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Thrackles containing a standard musquash

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Abstract

A thrackle is a drawing of a graph in which each pair of edges meets precisely once. Conway's Thrackle Conjecture asserts that a planar thrackle drawing of a graph cannot have more edges than vertices, which is equivalent to saying that no connected component of the graph contains more than one cycle. We prove that a thrackle drawing containing a standard musquash (standard n-gonal thrackle) cannot contain any other cycle of length three or five.

1 Introduction

Let G be a finite simple graph with n vertices and m edges. A *thrackle drawing* of G on the plane is a drawing $\mathcal{T} : G \to \mathbb{R}^2$, in which every pair of edges meets precisely once, either at a common vertex or at a point of proper crossing (see [11] for definitions of a drawing of a graph and a proper crossing). The notion of thrackle was introduced in the late sixties by John Conway, in relation with the following conjecture.

Conway's Thrackle Conjecture. *For a thrackle drawing of a graph on the plane, one has* $m \le n$.

Despite considerable effort, the conjecture remains wide open. At present, there are three main approaches to investigating the Conjecture. The first one, which was pioneered in [11], is to relax the definition of the thrackle: instead of requiring that the edges meet exactly once, one requires that every pair of edges meets an odd number of times (either at a proper crossing or at a common vertex). The resulting graph drawing is called a *generalised thrackle*. Generalised thrackles are much more flexible than "genuine" thrackles and are easier to study (in particular, one can use methods of low-dimensional homology theory as in [4, 5, 6]). This approach can produce upper bounds for the number of edges; the best known one is currently

$m < 1.4n - 1.4$ obtained in [10]. This improves earlier upper bounds of [5, 9, 11]. However, it seems unlikely that this method alone could lead to the full resolution of the Conjecture, since generalised thrackles are much more flexible than thrackles.

The second approach is to prove the Conjecture within specific classes of drawings: straight line thrackles ([8, §4]; see also an elegant proof by Perles in [14]), monotone thrackles [14], outerplanar and alternating thrackles [7], and spherical thrackles [3]. The philosophy of this approach is the fact that sometimes topological results can be proved using geometry. This leads to the natural question, *what is the best thrackle drawing of a given graph?* The answer to this cannot be a straight-edge drawing, as any even cycle of length at least six can be thrackled, but no such cycle has straight-line thrackle drawing. However, we know no example of a thrackle which cannot be deformed to a spherical thrackle (a thrackle on the sphere whose edges are arcs of great circles). Moving further in this direction, one can show that any thrackle can be drawn on the punctured sphere endowed with the hyperbolic metric, with the vertices at infinity, and with *geodesic* edges.

The third approach is the study of thrackles with small number of vertices. A folklore fact is that the Thrackle Conjecture is true for graphs having at most 11 vertices. In [9] it is shown that no bipartite graph of up to 11 vertices (in particular, no graph containing two non-disjoint 6-cycles) can be thrackled. It is further proved that for any $\varepsilon > 0$, the inequality $m < (1+\varepsilon)n$ for a thrackled graph would follow from the claim that a finite number (depending on ε) of certain graphs (dumbbells) cannot be thrackled.

Note that a complete classification of graphs that can be drawn as thrackles, *assuming* Conway's Thrackle Conjecture, was given in [15].

The simplest example of a thrackled cycle is the *standard n-musquash*, where $n \geq 3$ is odd: distribute n vertices evenly on a circle and then join by an edge every pair of vertices at the maximal distance from each other. This defines a musquash in the sense of Woodall [15]: *n-gonal musquash* is a thrackled n-cycle whose successive edges $e_0, \dots, e_{n-1}$ intersect in the following manner: if the edge e_0 intersects the edges $e_{k_1}, \dots, e_{k_{n-3}}$ in that order, then for all $j = 1, \dots, n-1$, the edge e_j intersects the edges $e_{k_1+j}, \dots, e_{k_{n-3}+j}$ in that order, where the edge subscripts are computed modulo n. A complete classification of musquashes was obtained in [1, 2]: every musquash is either isotopic to a standard n-musquash, or is a thrackled 6-cycle.

Conway's Thrackle Conjecture is equivalent to the fact that no connected component of a thrackled graph G may contain more than one cycle (and to the fact that no figure-eight graph can be thrackled). We prove the following.

Theorem. *Let $\mathcal{T}(G)$ be a thrackle drawing of a graph G such that the drawing of a cycle $c \subset G$ is a standard musquash. Then G contains no 3- and no 5-cycles (except possibly for c itself).*

Note that a thrackled graph can never contain a 4-cycle [11]. So the theorem may be rephrased as follows: if there is a counter-example to Conway's thrackle conjecture that is a figure-eight graph comprised of a standard n-musquash and an

m-cycle sharing a common vertex, then m is at least 6.

In this paper, we will consider thrackles up to isotopy, which is to be understood as follows. We regard graph drawing as being drawn on the 2-sphere S^2. Then two drawings $\mathcal{T}_1(G)$, $\mathcal{T}_2(G)$ of a graph G are *isotopic* if there is a homeomorphism h of S^2 with $\mathcal{T}_2(G) = h(T_1(G))$. Hence an isotopy amounts to a continuous deformation in the plane, combined eventually with an inversion. This notion is more convenient than simple planar deformation as it allows statements such as the following: up to isotopy, the only thrackle drawing of the 5-cycle is the standard 5-musquash.

2 Proof of the Theorem

In this section we give the proof of the Theorem assuming some technical lemmas that we establish later in Section 3.

Suppose that in a thrackle drawing $\mathcal{T}(G)$ of a graph G, a cycle c is thrackled as a standard musquash. Then $n := l(c)$ is odd, where $l(c)$ denotes the length of c. Suppose that the graph G contains another cycle c' with $l(c') = 3$ or $l(c') = 5$. For convenience, we can remove from G all the other edges and vertices; that is, we may assume that $G = c \cup c'$.

The first step is to reduce the proof to the case when G is a figure-eight graph consisting of cycles c and c', of the same lengths as before, sharing a common vertex and such that $\mathcal{T}(c)$ is still a standard musquash. As both c and c' are odd, they cannot be disjoint in G [11, Lemma 2.1(ii)], and so $c \cap c'$ is a nonempty union of vertices and paths. Repeatedly using the vertex-splitting operation shown in Figure 1(a), we obtain a new thrackle drawing such that $c \cap c'$ is a union of vertices, without changing the lengths of the cycles c, c' and without violating the fact that c is thrackled as a standard musquash. Next, we can perturb the drawing in a neighbourhood of every vertex at which $\mathcal{T}(c)$ and $\mathcal{T}(c')$ meet without crossing, to replace a vertex of degree four by four crossings (Figure 1(b)).

