a_i \in V(C)$ and $x_j \in V(L)$. To see this, say $e(a_i,L) \geq 6$ for some $a_i \in V(C)$. Then $[L-x_r-x_{r+1}+a_i]$ contains a feasible cycle and so $[C-a_i+x_r+x_{r+1}]$ does not contain a feasible cycle of order at most p for all $r \in \{1,\dots,q\}$. By Lemma 2.2(c), this implies that $e(x_rx_{r+1}, C-a_i) \leq 4$ and so $e(x_rx_{r+1},C) \leq 6$ for all $r \in \{1,\dots,q\}$. Consequently, $e(C,L) \leq 3q$, a contradiction. Hence $e(a_i,L) \leq 5$ for all $a_i \in V(C)$. Similarly, $e(x_j,C) \leq 5$ for all $x_j \in V(L)$.

Say $e(x_1,C) \geq e(x_i,C)$ for all $x_i \in V(L)$. As $e(C,L) \geq 3q+1$, $e(x_1,C) \geq 4$. We divide the proof into the following two cases.

Case 1. $p=5$.

As $e(x_1, C) \geq 4$, $x_1 \to (C, a_i)$ for some $a_i \in V(C)$. Thus $[L - x_1 + a_i]$ does not contain a feasible cycle of order $\leq q-1$. This implies that $e(a_i, L-x_1) \leq 3$ by Lemma 2.2(b). First, suppose that $e(x_1, C) = 5$. Then $x_1 \to C$ and so $e(a_j, L - x_1) \leq 3$ for all $1 \leq j \leq 5$. Thus $e(C, L) = e(x_1, C) + e(C, L - x_1) \leq 5 + 5 \cdot 3 = 20$. As $20 \geq e(C, L) \geq 3q + 1 \geq 19$, it follows that $q = 6$. We may assume without loss of generality that $e(a_j, L - x_1) = 3$ for $1 \leq j \leq 4$. As $[C, L] \not\supseteq 2C_5$, $e(a_r, x_2x_5) \leq 1$ and $e(a_r, x_3x_6) \leq 1$ for all $1 \leq r \leq 5$. It follows that $e(a_r, x_2x_5) = 1$, $e(a_r, x_3x_6) = 1$ and $a_rx_4 \in E$ for all $1 \leq r \leq 4$. Assume for the moment that $e(a_1, x_2x_6) > 0$. Say without loss of generality that $a_1x_2 \in E$. Then $[x_1a_4a_5a_1x_2] \supseteq C_5$ and so $[x_4x_5x_6, a_2a_3] \not\supseteq C_5$. This implies that $e(x_6, a_2a_3) = 0$ and so $e(x_3, a_2a_3) = 2$. Then $[a_1a_2a_3x_3x_2] \supseteq C_5$ and $[a_4x_1x_6x_5x_4] \supseteq C_5$, a contradiction. Therefore $e(a_1, x_2x_6) = 0$ and so $e(a_1, x_3x_5) = 2$. Similarly, $e(a_4, x_3x_5) = 2$. Thus $[x_3x_4a_1a_5a_4] \supseteq C_5$ and so $[x_1x_6x_5, a_2a_3] \not\supseteq C_5$. This implies that $e(x_5, a_2a_3) = 0$. Similarly, $[x_5x_4a_1a_5a_4] \supseteq C_5$ and so $e(x_3, a_2a_3) = 0$. Thus $e(x_2x_6, a_2a_3) = 4$. Therefore $[x_2x_1x_6a_2a_3] \supseteq C_5$ and so $[C, L] \supseteq 2C_5$, a contradiction.

Hence $e(x_1, C) = 4$ and so $e(x_i, C) \leq 4$ for all $x_i \in V(L)$. Say $N(x_1, C) = \{a_1, a_2, a_3, a_4\}$. Then $x_1 \to (C, a_i)$ and so $e(a_i, L - x_1) \leq 3$ for $i \in \{2, 3, 5\}$. Then $10 \geq e(a_1a_4, L) \geq 3q + 1 - 3 \cdot 3 - 2$. This implies that $q = 6$ and $e(a_1a_4, L) \geq 8$. We claim that $e(a_1, L) = e(a_4, L) = 4$. If this is not true, say without loss of generality that $e(a_1, L) = 5$. Label $L = z_1z_2z_3z_4z_5z_6z_1$ with $e(a_1, L - z_6) = 5$. Then $[a_1, L - z_1 - z_6] \supseteq C_5$ and so $e(z_1z_6, C - a_1) \leq 4$ by Lemma 2.2(c). Similarly, $e(z_3z_4, C - a_1) \leq 4$. It follows that $e(z_2z_5, C) \geq 19 - 2 \cdot 4 - 3 = 8$ and so $e(z_2, C) = e(z_5, C) = 4$. Consequently, $e(z_1z_6, C - a_1) = 4$ and $e(z_3z_4, C - a_1) = 4$. Similarly, we shall have that $e(z_5z_6, C - a_1) = 4$, $e(z_2z_3, C - a_1) = 4$ and $e(z_1, C) = e(z_4, C) = 4$. Consequently, $e(z_6, C - a_1) = e(z_3, C - a_1) = 1$. As $[C, L] \not\supseteq 2C_5$, $z_i \not\to (C, a_1)$ and so $e(z_i, a_2a_5) \leq 1$ for all $i \in \{1, 2, 4, 5\}$. Thus $e(z_i, a_2a_5) = 1$ and $e(z_i, a_3a_4) = 2$ for all $i \in \{1, 2, 4, 5\}$. Then we see that $[z_6, z_5, a_2, a_3, a_4, a_5] \not\supseteq C_5$ and so $e(z_6, a_2a_5) = 0$. Thus $e(z_6, a_3a_4) = 1$, say $z_6a_3 \in E$. Then $[a_1, a_2, a_3, z_6, z_5] \supseteq C_5$ and $[a_4, z_1, z_2, z_3, z_4] \supseteq C_5$, a contradiction.

Hence $e(a_1, L) = e(a_4, L) = 4$. It follows that $e(a_i, L) = 4$ for $i \in \{1, 2, 3, 4\}$ and $e(a_5, L) = 3$. We now go back to the labelling $L = x_1x_2x_3x_4x_5x_6x_1$. As $[C, L] \not\supseteq 2C_5$, we see that $e(a_i, x_2x_5) \leq 1$ and $e(a_i, x_3x_6) \leq 1$ for all $i \in \{2, 3, 5\}$. It follows that $e(a_2a_3a_5, x_4) = 3$. Then for each $i \in \{1, 4\}$, $x_4 \to (C, a_i)$ and so $[L - x_4 + a_i] \not\supseteq C_5$. By Lemma 2.3($b$), this implies that $e(a_i, L - x_4) \leq 3$ and so $a_ix_4 \in E$ for each $i \in \{1, 4\}$. Thus $e(x_4, C) = 5$, a contradiction.

Case 2. $p \geq 6$.

