A note on the restricted arc connectivity of oriented graphs of girth four

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Abstract
Let $D$ be a strongly connected digraph. An arc set $S$ of $D$ is a restricted arc-cut of $D$ if $D - S$ has a non-trivial strong component $D_1$ such that $D - V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ of a digraph $D$ is the minimum cardinality over all restricted arc-cuts of $D$. A strongly connected digraph $D$ is $\lambda'$-connected when $\lambda'(D)$ exists. This paper presents a family $\mathcal{F}$ of strong digraphs of girth four that are not $\lambda'$-connected and for every strong digraph $D \notin \mathcal{F}$ with girth four it follows that it is $\lambda'$-connected. Also, an upper and lower bound for $\lambda'(D)$ are given.

1 Terminology and introduction

All the digraphs considered in this work are finite oriented graphs; that is, they are digraphs with no symmetric arcs or loops. Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. If $v$ is a vertex of $D$, the sets of out-neighbors and in-neighbors of $v$ are denoted by $N^+(v)$ and $N^-(v)$, respectively. If $(u, v)$ is an arc of $D$, then it is said that $u$ dominates $v$ (or $v$ is dominated by $u$) and this is denoted by $u \rightarrow v$. Two vertices $u$ and $v$ of a digraph are adjacent if $u \rightarrow v$ or $v \rightarrow u$. The numbers

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$d^+(v) = |N^+(v)|$ and $d^-(u) = |N^-(u)|$ are the out-degree and the in-degree of the vertex $v$. By a cycle of a digraph we mean a directed cycle. A $p$-cycle is a cycle of length $p$. The minimum $p$ for which $D$ has a $p$-cycle is the girth of $D$, denoted by $g(D)$. Given a digraph $D$, the subdigraph of $D$ induced by a set of vertices $X$ is denoted by $D[X]$. For any subset $S$ of $A(D)$, the subdigraph obtained by deleting all the arcs of $S$ is denoted by $D - S$. A digraph $D$ is strongly connected or simply strong if for every pair $u, v$ of vertices there exists a directed path from $u$ to $v$ in $D$.

A strong component of a digraph $D$ is a maximal induced subdigraph of $D$ which is strong. A digraph $D$ is called $k$-arc-connected if for any set $S$ of at most $k - 1$ arcs the subdigraph $D - S$ is strong. The arc-connectivity $\lambda(D)$ of a digraph $D$ is the largest value of $k$ such that $D$ is $k$-arc-connected. For a pair $X, Y$ of vertex sets of a digraph $D$, we define $(X, Y) = \{x \rightarrow y \in A(D) : x \in X, y \in Y\}$. Let $X^c$ be the complement of $X$. If $Y = X^c$ we write $(X, X^c)$ as $\partial^+(X)$ or $\partial^-(Y)$. Let $D$ be a digraph with girth $g$. If $C = (v_1, v_2, \ldots, v_g)$ is a $g$-cycle of $D$, then let

$$\xi(C) = \min\left\{\sum_{i=1}^{g} d^+(v_i) - g, \sum_{i=1}^{g} d^-(v_i) - g\right\}$$

and

$$\xi(D) = \min\{\xi(C) : C \text{ is a } g\text{-cycle of } D\}.$$


As is well known, a digraph is a mathematical object modeling networks. An important parameter in the study of networks is the fault tolerance: it is desirable that if some nodes (respectively links) are unable to work, the message can still be always transmitted. There are measures that indicate the fault tolerance of a network (modeled by a digraph $D$); for instance, the arc-connectivity of $D$ measures how easily and reliably a packet sent by a vertex can reach another vertex. Since digraphs with the same arc-connectivity can have large differences in the fault tolerance of the corresponding networks, one might be interested in defining more refined reliability parameters in order to provide a more accurate measure of fault tolerance in networks than the arc-connectivity (see [6]). In this context, Volkmann [11] introduced the concept of restricted arc-connectivity of a digraph, which is closely related to the similar concept of restricted edge-connectivity in graphs proposed by Esfahanian and Hakimi [7].

**Definition 1 (Volkmann [11])** Let $D$ be a strongly connected digraph. An arc set $S$ of $D$ is a restricted arc-cut of $D$ if $D - S$ has a non-trivial strong component $D_1$ such that $D - V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ of $D$ is the minimum cardinality over all restricted arc-cuts. A strongly connected digraph $D$ is said to be $\lambda'$-connected if $\lambda'(D)$ exists.