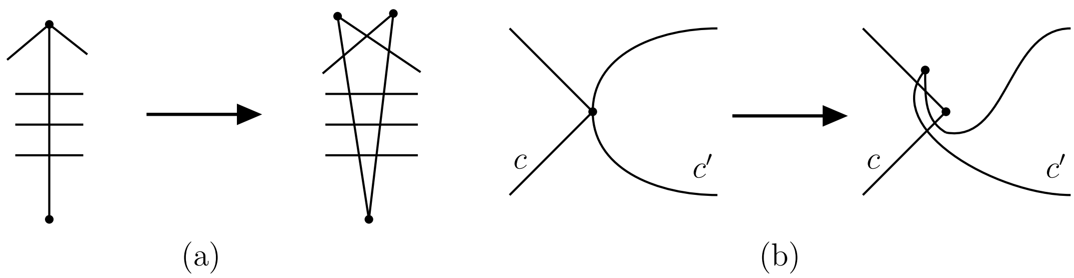

Figure 1: (a) vertex splitting; (b) replacing a vertex of degree 4 by four crossings.

Now $c \cap c'$ is a set of vertices at each of which the drawings of the cycles c, c' cross. Counting the number of crossings of the closed curves $\mathcal{T}(c)$ and $\mathcal{T}(c')$ we find that the number of such vertices must be odd [11]. If there are at least three of them, then since $l(c') \leq 5$, we arrive at a contradiction with [7, Theorem 3]. So $c \cap c'$ is a single vertex, hence $G = c \cup c'$ is a figure-eight graph, with common vertex v say.

In the second step, we fix a positive orientation on $\mathcal{T}(c)$, and to every domain D of the complement $\mathbb{R}^2 \setminus \mathcal{T}(c)$, assign a non-negative integer label, the rotation number of the closed oriented curve $\mathcal{T}(c)$ around a point of D. The labels change from 0 for the outer domain to $\frac{1}{2}(n-1)$ for the innermost n-gonal domain (Figure 2).

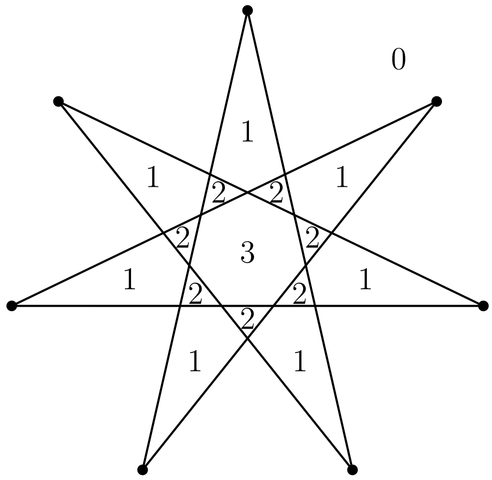

Figure 2: Labels (rotation numbers) for a standard 7-musquash.

We need the following two lemmas in which we consider the mutual position of a standard musquash and an edge or a 2-path attached to it.

Lemma 2.1. *The endpoints of a curve crossing all the edges of a standard musquash exactly once and not passing through the vertices (so that the resulting drawing is a thrackle) lie in domains whose labels differ by one.*

For the purposes of the proof of the Theorem, we will not need Lemma 2.1 in its full generality; we will only require special cases. The proof of Lemma 2.1 is quite involved, and the proof of the conclusions we require could be simpler. On the other hand, Lemma 2.1 may be of a certain independent interest and may be useful for the further study of thrackles containing musquashes. We also observe that Lemma 2.1 generalises Theorem 3 of [7].

Consider a graph G' consisting of an odd cycle c and an edge $e = vu$ attached to a vertex $v \in c$. Suppose that G' is thrackled in such a way that $\mathcal{T}(c)$ is a standard musquash. There are two possibilities for the starting segment of $\mathcal{T}(vu)$: its interior may either lie in a domain with the label 1, or in the outer domain of $\mathcal{T}(c)$. In the first case, as $\mathcal{T}(vu)$ cannot cross the drawings of the two edges of c incident to v, but has to cross the drawings of all the other edges of c, the vertex u lies in the outer domain (and what is more, by isotopy and Reidemeister moves we can transform $\mathcal{T}(e)$ to a straight line segment, as on the left in Figure 3). In the second case, we can perturb the drawing of the edge e in a neighbourhood of v to obtain a thrackle drawing of the disjoint union of c and e, with $\mathcal{T}(c)$ still being the standard musquash, and the starting point of $\mathcal{T}(e)$ lying in the outer domain (so that v is no longer a vertex of c), as on the right in Figure 3. Then by Lemma 2.1, the other endpoint u lies in a domain with the label 1.

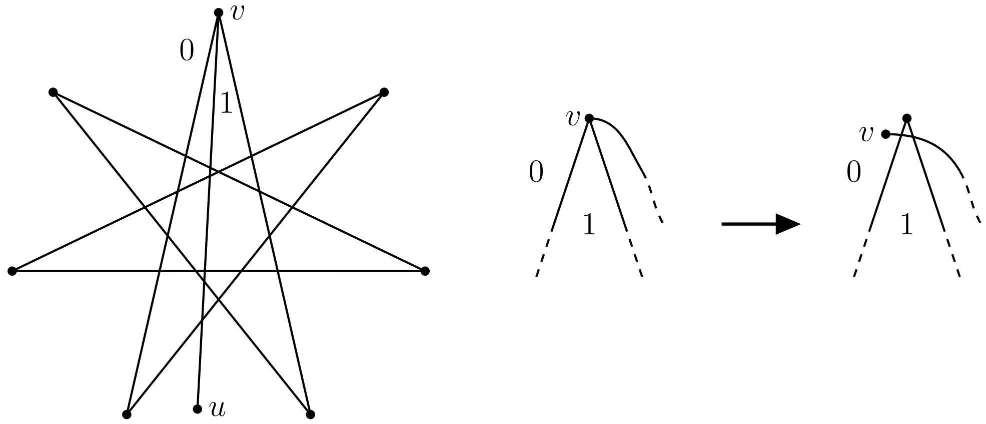

Figure 3: Two possible cases for attaching an edge to a musquash.

Now consider a graph G' consisting of an odd cycle c and a 2-path $p = vuw$ attached to a vertex $v \in c$. We suppose that G' is thrackled in such a way that $\mathcal{T}(c)$ is a standard musquash and that the starting segment of $\mathcal{T}(vu)$ lies in the outer domain of $\mathcal{T}(c)$. A possible thrackle drawing is shown in Figure 4. Perturbing the drawing of the edge vu in a neighbourhood of v, as on the right in Figure 3, we obtain a thrackle drawing of the disjoint union of c and p, with the starting point of $\mathcal{T}(p)$ lying in the outer domain. Then the point u lies in a domain labelled 1, and then w, the other endpoint of p, lies in a domain with the label either 0 or 2. The following lemma, which will be crucial for the proof of the Theorem for $l(c') = 5$, states that the second case cannot occur.