First, assume that $e(x_1, C) = 5$. Say the five vertices in $N(x_1, C)$ are a, b, c, d and g in order along C with $|C[g, a]| \geq 3$. Then $x_1C[a, d]x_1$ and $x_1C[d, a]x_1$ are two feasible cycles. By Lemma 2.2(c) and Lemma 2.3, this yields that $e(C(d, a), L - x_1) \leq 4 + r$ and $e(C(a, d), L - x_1) \leq 4 + r$ with $r \in \{0, 1\}$. It follows that $e(ad, L) \geq 3q + 1 - 2 \cdot (4 + r) - 3 = 3q - 10 - 2r$. As $C - a + x_1 \supseteq C_{\geq 5}$ and $C - d + x_1 \supseteq C_{\geq 5}$, we have $e(a, L - x_1) \leq 3$ and $e(d, L - x_1) \leq 3$ by Lemma 2.2(b). Thus $8 \geq 3q - 10 - 2r$. This yields $q = 6$ and so $p = 6$. Thus by Lemma 2.2(c), we may choose $r = 0$. It follows that $e(a, L) = 4$, $e(d, L) = 4$, $e(C(d, a), L-x_1) = 4$ and $e(C(a, d), L-x_1) = 4$. We may assume that $\{a, b, c, d, g\} = \{a_1, \dots, a_5\}$. Similarly, we shall have $e(a_5, L) = e(a_2, L) = 4$. It follows that $e(a_3, L-x_1) = e(a_6, L-x_1) = 1$. Clearly, $e(a_1, x_2x_6) \le 1$, for otherwise $[C, L] \supseteq C_5 \uplus C_6$. Say $a_1x_6 \notin E$. As $e(a_1a_5, L - x_1) = 6$, there exists x_ix_{i+1} on $L - x_1$ such that $\{a_1x_i, a_5x_{i+1}\} \subseteq E$. Thus $[x_i, x_{i+1}, a_5, a_6, a_1] \supseteq C_5$. Since $e(a_2, L - x_i - x_{i+1}) \ge 2$ and $e(a_4, L - x_i - x_{i+1}) \ge 2$, $[a_2a_3a_4, L - x_i - x_{i+1}] \supseteq C_{\ge 5}$, a contradiction. This proves that $e(x_1, C) = 4$. Similarly, we shall have $e(a_i, L) \le 4$ for all $a_i \in V(C)$. As $p \ge 6$, we see that there are two distinct vertices a_s and a_t in $N(x_1, C)$ such that both $x_1C[a_s, a_t]x_1$ and $x_1C[a_t, a_s]x_1$ are feasible cycles. By Lemma 2.2(c) and Lemma 2.3, we have $e(C(a_t, a_s), L-x_1) \le 4+r$ and $e(C(a_s, a_t), L-x_1) \le 4+r$ with $r \in \{0, 1\}$. Then $8 \ge e(a_sa_t, L) \ge 3q + 1 - 2 \cdot (4+r) - 2$. As above, we must have $q = 6$ and so $p = 6$. Then by Lemma 2.2(c), we may choose $r = 0$. Thus $8 \ge e(a_sa_t, L) \ge 3q + 1 - 2 \cdot 4 - 2 \ge 9$, a contradiction. ∎

Lemma 2.6 [Lemma 2.2, [9]] *Let D and L be two disjoint subgraphs of G such that $D \cong B$ and $L \cong C_5$. Say $D = x_0x_1x_2x_0x_3x_4x_0$. Suppose that $e(D - x_0, L) \ge 13$. Then $[D, L] \supseteq 2C_5$.*

Corollary 2.7 *Let $P = x_1x_2 \dots x_t$ be a path of order $t \ge 5$ and C a 5-cycle in G such that P and C are disjoint and $\{x_1x_h, x_tx_k\} \subseteq E$ for some $3 \le h \le k \le t-2$. If $e(x_ix_j, C) + e(x_qx_r, C) \ge 13$ for some $1 \le i < j \le h-1$ and $k+1 \le q < r \le t$ then $[P, C]$ contains two disjoint feasible cycles.*

3 Proof of Theorem 2

Let G be a graph of order $n \ge 5k$ with $k \ge 2$ and $\delta(G) \ge 3k$. Suppose, for a contradiction, that G does not contain k disjoint feasible cycles. By Theorem 1, $n \ge 5k + 1$. Let k_0 be the largest integer such that G contains k_0 disjoint feasible cycles. A *chain* of G is a sequence $(L_1, \dots, L_{k_0})$ of k_0 disjoint feasible cycles.

We use lexicographic order to order chains with respect to the lengths of feasible cycles in chains, that is, for two chains $(L_1, \dots, L_{k_0})$ and $(L'_1, \dots, L'_{k_0})$ in G, we write $(L_1, \dots, L_{k_0}) \prec (L'_1, \dots, L'_{k_0})$ if there exists $j \in \{1, \dots, k_0\}$ such that $l(L_i) = l(L'_i)$ for $i = 1, \dots, j$ and $l(L_{j+1}) < l(L'_{j+1})$. We say that $(L_1, \dots, L_{k_0})$ is a minimal chain if for any chain $(L'_1, \dots, L'_{k_0})$, $(L'_1, \dots, L'_{k_0}) \not\prec (L_1, \dots, L_{k_0})$. For any chain $\sigma = (L_1, \dots, L_{k_0})$, we use $V(\sigma)$ to denote $V(\cup_{i=1}^{k_0} L_i)$. We now choose a minimal chain $\sigma = (L_1, \dots, L_{k_0})$ such that

$$\text{The length of a longest path of } G - V(\sigma) \text{ is maximal.} \quad (1)$$

Let $H = \cup_{i=1}^{k_0} L_i$ and $D = G - V(H)$. Let $P = x_1 \dots x_t$ be a longest path of D. We shall prove the following two claims.

Claim 1. $t \ge 6$.

Proof of Claim 1. Assume first that $|D| \le 5$. Then $|L_{k_0}| \ge 6$ and by Lemma 2.2(a) and the minimality of σ, $e(D, L_{k_0}) \le 3|D|$. By Lemma 2.2(b), $e(x, L_{k_0} - x) \le 3$ for each $x \in V(L_{k_0})$. Thus $e(L_{k_0}, H-V(L_{k_0})) \geq 3k|L_{k_0}|-3|L_{k_0}|-3|D| \geq 3(k-2)|L_{k_0}|+3$. This implies that $e(L_{k_0}, L_i) \geq 3|L_{k_0}| + 1$ for some $1 \leq i \leq k-2$. By Lemma 2.5, $[L_i, L_{k_0}]$ contains two disjoint feasible cycles C' and L' such that either $l(C') < l(L_i)$ or $l(C') = l(L_i)$ and $l(L') < l(L_{k_0})$. Replacing L_i and L_{k_0} with C' and L', we obtain a chain $\sigma' \prec \sigma$, a contradiction. Therefore $|D| \geq 6$.

For a contradiction, suppose that $t \leq 5$. Let Q be a longest path in $D - V(P)$. Subject to (1), we choose σ and P in D such that $l(Q)$ is maximal. Say $Q = y_1y_2\dots y_s$. Let If D contains two distinct vertices x and y with $e(xy, D) \leq 5$, then $e(xy, L_i) \geq 7$ for some L_i in H since $e(xy, G) \geq 6k$. Then by Lemma 2.2(a) and the minimality of σ, we see that $|L_i| = 5$ and by Lemma 2.4, $[L_i + x + y] \supseteq C_5 \uplus P_2$. This argument shows that $t \geq 2$. If $t = 2$, this argument allows us to see that we may choose σ such that D contains two independent edges xu and yv. Then $e(xy, D) = 2$ and $e(xy, L_i) \geq 7$ for some L_i in H. As above, we see that $|L_i| = 5$ and so by Lemma 2.4, $[L_i, x, u, y, v] \supseteq C_5 \uplus P_3$, a contradiction. Hence $t \geq 3$.

