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Parking cars after a trailer

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Abstract

Recently, the authors extended the notion of parking functions to parking sequences, which include cars of different sizes, and proved a product formula for the number of such sequences. We give here a refinement of that result involving parking the cars after a trailer. The proof of the refinement uses a multi-parameter extension of the Abel–Rothe polynomial due to Strehl.

1 Introduction

Parking sequences were introduced in [3] as an extension of the classical notion of parking functions, where we now take into account parking cars of different sizes. This extension differs from other extensions of parking functions [1, 5, 6, 7, 11] since the parking sequences are not invariant under permuting the entries. The main result in [3] is that the number of parking sequences is given by the product

$$(y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdots (y_1 + \cdots + y_{n-1} + 2), \tag{1.1}$$

where the ith car has length y_i. Note that this reduces to the classical $(n+1)^{n-1}$ result of Konheim and Weiss [4] when setting $y_1 = y_2 = \cdots = y_n = 1$. The proof in [3] is an extension of the circular argument by Pollak; see [8].

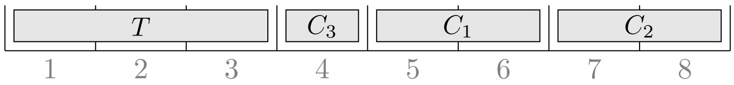
We now introduce a refinement of the result by adding a trailer.

Definition 1.1. *Let there be n cars $C_1, \ldots, C_n$ of sizes $y_1, \ldots, y_n$, where $y_1, \ldots, y_n$ are positive integers. Assume there are $z - 1 + \sum_{i=1}^{n} y_i$ spaces in a row, where the trailer occupies the $z - 1$ first spaces. Furthermore, let car C_i have the preferred spot c_i. Now let the cars in the order C_1 through C_n park according to the following rule:*

> *Starting at position c_i, car C_i looks for the first empty spot $j \geq c_i$. If the spaces j through $j + y_i - 1$ are empty, then car C_i parks in these spots. If any of the spots $j + 1$ through $j + y_i - 1$ is already occupied, then there will be a collision, and the result is not a parking sequence.*

Iterate this rule for all the cars $C_1, C_2, \ldots, C_n$. We call $(c_1, \ldots, c_n)$ a parking sequence *for $\vec{y} = (y_1, \ldots, y_n)$ if all n cars can park without any collisions and without leaving the $z - 1 + \sum_{i=1}^{n} y_i$ parking spaces.*

As an example, consider three cars of sizes $\vec{y} = (2, 2, 1)$, a trailer of size 3, that is $z = 4$, and the preferences $\vec{c} = (5, 6, 2)$. Then there are $2 + 2 + 1 = 5$ available parking spaces after the trailer, and the final configuration of the cars is



All cars are able to park, so this yields a parking sequence.

2 The result

We now have the main result. Observe that when setting $z = 1$, this expression reduces to equation (1.1).

Theorem 2.1. *The number of parking sequences $f(\vec{y}; z)$ for car sizes $\vec{y} = (y_1, \ldots, y_n)$ and a trailer of length $z - 1$ is given by the product*

$$f(\vec{y}; z) = z \cdot (z + y_1 + n - 1) \cdot (z + y_1 + y_2 + n - 2) \cdots (z + y_1 + \cdots + y_{n-1} + 1).$$

The first part of our proof comes from the following identity. Let $\dot{\cup}$ denote disjoint union of sets.

Lemma 2.2. *The number of parking sequences for car sizes $(y_1, \ldots, y_n, y_{n+1})$ and a trailer of length $z - 1$ satisfies the recurrence*

$$f(\vec{y}, y_{n+1}; z) = \sum_{L \dot{\cup} R = \{1, \ldots, n\}} \left(z + \sum_{l \in L} y_l \right) \cdot f(\vec{y}_L; z) \cdot f(\vec{y}_R; 1),$$

where $\vec{y}_S = (y_{s_1}, \ldots, y_{s_k})$ for $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq \{1, \ldots, n\}$.

Proof. Consider the situation required for the last car C_{n+1} to park successfully:

- Car C_{n+1} must see, to the left of its vacant spot, the trailer along with a subset of the cars labeled with indices L occupying the first $z - 1 + \sum_{l \in L} y_l$ spots. Hence, the restriction $\vec{c}_L$ of $\vec{c} = (c_1, c_2, \ldots, c_{n+1})$ to the indices in L must be a parking sequence for $\vec{y}_L$ and trailer of length $z - 1$. This can be done in $f(\vec{y}_L; z)$ possible ways.

– Car C_{n+1} must have a preference c_{n+1} that lies in the range $[1, z + \sum_{l \in L} y_l]$.

– Car C_{n+1} must see, to the right of its vacant spot, the complementary subset of cars labeled with indices $R = \{1, 2, \ldots, n\} - L$ occupying the last $\sum_{r \in R} y_r$ spots. These cars must have parked successfully with preferences $\vec{c}_R$ and no trailer, that is, $z = 1$. This is enumerated by $f(\vec{y}_R; 1)$.