Lemma 2.2. *Let a graph G' consist of an odd cycle c and a 2-path $p = vuw$ attached to a vertex $v \in c$. Let $\mathcal{T}(G')$ be a thrackle drawing, with $\mathcal{T}(c)$ a standard musquash, such that the starting segment of $\mathcal{T}(vu)$ lies in the outer domain of $\mathcal{T}(c)$. Then u lies in a domain labelled 1 and w lies in the outer domain of $\mathcal{T}(c)$.*

Remark 1. *Combining Lemma 2.1 and Lemma 2.2 one can generalise Lemma 2.2 to the case when p is a 3-path: if $p = vv_1v_2v_3$, then v_1 and v_3 lie in domains labelled 1, and v_2, in the outer domain. It would be very interesting to know, if the direct generalisation of this fact for longer paths p is still true: is it so that a path attached to a vertex of a standard musquash and starting at the outer domain cannot get "too deep" in the musquash (most optimistically, does it always ends in a domain labelled either 0 or 1)?*

Returning to the proof of the Theorem, we have a figure-eight graph G consisting of an odd cycle c and an odd cycle c' of length 3 or 5 that share a common vertex v, and a thrackle drawing $\mathcal{T}(G)$ such that $\mathcal{T}(c)$ is a standard musquash. From the above

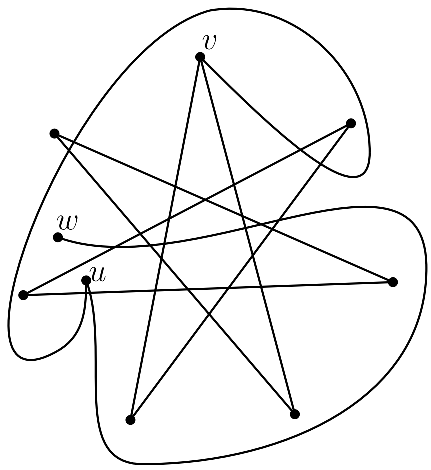

Figure 4: A standard 7-musquash with a 2-path attached.

argument, or by [11, Lemma 2.2], the drawings of c and c' cross at their common vertex v. Of the two edges of c' incident to v, one goes to the inner domain of the musquash $\mathcal{T}(c)$ (the interior of its starting segment lies in a domain labelled 1) and then ends in the outer domain, as on the left in Figure 3. The other edge goes to the outer domain. Hence by Lemma 2.1 and Lemma 2.2, because c' has length 3 or 5, we obtain the following key fact.

Corollary. *All the vertices of c' other than v lie in domains labelled 0 or 1, half in the outer domain and the other half, in the union of domains labelled 1.*

The third step in the proof of the Theorem is the operation of *edge removal* [7, Section 4], which will enable us to eventually shorten c to a thrackled cycle of the same length as c'.

The operation of edge removal is inverse to Woodall's edge insertion operation [15, Figure 14]. Let $\mathcal{T}(H)$ be a thrackle drawing of a graph H and let $v_1v_2v_3v_4$ be a 3-path in H such that $\deg v_2 = \deg v_3 = 2$. Let $A = \mathcal{T}(v_1v_2) \cap \mathcal{T}(v_3v_4)$. Removing the drawing of the edge v_2v_3, together with the segments Av_2 and Av_3 of $\mathcal{T}(v_1v_2)$ and $\mathcal{T}(v_3v_4)$ from the point A to $\mathcal{T}(v_2)$ and $\mathcal{T}(v_3)$, respectively, we obtain a drawing of a graph with a single edge v_1v_4 in place of the 3-path $v_1v_2v_3v_4$ (Figure 5). (In what follows, to make the notation less cumbersome, we will use the vertex name v_i to denote the point $\mathcal{T}(v_i)$, when there is no risk of confusion).

Unlike edge insertion, edge removal does not necessarily result in a thrackle drawing. Consider the triangular domain $\triangle$ bounded by the arcs $\mathcal{T}(v_2v_3)$, Av_2 and v_3A and not containing the vertices v_1 and v_4. We have the following Lemma.

Lemma 2.3 ([7, Lemma 3])**.** *Edge removal results in a thrackle drawing if and only if $\triangle$ contains no vertices of $\mathcal{T}(G)$.*

It follows that edge removal is always possible on (every edge of) a standard musquash. The resulting thrackled cycle is outerplanar (all the vertices lie on the

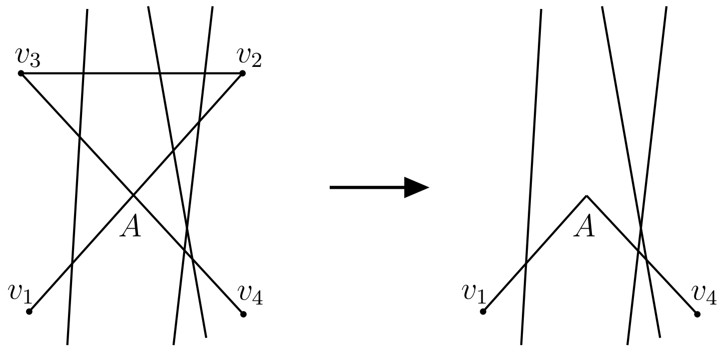

Figure 5: The edge removal operation.

boundary of a single domain) and as such, is Reidemeister equivalent to a standard musquash by [7, Theorem 1]. In fact, by the following lemma, it is even *isotopic* to a standard musquash (which will be important for the argument that follows) – see Figure 6.

Lemma 2.4. *The edge removal operation on a standard musquash of length* $n \geq 5$ *results in a standard* $(n-2)$*-musquash.*

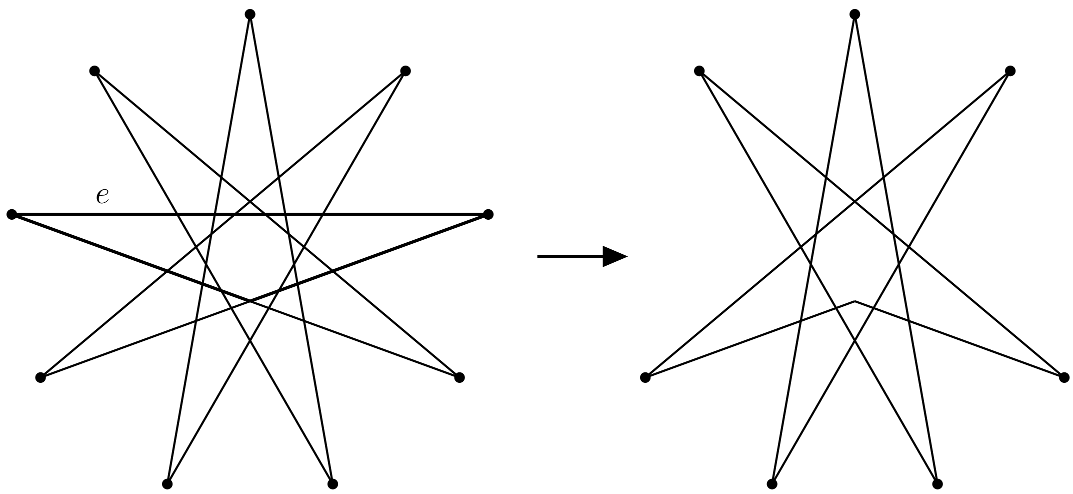

Figure 6: The edge removal operation on the edge e of the standard 9-musquash results in a standard 7-musquash.