First, suppose that $s \geq 2$. We claim that $e(x_1x_ty_1y_s, D) \leq 11$. To observe this, we readily see that $e(x_1x_t, P) \leq 6$ and $e(y_1y_s, P) \leq 2$ since $D \not\supseteq C_{\geq 5}$ and $t \leq 5$. Moreover, if $e(y_1y_s, P) > 0$ then $t = 5$, $s = 2$, $N(y_1y_2, P) = \{x_3\}$ and so $e(x_1x_ty_1y_s, D) < 11$. Suppose that $e(x_1x_ty_1y_s, D) \geq 12$. Then it is easy to see that $[P] \cong [Q] \cong K_4$. As $e(x_1x_4y_1y_4, G) \geq 12k$, $e(x_1x_4y_1y_4, L_i) \geq 12$ for some L_i in H. Say without loss of generality $e(x_1x_4, L_i) \geq 6$. By Lemma 2.2(a), $|L_i| \leq 6$ and we see that $[L_i, P] \supseteq C_5$. Thus $|L_i| = 5$ and so $[P, u] \supseteq C_5$ for some $u \in V(L_i)$. It follows that $e(Q, L_i - u) = 0$ by (1). Therefore $e(x_1x_4, L_i) = 10$ and $e(u, y_1y_4) = 2$. Consequently, $[L_i, P, Q] \supseteq 2C_5$, a contradiction. Therefore $e(x_1x_ty_1y_s, D) \leq 11$. As $e(x_1x_ty_1y_s, G) \geq 12k$, $e(x_1x_ty_1y_s, L_i) \geq 13$ for some L_i in H. By Lemma 2.2(a), we get $|L_i| = 5$. Say $L_i = u_1u_2u_3u_4u_5u_1$. Assume for the moment that $e(y_1y_s, L_i) \geq 7$. Say without loss of generality $e(y_1, L_i) \geq 4$ and $\{u_1, u_2, u_3, u_4\} \subseteq N(y_1)$. By (1), we see that $e(x_1x_t, u_2u_3u_5) = 0$. Thus $e(y_1y_s, L_i) \geq 13 - 4 = 9$. Thus $e(y_1, L_i) = 5$ or $e(y_s, L_i) = 5$ and so $e(x_1x_t, L_i) = 0$, a contradiction. Therefore $e(y_1y_s, L_i) \leq 6$ and so $e(x_1x_t, L_i) \geq 7$. If $t = 3$, let $u_r \in V(L_i)$ be such that $\{u_rx_1, u_{r+1}x_3\} \subseteq E$. Then by (1), $e(y_1y_s, u_{r+2}u_{r+3}u_{r+4}) = 0$. Thus $e(y_1y_s, L_i) \leq 4$ and so $e(x_1x_3, L_i) \geq 9$. Thus there exist four such vertices u_r and so $e(y_1y_s, L_i) = 0$, a contradiction. If $t = 4$, let $u_r \in I(x_1x_4, L_i)$. By (1), $e(y_1y_s, L_i - u_r) = 0$ and so $e(x_1x_4, L_i) \geq 13 - 2 = 11$, a contradiction. Hence $t = 5$. Say without loss of generality $e(x_1, L_i) \geq e(x_5, L_i)$. Then $e(x_1, L_i) \geq 4$. If $e(x_1, L_i) = 5$, then $I(x_5y_1, L_i) = \emptyset$ and so $e(y_s, L_i) \geq 3$. Thus $y_s \to (L_i, u_a)$ for some $u_a \in V(L_i)$ and $P + u_ax_1$ is longer than P, a contradiction. Hence $e(x_1, L_i) = 4$. Say $N(x_1, L_i) = \{u_1, u_2, u_3, u_4\}$. Then $I(x_5y_1, L_i) \subseteq \{u_1, u_4\}$. As $e(y_1y_s, L_i) \geq 13 - 2 \cdot 4 = 5$, say $e(y_1, L_i) \geq 3$. By (1), $y_1 \not\to (L_i, u_r)$ for all $r \in \{1, 2, 3, 4\}$ and this implies that $N(y_1, L_i) = \{u_1, u_5, u_4\}$. Thus $s \leq 4$ and $u_5x_5 \notin E$ for otherwise $[L_i, P, Q] \supseteq C_5 \uplus P_6$. If $s = 2$, then we readily see that $e(y_2, L_i - u_5) = 0$ for otherwise $[L_i, P, Q] \supseteq C_5 \uplus P_6$ and so $e(y_1y_2, L_i) + e(x_1x_5, L_i) \leq 4 + 8 = 12$, a contradiction. If $3 \leq s \leq 4$, then $e(y_s, u_1u_5u_4) = 0$ for otherwise $[L_i, P, Q] \supseteq C_5 \uplus P_6$. It follows that $e(y_s, u_2u_3) = 2$ and $e(x_t, u_1u_2u_3u_4) = 4$. Thus $[x_1, u_1, y_1, u_5, u_4] \supseteq C_5$ and $[P - x_1, u_2u_3] \supseteq P_6$, a contradiction.

Therefore $s = 1$. If $D - V(P)$ contains two distinct vertices x and y, then we readily see that $e(xy, D) < 6$ and so $e(xy, L_i) \geq 7$ for some L_i in H. Consequently, $[L_i, x, y] \supseteq C_5 \uplus P_2$ by Lemma 2.2(a) and Lemma 2.4, contradicting the maximality of Q. Hence $|D - V(P)| = 1$. Thus $t = 5$. In this case, we readily see as above that for some L_i in H, $[L_i, x_1, y_1] \supseteq C_5 \uplus P_2$. Thus G has a minimal chain σ' such that $G - V(\sigma') \supseteq P_4 \uplus P_2$. Say $\sigma' = (L'_1, L'_2, \ldots, L'_{k_0})$. Let $P' = z_1z_2z_3z_4$ and $Q' = v_1v_2$ be two disjoint paths in $G - V(\sigma')$. As $G - V(\sigma') \not\supseteq C_{\geq 5}$ and $G - V(\sigma') \not\supseteq P_6$, we see that $e(z_1z_4v_1v_2, P' \cup Q') \leq 8$. Thus $e(z_1z_4v_1v_2, L'_i) \geq 13$ for some $1 \leq i \leq k_0$. By Lemma 2.2(a), $|L'_i| = 5$. If there exists $u \in I(z_1z_4, L'_i)$ then $e(v_1v_2, u^-u^+) = 0$ by the maximality of P. Thus $e(v_1v_2, L'_i) \leq 6$ and so $e(z_1z_4, L'_i) \geq 7$. Then $i(z_1z_4, L'_i) \geq 2$ and we see that $e(v_1v_2, w) = 0$ for some $w \in V(L'_i) - \{u^-, u^+\}$ for the same reason. Thus $e(v_1v_2, L'_i) \leq 4$ and so $e(z_1z_4, L'_i) \geq 9$. Consequently, $e(v_1v_2, L'_i) = 0$ for the same reason, a contradiction. Hence $i(z_1z_4, L'_i) = 0$ and so $e(v_1v_2, L'_i) \geq 13 - 5 = 8$. Let uvw be a path on L'_i with $\{uv_1, wv_2\} \subseteq E$. Then $v_1uvwv_2v_1$ is a C_5 in G and so $e(z_1z_4, V(L'_i) - \{u, v, w\}) = 0$ by the maximality of P. Thus $e(z_1z_4, L'_i) \leq 3$ and so $e(v_1v_2, L'_i) = 10$. Then for the same reason, we see that $e(z_1z_4, L'_i) = 0$, a contradiction. ∎

Claim 2. $e(x_1, P) = 1$ or $e(x_t, P) = 1$.

Proof of Claim 2. On the contrary, say $e(x_1, P) > 1$ and $e(x_t, P) > 1$. Let h be maximal with $x_1x_h \in E$ and s be minimal with $x_tx_s \in E$. As $D \not\supseteq C_{\geq 5}$, $3 \leq h \leq s \leq t - 2$. Let a be the smallest integer and b be the largest integer such that $a \geq 2$, $b \leq t - 1$ and $\{x_1x_{a+1}, x_tx_{b-1}\} \subseteq E$. Set $R = \{x_1, x_a, x_b, x_t\}$. If $e(R, L_i) \geq 13$ for some L_i in H, then $|L_i| = 5$ by Lemma 2.2(a) and the minimality of σ, and consequently $[L_i, P]$ contains two disjoint feasible cycles by Corollary 2.7, a contradiction. Therefore $e(R, L_i) \leq 12$ for all L_i in H. By the maximality of P, $e(R, D - V(P)) = 0$. Thus $e(R, P) = e(R, D) \geq 12k - 12k_0 \geq 12$. As $D \not\supseteq C_{\geq 5}$, it follows that $e(x_i, P) = 3$ for all $x_i \in R$ and $[x_1, x_2, x_3, x_4] \cong [x_{t-3}, x_{t-2}, x_{t-1}, x_t] \cong K_4$. Thus $k_0 = k - 1$ and $e(R, L_j) = 12$ for all L_j in H. By Lemma 2.2(a), we readily see that $|L_{k-1}| = 5$. Say without loss of generality that $e(x_1x_2, L_{k-1}) \geq 6$. Let $u \in I(x_1x_2, L_{k-1})$. Then $[x_1, x_2, x_3, x_4, u] \supseteq C_5$. Thus $[L_{k-1} - u, x_{t-2}, x_{t-1}, x_t] \not\supseteq C_{\geq 5}$. Thus $e(x_i, L_{k-1} - u) \leq 3$ for each $i \in \{t-1, t\}$. In addition, if $e(x_i, L_{k-1} - u) > 0$ for all $i \in \{t-1, t\}$ then $e(x_{t-1}x_t, L_{k-1} - u) = 2$. Hence $e(x_{t-1}x_t, L_{k-1}) \leq 5$. Similarly, if $i(x_{t-1}x_t, L_{k-1}) \neq 0$ then $e(x_1x_2, L_{k-1}) \leq 5$ and so $e(R, L_{k-1}) \leq 10$, a contradiction. Thus $i(x_{t-1}x_t, L_{k-1}) = 0$ and in particular, $e(u, x_{t-1}x_t) \leq 1$ and so $e(x_{t-1}x_t, L_{k-1}) \leq 4$. Thus $e(x_1x_2, L_{k-1}) \geq 8$. Say without loss of generality $e(x_1, L_{k-1}) \geq e(x_2, L_{k-1})$. Then $e(x_1, L_{k-1}) \geq 4$. It follows that $x_1 \to (L_{k-1}, v)$ for some $i \in \{t-1, t\}$ and $v \in I(x_2x_i, L_{k-1})$, i.e., $[L_{k-1}, P] \supseteq C_5 \uplus C_{\geq 5}$, a contradiction. ∎