Now summing over all decompositions $L \dot{\cup} R = \{1, 2, \ldots, n\}$, the recursion follows. □

The next piece of the proof of Theorem 2.1 utilizes a multi-parameter convolution identity due to Strehl [10]. Let $\mathbf{x} = (x_{i,j})_{1 \le i < j}$ and $\mathbf{y} = (y_j)_{1 \le j}$ be two infinite sets of parameters. For a finite subset A of the positive integers, define the two sums

$$\mathbf{x}^A_{>a} = \sum_{j \in A, j > a} x_{a,j} \quad \text{and} \quad \mathbf{y}^A_{\le a} = \sum_{j \in A, j \le a} y_j.$$

Define the polynomials $t_A(\mathbf{x}, \mathbf{y}; z)$ and $s_A(\mathbf{x}, \mathbf{y}; z)$ by

$$t_A(\mathbf{x}, \mathbf{y}; z) = z \cdot \prod_{a \in A - \max(A)} (z + \mathbf{y}^A_{\le a} + \mathbf{x}^A_{>a}),$$
$$s_A(\mathbf{x}, \mathbf{y}; z) = \prod_{a \in A} (z + \mathbf{y}^A_{\le a} + \mathbf{x}^A_{>a}).$$

Note that, when A is the empty set, we set $t_A(\mathbf{x}, \mathbf{y}; z)$ to be 1. We directly have that

$$(z + \mathbf{y}^A_{\le \max(A)}) \cdot t_A(\mathbf{x}, \mathbf{y}; z) = z \cdot s_A(\mathbf{x}, \mathbf{y}; z). \tag{2.1}$$

Now Theorem 1, equation (6) in [10] states:

Theorem 2.3 (Strehl)**.** *The polynomials $s_L(\mathbf{x}, \mathbf{y}; z)$ and $t_R(\mathbf{x}, \mathbf{y}; w)$ satisfy the following convolution identity:*

$$s_A(\mathbf{x}, \mathbf{y}; z + w) = \sum_{L \dot{\cup} R = A} s_L(\mathbf{x}, \mathbf{y}; z) \cdot t_R(\mathbf{x}, \mathbf{y}; w). \tag{2.2}$$

Strehl first interprets $s_A(\mathbf{x}, \mathbf{y}; z)$ and $t_A(\mathbf{x}, \mathbf{y}; z)$ as sums of weights on functions, then translates these via a bijection to sums of weights on rooted, labeled trees where the $x_{i,j}$'s record ascents, and the y_j's record descents. The proof of (2.2) then follows from the structure inherent in splitting a tree into two. A similar result using the same bijection was discovered by Eğecioğlu and Remmel in [2].

Proof of Theorem 2.1. The proof follows from noticing that our proposed expression for $f(\vec{y}; z)$ is Strehl's polynomial $t_{\{1,2,\ldots,n\}}(\mathbf{1}, \mathbf{y}; z)$. By induction we obtain

$$\begin{aligned} f(\vec{y}, y_{n+1}; z) &= \sum_{L \dot{\cup} R = \{1,2,\ldots,n\}} \left(z + \sum_{l \in L} y_l \right) \cdot f(\vec{y}_L; z) \cdot f(\vec{y}_R; 1) \\ &= \sum_{L \dot{\cup} R = \{1,2,\ldots,n\}} (z + \mathbf{y}^L_{\leq \max(L)}) \cdot t_L(\mathbf{1}, \mathbf{y}; z) \cdot t_R(\mathbf{1}, \mathbf{y}; 1) \\ &= \sum_{L \dot{\cup} R = \{1,2,\ldots,n\}} z \cdot s_L(\mathbf{1}, \mathbf{y}; z) \cdot t_R(\mathbf{1}, \mathbf{y}; 1) \\ &= z \cdot s_{\{1,2,\ldots,n\}}(\mathbf{1}, \mathbf{y}; z+1) \\ &= t_{\{1,2,\ldots,n+1\}}(\mathbf{1}, \mathbf{y}; z), \end{aligned}$$

where we used the recursion in Lemma 2.2, equation (2.1) and Theorem 2.3. □

3 Concluding remarks

The polynomial $t_A(\mathbf{x}, \mathbf{y}; z)$ satisfies the following convolution identity; see [10, Equation (7)],

$$t_A(\mathbf{x}, \mathbf{y}; z + w) = \sum_{B \dot{\cup} C = A} t_B(\mathbf{x}, \mathbf{y}; z) \cdot t_C(\mathbf{x}, \mathbf{y}; w). \quad (3.1)$$

Hence it is suggestive to think of this polynomial as of binomial type and the polynomial $s_A(\mathbf{x}, \mathbf{y}; w)$ as an associated Sheffer sequence; see [9]. When setting all the parameters $\mathbf{x}$ to be constant and also the parameters $\mathbf{y}$ to be constant, we obtain the classical Abel–Rothe polynomials. Hence it is natural to ask if other sequences of binomial type and their associated Sheffer sequences have multi-parameter extensions. Since the Hopf algebra $\mathbf{k}[x]$ explains sequences of binomial type, one wonders if there is a Hopf algebra lurking in the background explaining equations (3.1) and (2.2).

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