What is more, for our thrackle drawing $\mathcal{T}(G)$, edge removal is always possible on $\mathcal{T}(c)$ as long as c is longer than c'. To see that, we first consider the case when $l(c') = 3$. We cannot remove either of the two edges of c incident to the common vertex v (as $\deg v = 4$) and neither can we remove the edges whose respective triangular domains contain the vertices of c', by Lemma 2.3. But by the Corollary, the two vertices of c' other than v lie in domains labelled 0 and 1, one in each. Clearly, the vertex lying in the outer domain does not belong to any triangular domain, while the vertex lying in a domain labelled 1 belongs to exactly two triangular domains, which prohibits the removal of two edges. Hence if $l(c) \geq 5$, there is always an edge of c which can be removed. By Lemma 2.3, after edge removal, the resulting drawing is again a thrackle drawing of the figure-eight graph consisting of an odd cycle c^* of length $l(c) - 2$ and the cycle c', and by Lemma 2.4, the drawing of c^* is again a standard musquash. Repeatedly using this argument we obtain a thrackle drawing of a figure-eight graph consisting of two 3-cycles, which is a contradiction (as can be seen by inspection or by [5, Lemma 5(b)]). This completes the proof in the case $l(c') = 3$. The proof in the case $l(c') = 5$ is almost identical, up to the second last step. This time by the Corollary, from among the four vertices of c' other than v, two lie in the outer domain of $\mathcal{T}(c)$, and the other two, in the union of domains labelled 1. Together with the two edges of c incident to v this gives at most six edges of c spared from removal. Therefore, by repeatedly edge-removing we get a thrackled figure-eight graph consisting of two 5-cycles.

The proof is then completed by the following Lemma.

Lemma 2.5. *A figure-eight graph consisting of two* 5*-cycles has no thrackle drawing.*

3 Proof of the Lemmas

Proof of Lemma 2.1: Let c be an odd cycle of length n and let $\mathcal{T}(c)$ be its standard musquash drawing. We assume the edges of $\mathcal{T}(c)$ to be straight line segments and the vertices to be the vertices of a regular n-gon inscribed in the unit circle C bounding the closed unit disc D. Let $\gamma = AB$ be a simple curve crossing every edge of $\mathcal{T}(c)$ exactly once and not passing through the vertices. We can assume that both endpoints of γ lie inside C, and that γ meets C in a finite collection of proper crossings $A_1, A_2, \dots, A_k$ labelled in the direction from A to B, where $k \geq 0$ is even. Then $\gamma = AA_1 \cup \bigcup_{j=1}^{k-1} A_jA_{j+1} \cup A_kB$, the arcs $AA_1, A_2A_3, \dots, A_kB$ lie in D, and the arcs $A_1A_2, A_3A_4, \dots, A_{k-1}A_k$, outside C. For each arc in this decomposition, consider the set of edges of $\mathcal{T}(c)$ it crosses. The arcs lying outside C do not meet $\mathcal{T}(c)$ at all. An arc lying in D crosses an edge of $\mathcal{T}(c)$ if and only if its endpoints lie on the opposite sides of that edge. Therefore the set of edges of $\mathcal{T}(c)$ such an arc is crossing depends only on its endpoints, and we lose no generality by replacing that arc by a straight line segment with the same endpoints. Note also that γ cannot completely lie inside C (so that $k \geq 1$), as no straight line crosses all the edges of $\mathcal{T}(c)$ (since one of the half-planes, determined by such a straight line, must contain more than the half of the vertices of $\mathcal{T}(c)$ and hence contain two vertices adjacent in c).

An arc A_jA_{j+1} lying in D, with both endpoints on C, crosses an *even number of consecutive edges* of $\mathcal{T}(c)$: the points A_j, A_{j+1} split C into two segments, hence splitting the set of vertices of $\mathcal{T}(c)$ into two subsets; the arc A_jA_{j+1} crosses all the edges incident to the vertices of the smaller of these two subsets. As a subset of c, the union of edges crossed by A_jA_{j+1} is an even path.

The picture is more complicated for the arcs AA_1, A_kB having one endpoint in the interior of D. Let XY be an arc lying in D, with exactly one endpoint Y on C, and let $S \subset c$ be the union of edges of $\mathcal{T}(c)$ which XY crosses. Then both S and its complement $c \setminus S$ is a finite collection of paths. Denote $O_1(XY)$ the number of paths (which are connected components) of odd length in S, and $O_0(XY)$ the number of paths of odd length in $c \setminus S$. We have the following key lemma.

Lemma 3.1. *If X lies in a domain labelled s, then $O_1(XY) = s$ and $O_0(XY) = s \pm 1$.*

Assuming Lemma 3.1, we can complete the proof of Lemma 2.1 as follows. Let S_1, S_2 be the unions of edges of c crossed by the arcs AA_1, BA_k, respectively. Since γ crosses every edge of $\mathcal{T}(c)$ exactly once, the (interiors of the) sets S_1, S_2 are disjoint, and the union of $S_1 \cup S_2$ and the sets of edges of $\mathcal{T}(c)$ which are crossed by the arcs A_jA_{j+1} is the whole cycle c. So, since the union of edges which are crossed by an arc A_jA_{j+1} is an even path in c (which can be empty), every connected component of the complement $c \setminus (S_1 \cup S_2)$ must be a path of even length. For this to be true, S_1 has to contain at least as many odd paths as $c \setminus S_2$ does, and vice versa, so $O_1(AA_1) \geq O_0(BA_k)$ and $O_1(BA_k) \geq O_0(AA_1)$. Thus, if the points A, B lie in domains labelled s_1, s_2, respectively, then by Lemma 3.1 we get $s_1 \geq s_2 - 1$ and $s_2 \geq s_1 - 1$, so $s_1 - 1 \leq s_2 \leq s_1 + 1$. As $s_1 \neq s_2$, since γ has an odd number of crossings with the closed curve $\mathcal{T}(c)$, we obtain $s_2 = s_1 \pm 1$, as claimed. □

Proof of Lemma 3.1: Our argument above shows that we can assume XY to be a straight line segment, so the proof of the lemma reduces to a question in plane geometry: we have to find the edges of $\mathcal{T}(c)$ which are crossed by XY. We can further assume that $s \neq 0$. Indeed, if $s = 0$ we can take X to also lie on C and the above arguments for the arcs A_jA_{j+1} show that the union of edges of $\mathcal{T}(c)$ crossed by XY is an even path of c (which can be empty), so $O_1(XY) = 0$ and $O_0(XY) = 1$.