For the proof of the theorem, we now choose, subject to (1), σ and $P = x_1 \ldots x_t$ in D with descending priorities such that the following two conditions hold:

$$r(P) \text{ is maximal;} \tag{2}$$

$$\sum_{i=1}^{k_0} \tau(L_i) \text{ is maximal.} \tag{3}$$

Let $R = \{x_1, x_2, x_{t-1}, x_t\}$. Clearly, $e(x_1x_t, D - V(P)) = 0$. If $x_2u \in E$ for some $u \in V(D) - V(P)$, then $e(x_1u, D) = e(x_1u, P) \leq 4$ as $D \not\supseteq C_{\geq 5}$. Consequently, $e(x_1u, H) \geq 6k - 4 = 6(k-1) + 2$ and so $e(x_1u, L_i) \geq 7$ for some L_i in H. By Lemma 2.2(a) and the minimality of σ, we see that $|L_i| = 5$. By Lemma 2.4, we see that $[L_i, P + u]$ contains a 5-cycle and a path of order $t + 1$ such that they are disjoint, contradicting the maximality of P. Hence $e(x_2, D - V(P)) = 0$. Similarly, $e(x_{t-1}, D - V(P)) = 0$.

As $D \not\supseteq C_{\geq 5}$, it is easy to see that $e(x_2, P) \leq 4$, $e(x_{t-1}, P) \leq 4$, $e(x_1x_2, P) \leq 6$ and $e(x_{t-1}x_t, P) \leq 6$. If $e(R, P) \geq 12$, then we would have that $e(x_1, P) \geq 2$ and $e(x_t, P) \geq 2$, contradicting Claim 2. Therefore $e(R, D) = e(R, P) < 12$. Thus $e(R, L_r) \geq 13$ for some L_r in H. By Lemma 2.2(a) and minimality of σ, we see that $|L_r| = 5$. Say without loss of generality that $L_r = L_1 = a_1a_2a_3a_4a_5a_1$. The following six properties will be used to complete our proof. For convenience in the following, we will resort to the definition of R_t^i in the introduction. Since $t \geq 6$ and $[L_1, P] \not\supseteq 2C_{\geq 5}$, we immediately have the following Property 1:

Property 1. *For each $u \in V(L_1)$, if $x_1 \to (L_1, u)$ then $e(u, x_2x_{t-1}) \leq 1$ and $e(u, x_2x_t) \leq 1$, and for each $v \in V(L_1)$, if $x_t \to (L_1, v)$ then $e(v, x_1x_{t-1}) \leq 1$ and $e(v, x_2x_{t-1}) \leq 1$.* ∎

Property 2. *There is no $i \in \{1, 2, 3, 4, 5\}$ such that $N(x_1x_tx_2, L_1) \subseteq \{a_i, a_{i+2}, a_{i+3}\}$ or $N(x_1x_tx_{t-1}, L_1) \subseteq \{a_i, a_{i+2}, a_{i+3}\}$.*

Proof of Property 2. On the contrary, say without loss of generality that $N(x_1x_tx_{t-1}, L_1) \subseteq \{a_1, a_3, a_4\}$. Since $e(R, L_1) \geq 13$, we see that $e(x_2, L_1) \geq 4$ and $8 \leq e(x_1x_tx_{t-1}, L_1) \leq 9$. It is easy to see that $x_2 \to (L_1, a_i)$ for some $a_i \in I(x_1x_t, \{a_1, a_3, a_4\})$. Thus $e(x_1, P) = 1$ for otherwise $[P - x_2 + a_i] \supseteq C_{\geq 5}$. It is also clear that $x_2 \to (L_1, a_j)$ for some $a_j \in I(x_{t-1}x_t, L_1)$. As $[P - x_2 + a_j] \not\supseteq C_{\geq 5}$, this implies that $r(P) \leq 3$, i.e., $x_tx_{t-3} \notin E$. Clearly, $[a_1, a_5, a_4, x_{t-1}, x_t] \supseteq C_5$ and $[a_1, a_2, a_3, x_{t-1}, x_t] \supseteq C_5$. Then neither of $[P - x_{t-1} - x_t, a_2a_3]$ and $[P - x_{t-1} - x_t, a_4a_5]$ contains R_t^4 by (2). This implies that $e(x_1x_2, a_2a_3) \leq 2$ and $e(x_1x_2, a_4a_5) \leq 2$. Consequently, $e(R, L_1) \leq 12$, a contradiction. ∎

Property 3. $e(x_1, L_1) < 5$ and $e(x_t, L_1) < 5$.

Proof of Property 3. Say $e(x_1, L_1) = 5$. Then $x_1 \to L_1$. By Property 1, $i(x_2x_{t-1}, L_1) = 0$ and $i(x_2x_t, L_1) = 0$. Thus $e(x_{t-1}, L_i) \geq 13 - 5 - e(x_2x_t, L_1) \geq 3$ and $e(x_t, L_i) \geq 13 - 5 - e(x_2x_{t-1}, L_1) \geq 3$. If $e(x_t, L_1) \geq 4$, then we readily see that $x_t \to (L_1, a_i)$ and $e(a_i, x_{t-1}x_1) = 2$ for some $a_i \in V(L_1)$, a contradiction. Hence $e(x_t, L_1) = 3$. First, assume that $N(x_t, L_1) = \{a_i, a_{i+1}, a_{i+2}\}$ for some $a_i \in V(L_1)$. Say $N(x_t, L_1) = \{a_1, a_2, a_3\}$. By Property 1, $e(x_2, a_1a_2a_3) = 0$ and $x_{t-1}a_2 \notin E$, and so $e(x_2x_{t-1}, L_1) \leq 4$. Consequently, $e(R, L_1) \leq 12$, a contradiction. Hence $N(x_t, L_1) = \{a_i, a_{i+2}, a_{i+3}\}$ for some $a_i \in V(L_1)$. Say $N(x_t, L_1) = \{a_1, a_3, a_4\}$. Then $e(x_2, a_1a_3a_4) = 0$ and $e(x_{t-1}, a_2a_5) = 0$. This implies that $e(x_2, a_2a_5) = 2$ and $e(x_{t-1}, a_1a_3a_4) = 3$. Then $[L_1, P] \supseteq 2C_{\geq 5} = \{x_1a_5a_1x_ta_4x_1, x_2 \ldots x_{t-1}a_3a_2x_2\}$, a contradiction. ∎

Property 4. $e(x_1, L_1) < 4$ and $e(x_t, L_1) < 4$.