Observe that $\lambda'(D)$ does not exist for every digraph with fewer than $g(D) + 2$ vertices. Volkmann [11] proved that each strong digraph $D$ of order $n \geq 4$ and girth $g(D) = 2$ or $g(D) = 3$ except for some families of digraphs is $\lambda'$-connected and satisfies $\lambda(D) \leq \lambda'(D) \leq \xi(D)$. Moreover, he proved the following characterization.
Theorem 1 [11] A strongly connected digraph $D$ with girth $g$ is $\lambda'$-connected if and only if $D$ has a cycle of length $g$ such that $D - V(C)$ contains an arc.

Concerning the arc-restricted connectivity of digraphs, Meierling, Volkmann and Winzen [10] studied the restricted arc-connectivity of generalizations of tournaments. Balbuena, García-Vázquez, Hansberg and Montejano [1, 2] studied the restricted arc-connectivity for some families of digraphs and introduced the concept of super-$\lambda'$ digraphs. Results on restricted arc-connectivity of digraphs can be found in, e.g. Balbuena and García-Vázquez [3], Chen, Liu and Meng [5], Grüter, Guo and Holtkamp [8], Grüter, Guo, Holtkamp and Ulmer [9] and Wang and Lin [12].

In this paper we present a family $\mathcal{F}$ of strong digraphs of girth four that are not $\lambda'$-connected and for every strong digraph $D \notin \mathcal{F}$ with girth four it follows that it is $\lambda'$-connected and $\lambda(D) \leq \lambda'(D) \leq \xi(D)$.

2 Main result

Let $D$ be a strong digraph of girth 4. In this section it is proved that $D$ is $\lambda'$-connected with the exception of the case that $D$ is a member of the following seven families (see Figure 1).

Let $H_1$ be the digraphs having the 4-cycle $(u, v, w, z, u)$ and the following vertex sets: $A = \{a_1, a_2, \ldots, a_p\}$, $B = \{b_1, b_2, \ldots, b_q\}$, $C = \{c_1, c_2, \ldots, c_r\}$ and $D = \{d_1, d_2, \ldots, d_s\}$ such that $u \to a_i \to v$, for $1 \leq i \leq p$, $v \to b_i \to w$, for $1 \leq i \leq q$, $w \to c_i \to z$, for $1 \leq i \leq r$ and $z \to d_i \to u$, for $1 \leq i \leq s$. The cases that $A$, $B$, $C$ or $D$ are empty sets are also allowed.

Let $H_2$ be the digraphs having the 4-cycles $(u, v, w, z, u)$ and $(u, v, w, x, u)$, and the vertex sets $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_q\}$ such that $w \to a_i \to u$, for $1 \leq i \leq p$ and $u \to b_i \to w$, for $1 \leq i \leq q$. The cases that $A$ or $B$ are empty sets are also allowed.

Let $H_3$ be the digraphs having the 4-cycles $(u, v, w, z, u)$ and $(u, v, w, x, u)$ and the vertex sets $A = \{a_1, a_2, \ldots, a_p\}$, $B = \{b_1, b_2, \ldots, b_q\}$ and $C = \{c_1, c_2, \ldots, c_r\}$ such that $u \to a_i \to v$, for $1 \leq i \leq p$, $v \to b_i \to w$, for $1 \leq i \leq q$ and $w \to c_i \to u$, for $1 \leq i \leq r$. The cases that $A$, $B$ or $C$ are empty sets are also allowed.

Let $H_4$ be the digraphs having the 4-cycles $(u, v, w, z, u)$ and $(u, v, w, x, u)$, a vertex $y$ such that $u \to y \to w$ and $y$ is adjacent to $v$, and the vertex set $A = \{a_1, a_2, \ldots, a_p\}$ such that $w \to a_i \to u$, for $1 \leq i \leq p$. The cases that $A$ is an empty set is also admissible.

Let $H_5$ be the digraphs having the 4-cycles $(u, v, w, z, u)$ and $(u, v, w, x, u)$ such that $x$ is adjacent to $z$, and the vertex set $A = \{a_1, a_2, \ldots, a_p\}$ such that $u \to a_i \to w$, for $1 \leq i \leq p$.

Let $H_6$ be the digraphs having the 4-cycles $(u, v, w, z, u)$ and $(u, v, w, x, u)$ such that $x$ is adjacent to $z$, and the vertex sets $A = \{a_1, a_2, \ldots, a_p\}$, and $B = \{b_1, b_2, \ldots, b_q\}$ such that $u \to a_i \to v$, for $1 \leq i \leq p$ and $v \to b_i \to w$, for $1 \leq i \leq q$. The cases that $A$ and $B$ are empty sets are also allowed.