Let $n = 2m+1$. We place the vertices of $\mathcal{T}(c)$ at the points $e^{\pi i/n}, e^{3\pi i/n}, \dots, e^{(2n-1)\pi i/n}$ in $\mathbb{C} = \mathbb{R}^2$ and label the edges of $\mathcal{T}(c)$ in such a way that the j^{th} edge joins the vertices $e^{(2mj+1)\pi i/n}$ and $e^{(2m(j+1)+1)\pi i/n}$ where $j = 0, 1, \dots, n-1$. From the symmetry and by a slight perturbation, we can assume that the point X lies in the open angle $\{z \in \mathbb{C} : \arg\ z \in (0, \frac{\pi}{n})\}$, as in Figure 7.

First consider the case when the point Y lies on the radius of C passing through X. Then $X = re^{i\alpha}$, $Y = e^{i\alpha}$, where $0 < r < 1$, $0 < \alpha < \frac{\pi}{n}$. The segment XY crosses the edge j of $\mathcal{T}(c)$ if and only if it crosses the line containing that edge, given by

$$\{z \in \mathbb{C} : \operatorname{Im}(z(e^{-(2m(j+1)+1)\pi i/n} - e^{-(2mj+1)\pi i/n})) + \sin(\pi/n) = 0\}.$$

Hence it crosses the edge j when we have $-\operatorname{Im}(e^{i\alpha-(2m(j+1)+1)\pi i/n} - e^{i\alpha-(2mj+1)\pi i/n}) \in (\sin(\frac{\pi}{n}), \frac{1}{r}\sin(\frac{\pi}{n}))$ which is equivalent to

$$\sin\frac{\pi}{n} < (-1)^j \left(\sin\left(\alpha + \frac{j\pi}{n}\right) + \sin\left(\alpha + \frac{(j-1)\pi}{n}\right)\right) < \frac{1}{r}\sin\frac{\pi}{n},$$

which is equivalent again to the following condition (obtained by dividing by $\cos\frac{\pi}{2n}$):

$$\sin\frac{\pi}{2n} < (-1)^j \sin\left(\alpha + \frac{(2j-1)\pi}{2n}\right) < \frac{1}{r}\sin\frac{\pi}{2n}\,. \tag{3.1}$$

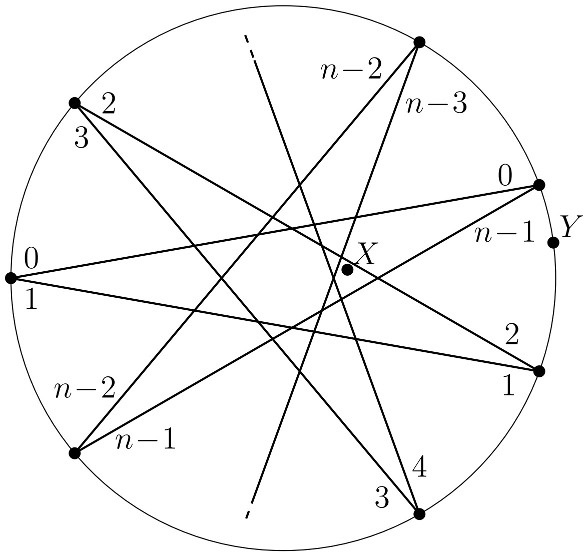

Figure 7: Labelling the edges of a musquash.

If $j = 0$, the left-hand inequality of (3.1) is false since $\alpha - \frac{\pi}{2n} \in (-\frac{\pi}{2n}, \frac{\pi}{2n})$. For all the other values of j, we have $\sin(\alpha + \frac{(2j-1)\pi}{2n}) > 0$, and so, to satisfy (3.1), j must be nonzero even. Denote $s(j) = \sin(\alpha + \frac{(2j-1)\pi}{2n})$. From the fact that $0 < \alpha < \frac{\pi}{n}$, it follows that

$$\sin\frac{\pi}{2n} < s(2m) < s(2) < s(2m-2) < s(4) < \cdots < s(2M) < 1,$$

where $M := \lfloor (m+1)/2 \rfloor$. As all the crossings of the radial segment XY with the edges of $\mathcal{T}(c)$ have the same orientation, the fact that X lies in a domain labelled s implies that there must be exactly s crossings. Hence the set of the values of j satisfying (3.1) is the set of the first s terms of the sequence $(2m, 2, 2m-2, 4, \ldots, 2M)$, that is, $\{2, 4, \ldots, 2\lfloor\frac{s}{2}\rfloor, 2m - 2\lfloor\frac{s-1}{2}\rfloor, \ldots, 2m-2, 2m\}$.

We now consider the general case when Y does not necessarily lie on the radius of C passing through X. For $Y \in C$, let $V(Y)$ be the n-dimensional row-vector over $\mathbb{Z}_2$ whose components are labelled from 0 to $n-1$, such that $V(Y)_j = 1$, if XY crosses the edge j of $\mathcal{T}(c)$, and $V(Y)_j = 0$ otherwise. As we have just shown, if XY lies on a radius of C, then

$$V(Y) = (0, (0,1)^{\lfloor s/2 \rfloor}, 0^{n-2s-1}, (0,1)^{\lfloor (s+1)/2 \rfloor}), \tag{3.2}$$

where the superscript denotes the number of consecutive repeats of the sequence.

The number $O_1(XY)$ (respectively, $O_0(XY)$) is the number of odd sequences of consecutive ones (respectively, zeros) in the vector $V(Y)$ (counted in the cyclic order, so that if $V(Y)_{n-1-a} = \cdots = V(Y)_{n-1} = V(Y)_0 = \cdots = V(Y)_b$ for some $a, b \geq 0$, we count it as a single sequence of length $a+b+2$). For the vector $V(Y)$ in (3.2) we have $O_1(XY) = s$ and $O_0(XY) = s-1$.

When $Y \in C$ moves in the positive direction, the vector $V(Y)$ only changes when Y passes through the vertices of $\mathcal{T}(c)$. Initially Y lies between the vertices $e^{(2n-1)\pi i/n}$

and $e^{\pi i/n}$, so when Y passes through the vertex $e^{\pi i/n}$, the resulting vector $V(Y)$ is obtained from the one in (3.2) by adding the vector $(1, 0, \dots, 0, 1)$, which gives $V(Y) = ((1, 0)^{\lfloor (s+2)/2 \rfloor}, 0^{n-2s-1}, (1, 0)^{\lfloor (s-1)/2 \rfloor}, 0)$, so $O_1(XY) = s$ and $O_0(XY) = s - 1$.