Proof of Property 4. Say $e(x_1, a_1a_2a_3a_4) = 4$. Then $x_1 \to (L_1, a_i)$ for each $i \in \{2, 3, 5\}$. Thus $i(x_2x_{t-1}, a_2a_3a_5) = 0$ and $i(x_2x_t, a_2a_3a_5) = 0$. This implies that $e(x_2x_{t-1}, L_1) \leq 7$ and $e(x_2t_t, L_1) \leq 7$. Consequently, $e(x_t, L_1) \geq 2$ and $e(x_{t-1}, L_1) \geq 2$.

First, assume that $e(x_t, L_1) = 4$. Then it is easy see that $[L_1 - a_i, x_1, x_t] \supseteq C_5$ for all $a_i \in V(L_1)$, and so $i(x_2x_{t-1}, L_1) = 0$. It follows that $e(x_2x_{t-1}, L_1) = 5$ and so $e(a_i, x_2x_{t-1}) = 1$ for all $a_i \in V(L_1)$. Thus by Property 1, for all $a_i \in V(L_1)$, if $e(a_i, x_1x_t) = 2$, then $x_1 \not\to (L_1, a_i)$ or $x_t \not\to (L_1, a_i)$. This implies that $a_rx_t \notin E$ for some $r \in \{2, 3\}$. Say $a_r = a_2$. Since $x_t \to (L_1, a_2)$ and $x_1 \to (L_1, a_3)$, it follows that $\{x_2a_2, x_{t-1}a_3\} \subseteq E$. Consequently, $[L_1, P] \supseteq 2C_{\geq 5} = \{x_1a_1x_ta_5a_4x_1, a_2x_2 \ldots x_{t-1}a_3a_2\}$, a contradiction.

Next, assume that $e(x_t, L_1) = 3$. Then $e(x_2x_{t-1}, L_1) \geq 6$ and by Property 1, $I(x_2x_{t-1}, L_1) \subseteq \{a_1, a_4\}$ and so $i(x_2x_{t-1}, a_1a_4) \geq 1$. Say $e(a_1, x_2x_{t-1}) = 2$. Then $[a_1, x_2, \ldots, x_{t-1}] \supseteq C_{\geq 5}$. Thus $[x_1, a_2, a_3, a_4, x_t] \not\supseteq C_5$ and so $e(x_t, a_2a_3a_4) \leq 1$. It follows that $e(x_t, a_1a_5a_4) = 3$ as $[L_1 - a_1, x_1, x_t] \not\supseteq C_5$. By Property 1, $x_2a_5 \notin E$. As $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$, $[x_3, \ldots, x_t, a_4, a_5] \not\supseteq C_{\geq 5}$. As $r(x_3 \ldots x_ta_4a_5) \geq 3$ and by (2), $r(P) \geq 3$. Assume first $e(x_t, P) \geq 2$. Then $[x_3, \ldots, x_t, a_1, a_5] \supseteq C_{\geq 5}$ and so $[x_1, x_2, a_2, a_3, a_4] \not\supseteq C_5$. This yields $e(x_2, a_2a_4) = 0$ and so $I(x_2x_{t-2}, L_1) \subseteq \{a_1\}$. As $e(x_2x_{t-1}, L_1) \geq 6$, it follows that $e(x_{t-1}, a_2a_4a_5) = 3$ and $e(a_3, x_2x_{t-1}) = 1$. Thus $[x_3, \ldots, x_t, a_4, a_5] \supseteq C_{\geq 5}$, a contradiction.

Therefore $e(x_t, P) = 1$ and so $e(x_1, P) \geq 2$. As $[x_t, a_4, a_5, a_1, x_{t-1}] \supseteq C_5$, $[a_3, a_2, x_1, \ldots, x_{t-2}] \not\supseteq C_{\geq 5}$ and so $e(x_2, a_2a_3) = 0$. As $[a_1, a_2, x_1, \ldots, x_{t-2}] \supseteq C_{\geq 5}$, $x_{t-1}a_3 \notin E$. As $e(x_2x_{t-1}, L_1) \geq 6$, it follows that $x_2a_4 \in E$ and $e(x_{t-1}, a_2a_4a_5) = 3$. Thus $[x_{t-1}, x_t, a_5, a_1, a_2] \supseteq C_5$ and $[a_3, a_4, x_1, \ldots, x_{t-2}] \supseteq C_{\geq 5}$, a contradiction.

Finally, $e(x_t, L_1) = 2$. In this situation, $e(a_i, x_2x_{t-1}) = 1$ for $i \in \{2, 3, 5\}$ and $e(a_1a_4, x_2x_{t-1}) = 4$. By Property 1, $x_1 \not\to L_1$ and this implies $\tau(a_5, L_1) = 0$. First, suppose $x_ta_5 \in E$. Then $x_2a_5 \notin E$. By Property 1, $e(x_t, a_2a_3) = 0$. Say without loss of generality $x_ta_4 \in E$. Clearly, $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$ and $r(x_3 \ldots x_ta_5a_4) = 4$. By (2), $r(P) = 4$. If $e(x_t, P) \geq 2$, then $[x_3, \ldots, x_t, a_4, a_5] \supseteq C_{\geq 5}$, a contradiction. Hence $e(x_t, P) = 1$ and so $x_1x_4 \in E$. Then we see that $[a_1, x_1 \ldots x_{t-2}] \supseteq C_5$ and so $e(x_{t-1}, a_2a_3) = 0$. Thus $e(x_2, a_2a_3) = 2$ and by(3), $\tau(L_1) \geq \tau(x_1x_2a_1a_2a_3x_1) \geq 4$ and so $\tau(a_5, L_1) > 0$, a contradiction. Hence $x_ta_5 \notin E$.

Next, suppose $e(x_t, a_1a_3) = 2$ or $e(x_t, a_2a_4) = 2$, say $e(x_t, a_2a_4) = 2$. Then $a_3x_{t-1} \notin E$ and $a_2x_2 \notin E$ by Property 1. Thus $a_3x_2 \in E$ and $a_2x_{t-1} \in E$. As $[x_{t-1}, x_t, a_4, a_5, a_1] \supseteq C_5$, $[x_1, \ldots, x_{t-2}, a_2a_3] \not\supseteq C_{\geq 5}$. This implies that $e(x_1, P) = 1$. As $r(x_{t-2} \ldots x_1a_2a_3) = 4$ and by (2), we obtain that $r(P) = 4$, i.e., $x_tx_{t-3} \in E$. Thus $[P - x_1, a_2] \supseteq C_5$ with $x_1 \to (L_1, a_2)$, a contradiction.

Next, suppose $e(x_t, a_1a_2) = 2$ or $e(x_t, a_3a_4) = 2$, say $e(x_t, a_3a_4) = 2$. Then $x_2a_3 \notin E$ by Property 1 and so $x_{t-1}a_3 \in E$. Since $[x_{t-1}, x_t, a_4, a_5, a_1] \supseteq C_5$ and $r(x_{t-2} \ldots x_1a_2a_3) \geq 3$, we see that either $e(x_1, P) \geq 2$ or $e(x_t, P) \geq 2$ by (2). If $e(x_t, P) \geq 2$ then $[x_3, \ldots, x_t, a_3, a_4] \supseteq C_{\geq 5}$ and so $[a_5, a_1, a_2, x_1, x_2] \not\supseteq C_5$. Consequently, $a_5x_2 \notin E$ and so $x_{t-1}a_5 \in E$. Then $[x_3, \ldots, x_t, a_4, a_5] \supseteq C_{\geq 5}$ and $[x_1, a_3, a_2, a_1, x_2] \supseteq C_5$, a contradiction. Hence $e(x_t, P) = 1$ and $e(x_1, P) \geq 2$. Then

$[x_{t-2}, \ldots, x_1, a_1, a_2] \supseteq C_{\geq 5}$ and so $[x_{t-1}, x_t, a_3, a_4, a_5] \not\supseteq C_5$. Thus $x_{t-1}a_5 \notin E$ and so $x_2a_5 \in E$. Thus $[x_{t-2}, \ldots, x_1, a_4, a_5] \supseteq C_{\geq 5}$ and $[a_1, a_2, a_3, x_t, x_{t-1}] \supseteq C_5$, a contradiction.