Let $H_7$ be the digraphs having the 4-cycles $(u, v, w, z, u)$ and $(u, v, w, x, u)$ such that $x$ is adjacent to $z$, and a vertex $y$ adjacent to $v$ such that $u \to y \to w$. 


Observe that by Theorem 1, the digraphs of $H_1, H_2, \ldots, H_7$ are not $\lambda'$-connected.

**Theorem 2** Let $D$ be a strong digraph of girth 4 and $|V(D)| \geq 6$. If $D$ is not isomorphic to a member of the families $H_1, H_2, \ldots, H_7$, then $D$ is $\lambda'$-connected and $\lambda(D) \leq \lambda'(D) \leq \xi(D)$.

**Proof.** To prove the left inequality, since every restricted cut is a cut, it follows that $\lambda(D) \leq \lambda'(D)$.

Next, we prove the right hand inequality. Let $C = (u, v, w, z, u)$ be a 4-cycle of $D$ such that $\xi(D) = \xi(C)$. Suppose without loss of generality that $\xi(C) = d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4$. If $D - \{u, v, w, z\}$ contains an arc, then $D$ is $\lambda'$-connected and $\lambda'(D) \leq \xi(D)$. Hence suppose that $D - \{u, v, w, z\}$ consists of a set of isolated vertices. Since $D$ is not isomorphic to a member of $H_1$, $D$ has to contain a 4-cycle $C'$ containing two arcs of $C$. Let $C' = (u, v, x, u)$. We continue the proof by distinguishing three cases.
Case 1  Assume that $d^+(x) = d^-(x) = 1$.

**Subcase 1.1** If $d^+(z) = d^-(z) = 1$. Since $D$ is not isomorphic to any member of $H_2$, $H_3$ and $H_4$, it follows that $|V(D)| \geq 7$ implying that there exists a set of vertices $a_1, a_2, \ldots, a_m$, $m \geq 2$, such that $a_i \notin \{u, v, w, x, z\}$ for $1 \leq i \leq m$. If $d^+(v) = d^-(v) = 1$. Since $D$ is strong, it follows that $d^+(a_i) = d^-(a_i) = 1$ for every $1 \leq i \leq m$ implying that $D$ is isomorphic to a member of $H_2$, a contradiction. Therefore, either $d^+(v) \geq 2$ or $d^-(v) \geq 2$. Suppose that $d^+(v) \geq 2$ and $d^-(v) = 1$, then there exists a vertex $a_1$ such that $v \rightarrow a_1$. Since $D$ is strong and has girth 4, it follows that $a_1 \rightarrow w$. Moreover, since $D$ is not a member of the families $H_3$ and $H_4$, there exists a vertex $a_2$, $a_2 \neq a_1$ such that $u \rightarrow a_2 \rightarrow w$. Also, as $d^-(v) = 1$, it follows that $d^+(a_2) = 1$. Consider the 4-cycle $C_1 = (u, a_2, w, z, u)$, therefore

$$
\xi(C_1) \leq d^+(u) + d^+(a_2) + d^+(w) + d^+(z) - 4
$$

giving a contradiction. Hence $d^+(v) = 1$ and $d^-(v) \geq 2$ or $d^+(v) \geq 2$ and $d^-(v) \geq 2$.

First suppose that $d^+(v) = 1$ and $d^-(v) \geq 2$, then there exists a vertex $a_1$ such that $a_1 \rightarrow v$. Further, since $D$ is strong and has girth 4, it follows that $u \rightarrow a_1$. As $D$ is not isomorphic to any member of families $H_3$ and $H_4$, there exists a vertex $a_2$, $a_2 \neq a_1$ such that $u \rightarrow a_2 \rightarrow w$. Let $S = \{ua_1, uv, wx\} \subset A(D)$. The digraph $D - S$ has a strong component $D_1$ containing the 4-cycle $(u, a_2, w, z, u)$ and $D - S$ contains the arc $a_1 v$. Therefore $D$ is $\lambda'$-connected and

$$
\lambda'(D) \leq |S| \leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).
$$

Now, suppose that $d^+(v) \geq 2$ and $d^-(v) \geq 2$, then there exist two vertices $a_1, a_2$, such that $a_1 \rightarrow v$ and $v \rightarrow a_2$. Since $D$ is strong and has girth 4, $u \rightarrow a_1$ and $a_2 \rightarrow w$. Since $D$ is not isomorphic to any member of the family $H_3$, there exists a vertex $a_3$ such that $u \rightarrow a_3 \rightarrow w$. Let $S = \partial^+(\{u, a_3, w, z\})$, then $S$ is a restricted arc-cut of $D$ and

$$
\lambda'(D) \leq |S| \leq d^+(u) + d^+(a_3) + d^+(w) + d^+(z) - 4
\leq d^+(u) + 2 + d^+(w) + d^+(z) - 4
\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),
$$