When Y keeps moving, every time when it passes through the vertex $e^{(2j+1)\pi i/n}$, $j = 1, \dots, n-1$, we add to $V(Y)$ the vector $(0^{n-2j-1}, 1^2, 0^{2j-1})$ for $j = 1, \dots, \frac{n-1}{2}$, and the vector $(0^{2n-2j-1}, 1^2, 0^{2j-n-1})$ for $j = \frac{n+1}{2}, \dots, n-1$, as the number of crossings of XY with the two edges incident to that vertex changes from 0 to 1 or vice versa. Then a routine check shows that $O_1(XY)$ and $O_0(XY)$ take the values shown in

Interval for j	$V(Y)$	O_1	O_0
$\left[1, \lfloor \frac{s-1}{2} \rfloor\right]$	$((1,0)^{\lfloor \frac{s+2}{2} \rfloor}, 0^{n-2s-1}, (1,0)^{\lfloor \frac{s-1}{2} \rfloor - j}, (0,1)^j, 0)$	s	$s-1$
$\left[\lfloor \frac{s-1}{2} \rfloor + 1, \frac{n-3}{2} - \lfloor \frac{s}{2} \rfloor\right]$	$((1,0)^{\lfloor \frac{s+2}{2} \rfloor}, 0^{n-2j-3-2\lfloor \frac{s}{2} \rfloor}, 1^{2(j - \lfloor \frac{s-1}{2} \rfloor)}, (0,1)^{\lfloor \frac{s-1}{2} \rfloor}, 0)$	s	$s+1$
$\left[\frac{n-3}{2} - \lfloor \frac{s}{2} \rfloor + 1, \frac{n-3}{2}\right]$	$((1,0)^{\frac{n-1}{2} - j}, (0,1)^{j - \frac{n-3}{2} + \lfloor \frac{s}{2} \rfloor}, 1^{n-2s-1}, (0,1)^{\lfloor \frac{s-1}{2} \rfloor}, 0)$	s	$s-1$
$\frac{n-1}{2}$	$(0, (1,0)^{\lfloor \frac{s}{2} \rfloor}, 1^{n-2s-1}, (1,0)^{\lfloor \frac{s+1}{2} \rfloor})$	s	$s-1$
$\left[\frac{n+1}{2}, \frac{n-1}{2} + \lfloor \frac{s+1}{2} \rfloor\right]$	$(0, (1,0)^{\lfloor \frac{s}{2} \rfloor}, 1^{n-2s-1}, (1,0)^{\lfloor \frac{s+1}{2} \rfloor + \frac{n-1}{2} - j}, (0,1)^{j - \frac{n-1}{2}})$	s	$s-1$
$\left[\frac{n+1}{2} + \lfloor \frac{s+1}{2} \rfloor, n-2-\lfloor \frac{s}{2} \rfloor\right]$	$(0, (1,0)^{\lfloor \frac{s}{2} \rfloor}, 1^{2(n - \lfloor \frac{s}{2} \rfloor - 1 - j)}, 0^{2j-n+1-2\lfloor \frac{s+1}{2} \rfloor}, (0,1)^{\lfloor \frac{s+1}{2} \rfloor})$	s	$s+1$
$\left[n-1-\lfloor \frac{s}{2} \rfloor, n-1\right]$	$(0, (1,0)^{n-1-j}, (0,1)^{\lfloor \frac{s}{2} \rfloor - n+1+j}, 0^{n-2s-1}, (0,1)^{\lfloor \frac{s+1}{2} \rfloor})$	s	$s-1$

Table 1: $O_1(XY)$ and $O_0(XY)$.

Table 1 and the required result follows. Note that the last three rows follow from the first three by symmetry. □

PROOF OF LEMMA 2.4: Let c be an odd cycle of length n. Choose a direction on c and label the edges $0, 1, \dots, n-1$ in consecutive order. According to [2], the standard musquash is uniquely determined by its *crossing diagram*, that is, by the order of crossings on every edge with the other edges. For the standard musquash $\mathcal{T}(c)$, this order on the edge labelled i is

$$i+n-3, i+n-5, \dots, i+4, i+2, i+n-2, i+n-4, \dots, i+5, i+3, \tag{3.3}$$

where the labels are computed modulo n [2]. By Lemma 2.3, the edge removal operation on any edge results in a thrackle drawing $\mathcal{T}(c^*)$ of a cycle of length $n-2$. Without loss of generality we assume that we remove the edge labelled $n-2$. We keep the labels $0, 1, \dots, n-4$ for the edges of c^* which are unaffected by the removal, and we label $n-3$ the single edge of c^* formed by the segments of edges $n-3$ and $n-1$ of c as the result of the edge removal.

The proof now is just a routine verification that the order of crossings for every edge of $\mathcal{T}(c^*)$ is the same as that given by (3.3), with n replaced by $n-2$, and with the labels computed modulo $n-2$. We consider three cases.

Suppose an edge i of $\mathcal{T}(c)$ crosses all the three edges $n-3, n-2, n-1$ (that is, $1 \le i \le n-5$). If i is even, then from (3.3) the order of crossings is $i-3, i-5, \dots, 1, n-1, n-3, \dots, i+4, i+2, i-2, i-4, \dots, 2, 0, n-2, n-4, \dots, i+5, i+3$, so the crossings with $n-3$ and $n-1$ are consecutive. Hence the crossing order on the edge i of $\mathcal{T}(c^*)$ is obtained by deleting the labels $n-1$ and $n-2$, which results in the same sequence as in (3.3), with n replaced by $n-2$. If i is odd, then the order of crossings is $i-3, i-5, \dots, 2, 0, n-2, \dots, i+4, i+2, i-2, i-4, \dots, 1, n-1, n-3, \dots, i+5, i+3$, and the proof follows by a similar argument.

Suppose now an edge i of $\mathcal{T}(c)$ crosses only two of the three edges $n-3, n-2, n-1$, so that $i = 0$ or $i = n-4$. When $i = 0$, by (3.3) the crossing order is $n-3, n-5, \dots, 4, 2, n-2, n-4, \dots, 5, 3$. The crossing order on the edge 0 of $\mathcal{T}(c^*)$ is obtained by deleting the labels $n-3$ and $n-2$, which results in the same sequence, with n replaced by $n-2$. Similarly, for $i = n-4$, (3.3) gives $n-7, n-9, \dots, 0, n-2, n-6, n-8, \dots, 1, n-1$. The crossing order on the edge $n-4$ of $\mathcal{T}(c^*)$ is obtained by deleting the labels $n-1$ and $n-2$. The resulting sequence is the same as that obtained from (3.3) by replacing n by $n-2$, and then reducing modulo $n-2$.