Next, suppose $e(x_t, a_2a_3) = 2$. Then $e(x_2, a_2a_3) = 0$ by Property 1 and so $e(x_{t-1}, a_2a_3) = 2$. Since $[x_1, x_2, a_1, a_5, a_4] \supseteq C_5$, we have $[x_3, \ldots, x_t, a_2, a_3] \not\supseteq C_{\geq 5}$. Since $r(x_3 \ldots x_ta_2a_3) = 4$ and by (2), $r(P) = 4$, it follows that $e(x_t, P) = 1$ and $x_1x_4 \in E$. Thus $[a_1, x_1, x_4, x_3, x_2] \supseteq C_5$ and $[x_{t-1}, x_t, a_2, a_3, a_4] \supseteq C_5$, a contradiction.

Finally, suppose $e(x_t, a_1a_4) = 2$. Since $[x_{t-1}, x_t, a_1, a_5, a_4] \supseteq C_5$, it follows that $[x_1, \ldots, x_{t-2}, a_2, a_3] \not\supseteq C_{\geq 5}$. As $r(x_{t-2} \ldots x_1a_2a_3) \geq 3$, we have $r(P) \geq 3$ by (2). Assume first that $e(x_1, P) \geq 2$. By Claim 2, $e(x_t, P) = 1$. Then $e(x_2, a_2a_3) = 0$ because $[x_1, \ldots, x_{t-2}, a_2, a_3] \not\supseteq C_{\geq 5}$. Thus $e(x_{t-1}, a_2a_3) = 2$. Consequently, $[x_{t-1}, x_t, a_4, a_3, a_2] \supseteq C_5$ and $[x_{t-2}, \ldots, x_1, a_4, a_5] \not\supseteq C_{\geq 5}$, which implies $a_5x_2 \notin E$. Thus $x_{t-1}a_5 \in E$. This yields that $r(x_3 \ldots x_ta_4a_5) = 4$. As $[x_1, a_3, a_2, a_1, x_2] \supseteq C_5$, it follows by (2) that $x_1x_4 \in E$. Thus $[x_1, x_4, x_3, x_2, a_1] \supseteq C_5$, a contradiction. Hence $e(x_t, P) \geq 2$ and $e(x_1, P) = 1$. Since $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$, it follows that $[x_3, \ldots, x_t, a_4, a_5] \not\supseteq C_{\geq 5}$. This implies that $a_5x_{t-1} \notin E$ and $r(P) = 3$. Thus $a_5x_2 \in E$. Then $[x_1, x_2, a_5, a_1, a_2] \supseteq C_5$ and so $[x_3, \ldots, x_t, a_3, a_4] \not\supseteq C_{\geq 5}$. Thus $x_{t-1}a_3 \notin E$ and so $x_2a_3 \in E$. Then $r(x_{t-2} \ldots x_1a_2a_3) = 4$. By (2), $r(P) = 4$, a contradiction. ∎

Property 5. $e(x_1x_t, L_1) \geq 5$.

Proof of Property 5. Say $e(x_1x_t, L_1) \leq 4$. Since $e(x_2x_{t-1}, L_1) \geq 13 - e(x_1x_t, L_1) \geq 9$, we may assume without loss of generality that $e(x_2, L_1) = 5$ and $e(x_{t-1}, L_1) \geq 4$. Say $e(x_{t-1}, a_1a_2a_3a_4) = 4$. We have $x_2 \to L_1$. We claim that $e(x_1, P) = 1$. If this is false, say $e(x_1, P) \geq 2$. Then $[P - x_2 + a_i] \not\supseteq C_{\geq 5}$ for all $a_i \in V(L_1)$. This implies $i(x_1x_t, L_1) = 0$. Moreover, if $i(x_1x_{t-1}, L_1) \neq 0$, then $t = 6$ and $x_1x_4 \in E$. Assume for the moment that $i(x_1x_{t-1}, L_1) \neq 0$. As $x_5 \to (L_1, a_i)$ for $i \in \{2, 3, 5\}$, $[P - x_5, a_i] \not\supseteq C_{\geq 5}$ for $i \in \{2, 3, 5\}$ and so $e(x_1, a_2a_3a_5) = 0$. Thus $e(x_1, a_1a_4) > 0$. Say $x_1a_1 \in E$. Then $[a_1, x_1, x_4, x_3, x_2] \supseteq C_5$ and so $[x_5, x_6, a_2, a_3, a_4, a_5] \not\supseteq C_{\geq 5}$. Thus $e(x_6, a_2a_4a_5) = 0$. It follows that $e(x_1x_6, L_1) = 3$ and $e(x_5, L_1) = 5$ with $e(x_1, a_1a_4) = 2$ and $x_6a_3 \in E$ and we readily see that $[L_1, P] \supseteq 2C_5$, a contradiction. Therefore $i(x_1x_{t-1}, L_1) = 0$. As $e(P, L_1) \geq 13$, $e(x_t, L_1) \geq 3$. By Property 1, we see that $N(x_t, L_1) = \{a_1, a_5, a_4\}$ and $x_{t-1}a_5 \notin E$. Thus $i(x_1x_{t-1}, L_1) \neq 0$ or $i(x_1x_t, L_1) \neq 0$, a contradiction. Hence $e(x_1, P) = 1$.

Next, we claim that $e(x_{t-1}, L_1) = 4$. If this is false, say $e(x_{t-1}, L_1) = 5$. Then we also have $e(x_t, P) = 1$. By Property 1, $x_1 \not\to (L_1, a_i)$ and $x_t \not\to (L_1, a_i)$ for $a_i \in V(L_1)$, which implies that $e(x_1, L_1) \leq 2$ and $e(x_t, L_1) \leq 2$. Say without loss of generality $x_1a_1 \in E$. Then $[x_1, a_1, a_2, a_3, x_2] \supseteq C_5$. Then $e(x_t, a_4a_5) = 0$ for otherwise $[x_3, \ldots, x_t, a_4, a_5] \supseteq R_t^4$, contradicting (2). Similarly, $e(x_t, a_2a_3) = 0$. Thus $e(x_1, L_1) \geq 2$. Then $x_1a_j \in E$ for some $j \neq 1$. With a_j in place of a_1 in the above, we see that $x_ta_1 \notin E$. a contradiction.

Therefore $e(x_{t-1}, L_1) = 4$ and so $e(x_1x_t, L_1) = 4$. Suppose that $x_1a_5 \in E$. Then $[x_1, a_5, a_1, a_2, x_2] \supseteq C_5$. If $e(x_t, a_3a_4) > 0$ then $[x_3, \ldots, x_t, a_3, a_4] \supseteq R_t^4$. By (2), $r(P) = 4$, i.e., $x_tx_{t-3} \in E$. Thus $[x_3, \ldots, x_t, a_3, a_4] \supseteq C_{\geq 5}$, a contradiction. Hence