**Subcase 1.2** Assume that either $d^+(z) \geq 2$ or $d^-(z) \geq 2$. This implies that there exists a vertex $a$, different from $u, v, w, x$ in $N^+(z) \cup N^-(z)$. Suppose first that $z \rightarrow a$. Therefore

$$
\xi((u, v, w, x, u)) \leq d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4
\leq d^+(u) + d^+(v) + d^+(w) + 2 - 4 \leq \xi(D),
$$
giving a contradiction. Now suppose that \( a \to z \). Let \( S = \partial^+([u,v,w,x]) \). Note that \( D - S \) has a strong component \( D_1 \) containing the 4-cycle \((u,v,w,x,u)\) and \( D - V(D_1) \) contains the arc \( az \). Hence \( S \) is a \( \lambda' \)-restricted arc cut and

\[
\lambda'(D) \leq |S| \leq d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4
\]

\[
= d^+(u) + d^+(v) + d^+(w) + 1 - 4
\]

\[
\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),
\]

and the result follows.

**Case 2** Assume that \( d^+(x) = 1 \) and \( d^-(x) = 2 \). This implies that \( z \to x \) and therefore \( d^+(z) \geq 2 \). Since \((u,v,w,x,u)\) is a 4-cycle, it follows that

\[
\xi((u,v,w,x,u)) \leq d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4
\]

\[
< d^+(u) + d^+(v) + d^+(w) + 2 - 4
\]

\[
\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),
\]

yielding a contradiction.

**Case 3** Assume that \( d^+(x) = 2 \) and \( d^-(x) = 1 \). This implies that \( x \to z \).

**Subcase 3.1** If \( d^+(z) = 1 \) and \( d^-(z) = 2 \). Suppose first that \( d^+(v) = d^-(v) = 1 \). Since \( D \) is not isomorphic to any member of the family \( H_5 \), it follows that there exists a vertex \( a_1 \) such that \( w \to a_1 \to u \). Let \( S = \partial^+([u,v,w,a_1]) \). The digraph \( D - S \) has a strong component \( D_1 \) containing the 4-cycle \((u,v,w,a_1,u)\) and \( D - V(D_1) \) contains the arc \( xz \). Hence \( D \) is \( \lambda' \)-connected and

\[
\lambda'(D) \leq d^+(u) + d^+(v) + d^+(w) + d^+(a_1) - 4
\]

\[
= d^+(u) + d^+(v) + d^+(w) + 1 - 4
\]

\[
= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).
\]

Now, suppose that either \( d^+(v) \geq 2 \) or \( d^+(v) \geq 2 \). If \( d^+(v) \geq 2 \) and \( d^-(v) = 1 \), then there exists a vertex \( a_1 \) such that \( v \to a_1 \). Further, as \( D \) is strong, it follows that \( a_1 \to w \). Since \( D \) is not isomorphic to any member of the families \( H_6 \) and \( H_7 \), the order of \( D \) is at least 7 and there exists a vertex \( a_2 \) adjacent to \( u \) and \( w \). If \( u \to a_2 \to w \), then \( a_2 \) is not adjacent to \( v \) and \( d^+(a_2) = 1 \). Since \((u,a_2,w,z,u)\) is a 4-cycle, it follows that

\[
\xi((u,a_2,w,z,u)) \leq d^+(u) + d^+(a_2) + d^+(w) + d^+(z) - 4
\]

\[
= d^+(u) + 1 + d^+(w) + d^+(z) - 4
\]

\[
< d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),
\]
giving a contradiction.

If that $w \to a_2 \to u$. Let $S = \partial^+(\{u, v, w, a_2\})$. The digraph $D - S$ has a strong component $D_1$ containing the 4-cycle $(u, v, w, a_2, u)$ and $D - V(D_1)$ has the arc $xz$. Therefore $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq d^+(u) + d^+(v) + d^+(w) + d^+(a_2) - 4$$
$$= d^+(u) + d^+(v) + d^+(w) + 1 - 4$$
$$= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