And finally, the crossing order on the edge $n-3$ of $\mathcal{T}(c^*)$ is the crossing order on the edge $n-3$ of $\mathcal{T}(c)$, up to but excluding the crossing with the edge $n-1$, followed by the crossing order on the edge $n-1$ of $\mathcal{T}(c)$ starting from but excluding the crossing with the edge $n-3$. From (3.3) we obtain the sequence $n-6, n-8, \dots, 1, n-5, \dots, 4, 2$. This sequence is the same as that obtained from (3.3) by replacing n by $n-2$, and then reducing modulo $n-2$. □

Proof of Lemma 2.2: The fact that u lies in a domain labelled 1 follows from Lemma 2.1. Then, again by Lemma 2.1, w lies either in the outer domain, or in a domain labelled 2. Arguing by contradiction, suppose that w lies in a domain labelled 2. Our approach is to shorten the musquash $\mathcal{T}(c)$ to a standard musquash of length at most 7 using the edge removal operation. In view of Lemma 2.3, the edge removal operation on c is forbidden on the following edges: on the two edges incident to v as $\deg v > 2$, on the two edges whose corresponding triangular domains $\triangle$ contain the vertex u, and on the four edges whose corresponding triangular domains $\triangle$ contain the vertex w (it is not hard to see that if a point lies in a domain with label $i < \frac{n-1}{2}$, then it is contained in $2i$ triangular domains, hence forbidding the removal of $2i$ edges; a point lying in a domain with label $\frac{n-1}{2}$, the innermost n-gon of the musquash, lies in all the triangular domains, hence not permitting any edge removal at all). This gives no more than 8 edges in total. So edge removal on c is always possible as long as the cycle c has at least 9 edges. By Lemma 2.4, edge removal results in a standard musquash, and what is more, the vertices u and w still lie in domains labelled 1 and 2 respectively, of the complement to that new musquash. Repeating edge removal, we come to a thrackle consisting of a standard 7-musquash, with a 2-path attached to its vertex. Note that for some pairs of domains containing u and w, it could happen that the sets of forbidden edges overlap, which could make further edge removal possible, hence resulting in a standard 5-musquash, with a 2-path attached to its vertex. Lemma 2.2 is obvious when c is a 3-cycle. So Lemma 2.2 follows from the following result. □

Lemma 3.2. *The claim of Lemma 2.2 holds when c is a 5-cycle or a 7-cycle.*

In order to complete the proof of the Theorem, it remains to prove Lemmas 3.2 and 2.5, which deal with thrackles that are sufficiently small that they can be treated by computer (which is why we separated Lemma 3.2 from Lemma 2.2). We give computer-assisted proofs of the both Lemmas (for formal proofs not relying on computer the interested reader is referred to [12]). As we mentioned in the Introduction, it is "folkloric" that the Thrackle Conjecture has been verified for graphs having up to 11 vertices, and were this claim true, Lemma 2.5 would follow. However, the verification up to 11 vertices has not appeared in the literature, to our knowledge. For this reason we give details for both Lemma 2.5 and Lemma 3.2.

Every thrackled path is uniquely, up to isotopy, determined by two pieces of combinatorial data: the crossing diagram and the orientation diagram. If we choose a direction on the given l-path and label the edges consecutively from 1 to l, then the crossing diagram is the $l \times (l-2)$ table whose i-th row is the ordered list of crossings on edge i with the other edges (as in the proof of Lemma 2.4). Note that the crossing diagram has $l-2$ entries in the rows 1 and l, and $l-3$ entries in all the other rows. The orientation diagram is the table of the same size with the entries ± 1 depending of whether the corresponding crossing occurs with positive or negative orientation respectively. The crossing diagram of an l-path defines a thrackled cycle if and only if the first crossing on the edge 1 occurs with the edge l and the last crossing on the edge l, with the edge 1; we can replace that crossing in the thrackle drawing by a vertex and remove the portion of the edge 1 before that vertex and the portion of the edge l after it to obtain a thrackled l-cycle. We can also define both the crossing and the orientation diagram for a *partial thrackled path*, the drawing consisting of a thrackled $(l-1)$-path and a portion of the l-th edge which does not cross the edge $l-1$ and crosses all the other edges no more than once. The algorithm described below enables one starting from (the crossing and the orientation diagrams of) a partial thrackled path to obtain (the crossing and the orientation diagrams of) all the thrackled paths of length $n \geq l$ to which that path can be extended, and then to find among them all the thrackled cycles of length $n \geq l$ using the simple characterisation above. In principle, one can even start with an embedded 2-path and produce all the thrackled paths and cycles of a given length $n \geq 2$.

Proof of Lemma 2.5: Let G be a figure-eight graph comprised of two 5-cycles c_1 and c_2 sharing a vertex v, and assume that there exists a thrackle drawing $\mathcal{T}(G)$. Any thrackle drawing of a 5-cycle is a standard musquash, and by [11, Lemma 2.2], the drawings of c_1 and c_2 cross at v. Take a drawing of c_1 with the vertices at the vertices of a regular pentagon. Let vw and vu be the edges of c_2 such that the starting segment of $\mathcal{T}(vw)$ lies in the outer domain determined by $\mathcal{T}(c_1)$, and the starting segment of $\mathcal{T}(vu)$, in a domain labelled 1. Up to reflection and isotopy the drawing of c, vu and the starting segment of vw is as given on the left in Figure 8.

We can now change the drawing in a small neighbourhood of vertex v replacing that vertex by two vertices of degree two as shown on the right in Figure 8. The resulting drawing is a thrackle drawing of a 10-cycle, which additionally satisfies the following conditions: on edge 1, there are no crossings between the crossings with

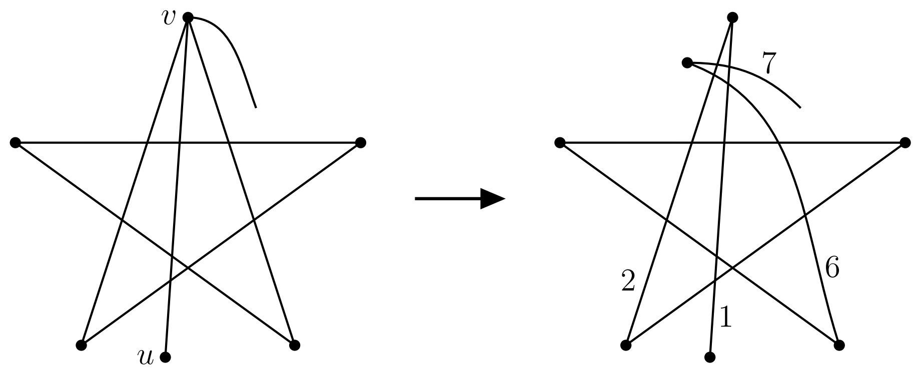

Figure 8: Changing the drawing in a neighbourhood of the vertex v of $\mathcal{T}(G)$.

edges 6 and 7 and there are no crossings after the crossing with edge 7; on edge 2, there are no crossings between the crossings with edges 7 and 6 and there are no crossings before the crossing with edge 7; on edge 6, there are no crossings between the crossings with edges 1 and 2 and there are no crossings after the crossing with edge 2; and on edge 7, there are no crossings between the crossings with edges 2 and 1 and there are no crossings before the crossing with edge 2 (where we refer to the labelling of the edges given on the right in Figure 8). Starting with the partial thrackled path consisting of the drawing of the first six edges and a portion of the seventh edge shown on the right in Figure 8 we use the computer code to produce all the thrackled 10-paths extending that partial thrackled path and satisfying these conditions. We then check that neither of them gives a thrackled 10-cycle, hence proving the lemma. The program produced a total of $132,039$ thrackled 10-paths in 473 seconds on a desktop computer. The Maple code and the output `Lemma5` is available on the journal website [13]. □

Proof of Lemma 3.2: Suppose $\mathcal{T}(G')$ is a thrackle drawing of the graph G' which is the union of the cycle c with $l(c) \in \{5, 7\}$ and a 2-path $p = vuw$ attached to a vertex $v \in c$. We already know that the vertex u lies in a domain labelled 1, and so to prove that w lies in the outer domain we have to show that the sum of orientation of all the crossings on the (directed) edge uw is $+1$.