$e(x_t, a_3a_4) = 0$. Similarly, $e(x_t, a_1a_2) = 0$. Thus $a_5x_t \in E$ and $e(x_1, L_1) = 3$. By Property 1, $x_1 \not\to (L_1, a_i)$ for each $i \in \{1, 2, 3, 4\}$ and so $e(x_1, a_1a_5a_4) = 3$. Thus $x_1 \to (L_1, a_5)$ and $[P - x_1, a_5] \supseteq C_t$, a contradiction. Hence $x_1a_5 \notin E$. Thus $e(x_1, L_1 - a_5) \geq 1$ and so either $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$ or $[x_1, x_2, a_2, a_3, a_4] \supseteq C_5$. Say without loss of generality $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$. If $x_ta_5 \in E$, then $r(x_3 \ldots x_ta_5a_4) = 4$. Consequently, $r(P) = 4$ by (2) and so $[x_3, \ldots, x_t, a_5, a_4] \supseteq C_{\geq 5}$, a contradiction. Hence $x_ta_5 \notin E$. As $x_1 \not\to (L_1, a_i)$ and $x_t \not\to (L_1, a_i)$ for each $i \in \{1, 2, 3, 4\}$, it follows that $e(x_1, L_1 - a_5) = e(x_t, L_1 - a_5) = 2$. Assume for the moment that $e(x_1, a_1a_4) > 0$. Say without loss of generality that $x_1a_1 \in E$. Then $[x_1, x_2, a_4, a_5, a_1] \supseteq C_5$. This implies that $e(x_t, a_2a_3) = 0$ for otherwise $[x_3, \ldots, x_t, a_2, a_3] \supseteq R_t^4$, which implies that $x_tx_{t-3} \in E$ by (2) and so $[x_3, \ldots, x_t, a_2, a_3] \supseteq C_{\geq 5}$, a contradiction. Hence $e(x_t, a_1a_4) = 2$. Thus $[a_2, a_3, a_4, x_{t-1}, x_t] \supseteq C_5$ and $r(x_{t-2}x_{t-3} \ldots x_1a_1a_5) = 4$. By (2), $x_tx_{t-3} \in E$. Consequently, $[a_4, x_t, x_{t-3}, x_{t-2}, x_{t-1}] \supseteq C_5$ and $[x_1, a_1, a_2, a_3, x_2] \supseteq C_5$, a contradiction. Hence $e(x_1, a_1a_4) = 0$ and so $e(x_1a_2a_3) = 2$. As $e(x_t, L_1 - a_5) = 2$, say without loss of generality $e(x_t, a_3a_4) \geq 1$. Then $[x_3 \ldots x_t, a_3a_4] \supseteq R_t^4$. As $[x_1, a_2, a_1, a_5, x_2] \supseteq C_5$ and by (2), $r(P) = 4$, i.e., $x_tx_{t-3} \in E$ and so $[x_3 \ldots x_t, a_3a_4] \supseteq C_{\geq 5}$, a contradiction. ∎

By the above properties, we may assume that $e(x_1, L_1) = 3$ and $2 \leq e(x_t, L_1) \leq 3$. Then $e(x_2x_{t-1}, L_1) \geq 13 - e(x_1x_t, L_1) \geq 7$.

Property 6. $N(x_1, L_1) = \{a_i, a_{i+2}, a_{i+3}\}$ for some $a_i \in V(L_1)$.

Proof of Property 6. On the contrary, say the property does not hold. Then $N(x_1, L_1)$ contains three consecutive vertices of L_1. Say $N(x_1, L_1) = \{a_1, a_2, a_3\}$. By Property 1, $e(a_2, x_2x_{t-1}) \leq 1$ and $e(a_2, x_2x_t) \leq 1$. Thus $e(x_2x_{t-1}, a_4a_5) \geq 7 - e(x_2x_{t-1}, a_1a_2a_3) \geq 2$. As $e(x_t, L_1) \geq 2$, either $N(x_t, L_1) \supseteq \{a_i, a_{i+2}\}$ for some $a_i \in V(L_1)$ or $N(x_t, L_1) = \{a_i, a_{i+1}\}$ for some $a_i \in V(L_1)$. We divide the proof into the following cases.

Case 1. For some $a_i \in V(L_1)$, $\{a_i, a_{i+2}\} \subseteq N(x_t)$.

First, assume that $\{a_1, a_3\} \subseteq N(x_t)$. Then $x_t \to (L_1, a_2)$ and $[x_1, a_1, x_t, a_3, a_2] \supseteq C_5$. Thus $x_{t-1}a_2 \notin E$ and either $e(x_2, a_4a_5) = 0$ or $e(x_{t-1}, a_4a_5) = 0$. It follows that $e(x_2x_{t-1}, a_1a_3) = 4$, $x_2a_2 \in E$, $e(x_2x_{t-1}, a_4a_5) = 2$ and $e(x_t, L_1) = 3$. Clearly, $x_ta_2 \notin E$ as $x_2a_2 \in E$. Thus $e(x_t, a_4a_5) = 1$. Say without loss of generality that $x_ta_4 \in E$. As $[x_{t-1}, a_1, a_5, a_4, x_t] \supseteq C_5$ and $r(x_{t-2} \ldots x_1a_3a_2) = 4$ we have $r(P) = 4$ by (2). This implies that $x_tx_{t-3} \in E$ for otherwise $[x_1x_4x_3x_2a_2] \supseteq C_5$. If $e(x_2, a_4a_5) > 0$, then $[x_1, a_1, a_5, a_4, x_2, a_2] \supseteq C_{\geq 5}$ and $[a_3, x_t, x_{t-3}, x_{t-2}, x_{t-1}] \supseteq C_5$, a contradiction. Hence $e(x_2, a_4a_5) = 0$ and so $e(x_{t-1}, a_4a_5) = 2$ as $e(x_2x_{t-1}, L_1) \geq 7$. Then $[a_4, x_t, x_{t-3}, x_{t-2}, x_{t-1}] \supseteq C_5$ and $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$, a contradiction.

Next, assume that $\{a_1, a_4\} \subseteq N(x_t)$ or $\{a_3, a_5\} \subseteq N(x_t)$. Say without loss of generality $\{a_3, a_5\} \subseteq N(x_t)$. Then $x_t \to (L_1, a_4)$ and so $e(a_4, x_2x_{t-1}) \leq 1$. As $e(a_2, x_2x_{t-1}) \leq 1$, $e(x_2x_{t-1}, a_1a_3a_5) \geq 7 - 2 = 5$. Thus $e(x_{t-1}, a_3a_5) \geq 1$ and so $[x_{t-1}, x_t, a_3, a_4, a_5] \supseteq C_5$. Clearly, $r(x_{t-2}x_{t-3} \ldots x_1a_i) \geq 3$ for $i \in \{1, 2\}$. By (2), $r(P) \geq 3$. For the moment, assume $e(x_1, P) \geq 2$. Then $e(x_2, a_1a_2) = 0$ for otherwise $[x_1, x_2, \ldots, x_{t-2}, a_1, a_2] \supseteq C_{\geq 5}$. This yields that $e(x_2, a_3a_5) = 2$, $e(x_{t-1}, a_1a_2a_3a_5) = 4$

and $e(a_4, x_2x_{t-1}) = 1$. Consequently, $e(x_t, L_1) = 3$. By Property 1, $x_t \not\to (L_1, a_i)$ for $i \in \{1, 2\}$ and it follows that $e(x_t, a_1a_2) = 0$. As $[a_1, a_5, x_1, x_2, \ldots, x_{t-2}] \supseteq C_{\geq 5}$, $[a_2, a_3, a_4, x_{t-1}, x_t] \not\supseteq C_5$ and so $x_ta_4 \notin E$. Thus $e(x_t, L_1) = 2$, a contradiction. Therefore $e(x_1, P) = 1$ and so $e(x_t, P) \geq 2$. As $e(x_2x_{t-1}, a_1a_3a_5) \geq 5$, we see that either $\{x_{t-1}a_3, x_2a_5\} \subseteq E$ or $\{x_{t-1}a_5, x_2a_3\} \subseteq E$. In the former situation $[x_1, x_2, a_5, a_1, a_2] \supseteq C_5$ and in the latter $[x_1, x_2, a_3, a_2, a_1] \supseteq C_5$. This means that $[a_3, x_t, x_{t-1}, \ldots, x_3] \not\supseteq C_{\geq 5}$ and $[a_5, x_t, x_{t-1}, \ldots, x_3] \not\supseteq C_{\geq 5}$. It follows that $e(x_t, P) = 2$ and $x_tx_{t-2} \in E$. As $[x_{t-1}, x_t, a_3, a_4, a_5] \supseteq C_5$ and by (2), we must have that $r(x_{t-2}x_{t-3} \ldots x_1a_1a_2) = 3$. This yields that $e(x_2, a_1a_2) = 0$. It follows that $e(x_2, a_3a_5) = 2$, $e(x_{t-1}, a_1a_2a_3a_5) = 4$, $e(a_4, x_2x_{t-1}) = 1$ and $e(x_t, L_1) = 3$. As $[x_1, x_2, a_3, a_2, a_1] \supseteq C_5$, we see that $[a_4, a_5, x_t, x_{t-1}, x_{t-2}] \not\supseteq C_5$ and so $a_4x_{t-1} \notin E$. Thus $x_2a_4 \in E$. Consequently, $[x_1, a_1, a_5, a_4, x_2] \supseteq C_5$ and $[a_2, x_{t-1}, x_{t-2}, x_t, a_3] \supseteq C_5$, a contradiction.