Now, suppose that $d^+(v) = 1$ and $d^-(v) \geq 2$, then there exists a vertex $a_1$ such that $a_1 \to v$, and since $D$ is strong it follows that $u \to a_1$. Since $D$ is not isomorphic to any member of the families $H_6$ and $H_7$, then $|V(D)| \geq 7$ and there exists a vertex $a_2$ such that $a_2$ and $w$ are adjacent. If $u \to a_2 \to w$, let $S = \{ua_1, vw, wx\} \subset A(D)$, then $D - S$ has a strong component $D_1$ containing the 4-cycle $(u, a_2, w, z, u)$ and $D - V(D_1)$ has the arc $a_1v$. Therefore $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq 3 \leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

If $w \to a_2 \to u$, let $S = \partial^+(\{u, v, w, a_2\}) \subset A(D)$, then $D - S$ is a restricted arc cut of $D$ such that $D - S$ has a strong component $D_1$ containing the 4-cycle $(u, v, w, a_2, u)$ and $D - V(D_1)$ has the arc $xz$. Therefore,

$$\lambda'(D) \leq |S| = d^+(u) + d^+(v) + d^+(w) + d^+(a_2) - 4$$
$$= d^+(u) + d^+(v) + d^+(w) + 1 - 4$$
$$= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

Now, suppose that $d^+(v) \geq 2$ and $d^-(v) \geq 2$, then there are two vertices $a_1$ and $a_2$ such that $a_1 \to v$ and $v \to a_2$. Since $D$ is strong and has girth 4 it follows that $u \to a_1$ and $a_2 \to w$ (note that this may be the case where $a_1 \to w$ or $u \to a_2$). Since $D$ is not isomorphic to any member of the family $H_6$ there exists a vertex $a_3$ adjacent to $u$ and $w$. If $u \to a_3 \to w$ (note that this may be the case where $a_3 = a_1$ or $a_3 = a_2$ or $a_3$ is adjacent to $v$). Let $S = \partial^+(\{u, a_3, w, z\})$. Then the digraph $D - S$ has a strong component $D_1$ containing the 4-cycle $(u, a_3, w, z, u)$ and $D - V(D_1)$ has the arc $a_1v$ or $va_2$, according to the case. Therefore $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq |S| = d^+(u) + d^+(a_3) + d^+(w) + d^+(z) - 4$$
$$\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4$$
$$\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

If $w \to a_3 \to u$. Let $S = \partial^+(\{u, v, w, a_3\}) \subset A(D)$, then the digraph $D - S$ has a strong component $D_1$ containing the 4-cycle $(u, v, w, a_3, u)$ and $D - V(D_1)$ has the arc $xz$. Therefore $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq |S| = d^+(u) + d^+(v) + d^+(w) + d^+(a_3) - 4$$
$$= d^+(u) + 2 + d^+(w) + 1 - 4$$
$$= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$
**Subcase 3.2** If $d^+(z) \geq 2$ or $d^-(z) \geq 3$. Then there exists a vertex $a \notin \{u,v,w,x\}$ such that $a$ and $z$ are adjacent. Suppose first that $z \rightarrow a$, then consider the set of arcs $S = \partial^+(\{u,v,w,x\})$. Therefore the digraph $D - S$ has a strong component $D_1$ containing de 4-cycle $(u,v,w,x,u)$ and $D - V(D_1)$ has the arc $az$. Consequently, $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq |S| = d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4$$

$$= d^+(u) + d^+(v) + d^+(w) + 2 - 4$$

$$\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

Now, suppose that $a \rightarrow z$. Since $D$ is strong it follows that either $v \rightarrow a$ or $w \rightarrow a$. Suppose first that $v \rightarrow a$ and let $S = \partial^+(\{u,v,a,z\})$. Therefore $D - S$ has a strong component $D_1$ containing de 4-cycle $(u,v,a,z,u)$ and $D - V(D_1)$ has the arc $wx$. Therefore $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq |S| = d^+(u) + d^+(v) + d^+(a) + d^+(z) - 4$$

$$\leq d^+(u) + d^+(v) + 2 + d^+(z) - 4$$

$$\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

Now, suppose that $w \rightarrow a$. If either $v \rightarrow a$ or there exists a vertex $a' \neq a$ such that $z \rightarrow a'$, then this case is reduced to one of the two previous subcases. Otherwise observe that the condition on the girth implies that neither $a \rightarrow v$ nor $u \rightarrow a$. Suppose that $a \rightarrow u$. Let $S = \{zu, au\}$, then the digraph $D - S$ has a strong component $D_1$ containing de 4-cycle $(u,v,w,x,u)$ and $D - V(D_1)$ has the arc $az$. Therefore $D$ is $\lambda'$-connected and

$$\lambda'(D) \leq 2 \leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),$$

concluding the proof. ■

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**References**


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