The proof in the cases $l(c) = 5$ and $l(c) = 7$ is similar and employs the same idea as in the proof of Lemma 2.5. Suppose $l(c) = 5$. Changing the drawing in a small neighbourhood of the vertex v we obtain a thrackle drawing of a 7-path whose starting partial thrackle path consisting of the thrackled 5-path and the starting segment of the sixth edge is shown on the right in Figure 9.

The resulting thrackled 7-path must satisfy the following conditions: on edge 1, there are no crossings between the crossings with edges 6 and 5 and there are no crossings before the crossing with edge 6; on edge 5, there are no crossings after the crossing with edge 1; and on edge 6, there are no crossings before the crossing with edge 1 (where we refer to the labelling of the edges given on the right in Figure 9). We now use the computer code to produce all the thrackled 7-paths which extend the partial

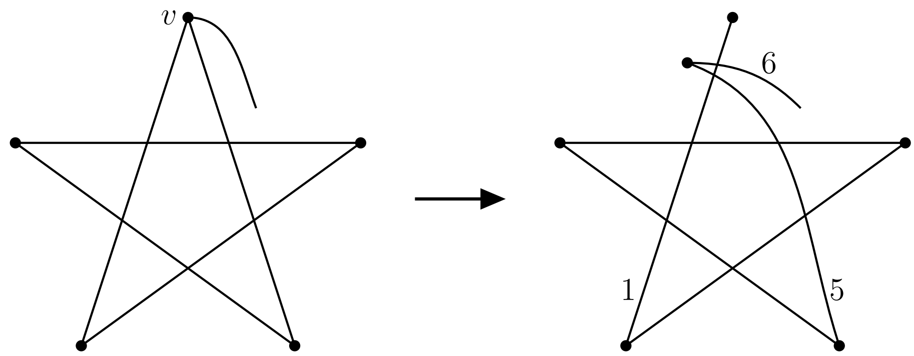

Figure 9: Changing the drawing in a neighbourhood of the vertex v of $\mathcal{T}(G')$.

thrackled path on the right in Figure 9 and satisfying these conditions. Using the orientation diagram of each of them we check that the sum of orientations of all the five crossings on the seventh edge (the edge uw of the original thrackle) is indeed $+1$, hence proving the claim in the case $l(c) = 5$. In the case $l(c) = 7$ the proof is similar: we change the drawing in a neighbourhood of the vertex v which produces a thrackled 9-path which satisfies similar conditions for the crossings on the first, the seventh and the eighth edges and then extend the partial thrackled path consisting of seven edges and the starting segment of the eighth edge to a thrackled 9-path. For every such extension, the sum of orientations on the ninth edge is $+1$, as required. The program produced a total of 50 thrackled 7-paths in 0.5 seconds, and a total of 556 thrackled 9-paths in 1.3 seconds on a desktop computer. The Maple codes and the outputs `Lemma7_7path.mw` and `Lemma7_9path.mw` are available on the journal website [13]. □

We now give a description of our algorithm. Suppose we are given (the crossing and the orientation diagrams of) a certain partial thrackled path consisting of a thrackled $(l-1)$-path and a segment of the l-th edge which does not cross the $(l-1)$-st edge and crosses all the other edges at most once. The complement of the drawing is the union of domains homeomorphic to discs (if we add the point at infinity to the unbounded domain), the boundary of each of which is the union of segments of the edges, either between two consecutive crossings or between a crossing and an endpoint. From the diagrams, we can find, for any such (directed) segment, the next segment on the boundary of the same domain, in the positive direction. We start with the last segment on the l-th edge and take the next segment on the boundary of the same domain, in the positive direction. If it is forbidden, that is, if it lies to the $(l-1)$-st edge, or on an edge which the edge l has already crossed before, or for some other reason (because of some particular conditions on the crossing diagram, as in the proofs of Lemmas 3.2 and 2.5 above), we skip that segment and continue to the next one. When we find one which is not forbidden, we extend the l-th edge to cross it (which computationally means adding an extra crossing to the crossing and the orientation diagrams), take the new last segment as our new starting segment, find the next one on the boundary of the new domain in the positive direction, and so on. If we get $l-2$ crossings on the edge l, we obtain a thrackled l-path. We can then add a starting segment of the $(l+1)$-st edge (the empty $(l+1)$-st row in the both diagrams) and continue similarly till we get a thrackled path of the required length n. We can then check if that path is in fact a thrackled n-cycle, as explained above. If at some stage, we cannot continue because either we made a full turn around the boundary of a domain and returned back to the last segment on the last edge, or we obtained a thrackled n-path and we do not want to increase the length of the path further, we go back. This means that we remove the last crossing on the last edge from our partial thrackled path (that is, from the diagrams). This brings us to the previous domain and we then try for the next crossing the next segment on the boundary of that domain in the positive direction after the one with the crossing which has just been removed. If it is forbidden, we go to the next one, and so on. If we cannot continue, we remove yet another crossing and again proceed as above. The execution stops when we return back to our original partial thrackled path and the next segment on the boundary of the domain containing the last segment of the l-th edge is that very segment, with the opposite direction. By that time, the program will have produced all the thrackled paths (and cycles) of the given length n.

4 Concluding remarks

In this paper we only considered standard musquashes, the odd ones. By [1] there is the only one even musquash, the thrackled 6-cycle. A direct generalisation of the Theorem to the case when c is a 6-cycle is false, because c and c' can be disjoint in G. The disjoint union of a 6-cycle and a 3- or a 5-cycle *can* be thrackled (which does not violate the Thrackle Conjecture) following the approach in [15, Section 2]: for both a 3- and a 5-thrackle, there is a curve which crosses every edge exactly once. We can take a thin strip around that curve and then place the 6-musquash inside that strip so that three vertices are close to one end, and the other three close to the other end, as in [15, Figure 6]. However, the figure-eight graph comprised of a 3- and a 6-cycle cannot be thrackled (by duplicating the 3-cycle we get the theta-graph Θ_3, with three paths of length 3 sharing common endpoints, which has no thrackle drawing by [11, Theorem 5.1]). However, to the best of our knowledge, the question of whether the figure-eight graph comprised of a 5- and a 6-cycle can be thrackled remains open.

Finally, we give a more precise statement of Problem 1 in [7], to which we know no counterexamples: *is it true that the sum of orientations of crossings on any edge of an odd thrackled cycle is* 0*, and on any edge of an even thrackled cycle is* ± 1*?*

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