Finally, assume that $\{a_2, a_4\} \subseteq N(x_t)$ or $\{a_2, a_5\} \subseteq N(x_t)$. Say without loss of generality $\{a_2, a_4\} \subseteq N(x_t)$. Then we readily see that $x_2a_2 \notin E$ and $x_{t-1}a_3 \notin E$. Thus $e(x_2x_{t-1}, L_1) \leq 8$. As $e(x_2x_{t-1}, L_1) \geq 7$, it follows that either $\{x_2a_5, x_{t-1}a_2\} \subseteq E$ or $\{x_2a_1, x_{t-1}a_5\} \subseteq E$. Then as above it is easy to see that $[L_1, P] \supseteq C_5 \uplus R_t^4$. By (2), $r(P) = 4$. Then in each of the two situations, we readily see that $[L_1, P] \supseteq C_5 \uplus C_{\geq 5}$, a contradiction.

Case 2. $N(x_t, L_1) = \{a_i, a_{i+1}\}$ for some $a_i \in V(L_1)$.

In this case, $e(x_2x_{t-1}, L_1) \geq 13 - 3 - 2 = 8$. First, assume that $N(x_t, L_1) \subseteq \{a_1, a_2, a_3\}$. Then $[x_1, x_t, a_1, a_2, a_3] \supseteq C_5$. Since $e(a_2, x_2x_{t-1}) \leq 1$, we see that $e(x_2, a_4a_5) \geq 1$ and $e(x_{t-1}, a_4a_5) \geq 1$ and so $[x_2, \ldots, x_{t-1}, a_4, a_5] \supseteq C_{\geq 5}$, a contradiction.

Next, asssume that $N(x_t, L_1) = \{a_1, a_5\}$ or $N(x_t, L_1) = \{a_3, a_4\}$. Say without loss of generality $N(x_t, L_1) = \{a_3, a_4\}$. As $e(a_2, x_2x_{t-1}) \leq 1$, we see that either $\{x_2a_5, x_{t-1}a_3\} \subseteq E$ or $\{x_2a_1, x_{t-1}a_5\} \subseteq E$. Thus $[L_1, P] \supseteq C_5 \cup R_t^4$ with x_1 and x_2 on the 5-cycle. Furthermore, we see that $e(x_t, P) = 1$ as $[P, L_1] \not\supseteq C_5 \uplus C_{\geq 5}$. By (2), $r(P) = 4$ and so $x_1x_4 \in E$. We also have either $\{x_2a_1, x_{t-1}a_5\} \subseteq E$ or $\{x_2a_3, x_{t-1}a_1\} \subseteq E$. With $x_1x_4 \in E$, we then readily see that $[L_1, P] \supseteq C_5 \uplus C_{\geq 5}$, a contradiction.

Finally, assume that $N(x_t, L_1) = \{a_4, a_5\}$. Clearly, $e(x_2, a_1a_3) \geq 1$ and $e(x_{t-1}, a_4a_5) \geq 1$. Then we readily see that $[x_1, x_2, a_1, a_2, a_3] \supseteq C_5$ and $[x_3, \ldots, x_t, a_4, a_5] \supseteq R_t^4$. Moreover, we see that $e(x_t, P) = 1$ as $[x_3, \ldots, x_t, a_3, a_4] \not\supseteq C_{\geq 5}$. By (2), we obtains $x_1x_4 \in E$. We have either $\{x_2a_1, x_{t-1}a_3\} \subseteq E$ or $\{x_2a_3, x_{t-1}a_1\} \subseteq E$. Then we see that $[L_1, P] \supseteq 2C_5$, a contradiction. ∎

We are ready to complete the proof of the theorem. By the above properties, we see that $e(x_1, L_1) = \{a_i, a_{i+2}, a_{i+3}\}$ for some $a_i \in V(L_1)$ and $2 \leq e(x_t, L_1) \leq 3$. Say without loss of generality that $N(x_1, L_1) = \{a_1, a_3, a_4\}$. Then $e(x_2x_{t-1}, L_1) \geq 7$, $e(a_2, x_2x_{t-1}) \leq 1$ and $e(a_5, x_2x_{t-1}) \leq 1$.

First, assume that $e(x_t, L_1) = 3$. With x_t playing the role of x_1, we see, by Property 6, that $N(x_t, L_1) = \{a_j, a_{j+2}, a_{j+3}\}$ for some $a_j \in V(L_1)$. If $j = 1$, then by Property 2, we see that $e(x_2, a_2a_5) \geq 1$, $e(x_{t-1}, a_2a_5) \geq 1$ and there are two independent edges between $\{a_2, a_5\}$ and $\{x_2, x_{t-1}\}$. Say $\{x_2a_2, x_{t-1}a_5\} \subseteq E$. Then $[L_1, P] \supseteq C_5 \cup R_t^4$. By (2), $r(P) = 4$ and consequently, $[L_1, P] \supseteq C_5 \cup C_{\geq 5}$, a contradiction. If $j \in \{2, 5\}$, say without loss of generality that $j = 2$. Then $e(a_1, x_2x_{t-1}) \leq 1$ and $e(a_3, x_2x_{t-1}) \leq 1$. Thus $e(x_2x_{t-1}, L_1) = e(x_2x_{t-1}, a_2a_5) + e(x_2x_{t-1}, a_1a_3) + e(x_2x_{t-1}, a_4) \leq 2 + 2 + 2 = 6$, a contradiction.

Therefore $j \in \{3, 4\}$. Say without loss of generality that $j = 3$. Then $e(a_4, x_2x_{t-1}) \leq 1$. It follows that $e(x_2x_{t-1}, a_1a_3) = 4$ and $e(a_p, x_2x_{t-1}) = 1$ for $p \in \{2, 4, 5\}$. Then $[a_1, a_2, x_2, \ldots, x_{t-1}] \supseteq C_{\geq 5}$ and $[x_1, a_3, x_t, a_5, a_4] \supseteq C_5$, a contradiction.

Therefore $e(x_t, L_1) = 2$ and so $e(x_2x_{t-1}, a_1a_3a_4) = 6$, $e(a_2, x_2x_{t-1}) = 1$ and $e(a_5, x_2x_{t-1}) = 1$. Assume for the moment that $N(x_t, L_1) \not\subseteq \{a_1, a_3, a_4\}$. Say without loss of generality that $a_2x_t \in E$. Then $[x_1, x_2, a_4, a_5, a_1] \supseteq C_5$ and $[x_3, \ldots, x_t, a_2, a_3] \supseteq R_t^4$. By (2), $r(P) = 4$, and consequently, $[L_1, P] \supseteq C_5 \uplus C_{\geq 5}$, a contradiction. Hence $e(x_t, L_1) \subseteq \{a_1, a_3, a_4\}$ and so $e(x_t, a_3a_4) \geq 1$. If $e(x_{t-1}, a_2a_5) = 2$, we readily see that $[L_1, P] \supseteq C_5 \cup R_t^4$. Consequently, $r(P) = 4$ and $[L_1, P] \supseteq C_5 \cup C_{\geq 5}$, a contradiction. Therefore $e(x_{t-1}, a_2a_5) \leq 1$ and so $e(x_2, a_2a_5) \geq 1$. We may assume without loss of generality that $a_2x_2 \in E$. As $e(x_t, a_3a_4) \geq 1$ and $e(a_5, x_2x_{t-1}) = 1$, we readily see that $[L_1, P] \supseteq C_5 \cup R_t^4$ and so $r(P) = 4$. Again, we see that $[L_1, P] \supseteq C_5 \cup C_{\geq 5}$, a contradiction. This proves the theorem.

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