AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(1)** (2018), Pages 28–51

Constructing internally disjoint pendant Steiner trees in Cartesian product networks*

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Abstract

The concept of pendant tree-connectivity was introduced by Hager in 1985. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. For an S-Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pendant S-Steiner tree*. Two pendant S-Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local pendant tree-connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pendant S-Steiner trees in G. For an integer k with $2 \leq k \leq n$, *pendant tree k-connectivity* is defined as $\tau_k(G) = \min\{\tau_G(S) \mid S \subseteq V(G), |S| = k\}$. In this paper, we prove that for any two connected graphs G and H, $\tau_3(G\square H) \geq \min\{3\lfloor \frac{\tau_3(G)}{2} \rfloor, 3\lfloor \frac{\tau_3(H)}{2} \rfloor\}$. Moreover, the bound is sharp.

1 Introduction

A processor network is expressed as a graph, where a node is a processor and an edge is a communication link. Broadcasting is the process of sending a message from the source node to all other nodes in a network. It can be accomplished by message dissemination in such a way that each node repeatedly receives and forwards messages. Some of the nodes and/or links may be faulty. However, multiple copies of messages can be disseminated through disjoint paths. We say that the broadcasting succeeds if all the healthy nodes in the network finally obtain the correct message

* Supported by the National Science Foundation of China (Nos. 11601254, 11551001, 11161037), the Science Found of Qinghai Province (Nos. 2016-ZJ-948Q, and 2014-ZJ-907), and the project on the key lab of IOT of Qinghai province (No. 2017-Z-Y21).

from the source node within a certain limit of time. A lot of attention has been devoted to fault-tolerant broadcasting in networks [10,15,17,36]. In order to measure the ability of fault-tolerance, the above path structure connecting two nodes are generalized into some tree structures connecting more than two nodes, see [19,21,25].

To show the properties of these generalizations clearly, we hope to start from the connectivity in graph theory. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

1.1 Connectivity and k-connectivity

All graphs considered in this paper are undirected, finite and simple. We refer to the book [2] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices, the set of edges and the minimum degree of G, respectively. Connectivity is one of the most basic concepts of graph-theoretic subjects, both in combinatorial sense and the algorithmic sense. It is well-known that the classical connectivity has two equivalent definitions. The *connectivity* of G, written $\kappa(G)$, is the minimum order of a vertex set $S \subseteq V(G)$ such that $G \setminus S$ is disconnected or has only one vertex. We call this definition the 'cut' version definition of connectivity. Menger theorem provides an equivalent definition of connectivity, which can be called the 'path' version definition of connectivity. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x,y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x,y) \,|\, x,y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G. For connectivity, Oellermann gave a survey paper on this subject; see [32].

Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings. So people want to generalize this concept. For the 'cut' version definition of connectivity, we find the above minimum vertex set without regard to the number of components of $G \setminus S$. Two graphs with the same connectivity may have differing degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n+1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. Chartrand et al. [4] generalized the 'cut' version definition of connectivity. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G of order n ($n \geq k$) produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about k-connectivity, we refer to [4, 33, 34].

1.2 Generalized connectivity

The generalized connectivity of a graph G, introduced by Hager [13], is a natural generalization of the 'path' version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$ a minimal S-Steiner tree is just a path connecting the two vertices of S. Two S-Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint S-Steiner trees in G, that is, we search for the maximum cardinality of edge-disjoint trees which include S and are vertex disjoint with the exception of S. For an integer k with $2 \leq k \leq n$, *generalized k-connectivity* (or *k-tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k-subsets of $V(G)$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. Note that the generalized k-connectivity and k-connectivity of a graph are indeed different. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on generalized connectivity; see the book [24] and the papers [5, 20–23, 25–29, 35].

The following Table 1 shows how the generalization proceeds.

	Classical connectivity	Generalized connectivity
Vertex subset	$S = \{x, y\} \subseteq V(G)$ ($\lvert S\rvert = 2$)	$S \subseteq V(G)$ ($\lvert S\rvert \geq 2$)
Set of Steiner trees	$\begin{cases} \mathscr{P}_{x,y} = \{P_1, P_2, \ldots, P_\ell\} \\ \{x, y\} \subseteq V(P_i) \\ E(P_i) \cap E(P_j) = \emptyset \\ V(P_i) \cap V(P_j) = \{x, y\} \end{cases}$	$\begin{cases} \mathscr{T}_S = \{T_1, T_2, \ldots, T_\ell\} \\ S \subseteq V(T_i) \\ E(T_i) \cap E(T_j) = \emptyset \\ V(T_i) \cap V(T_j) = S \end{cases}$
Local parameter	$\kappa(x, y) = \max \lvert\mathscr{P}_{x,y}\rvert$	$\kappa(S) = \max \lvert\mathscr{T}_S\rvert$
Global parameter	$\kappa(G) = \min_{x,y \in V(G)} \kappa(x, y)$	$\kappa_k(G) = \min_{S \subseteq V(G), \lvert S\rvert = k} \kappa(S)$

Table 1. Classical connectivity and generalized connectivity

In fact, Mader [30] studied an extension of Menger's theorem to independent sets of three or more vertices. We know that from Menger's theorem that if $S = \{u, v\}$ is a set of two independent vertices in a graph G, then the maximum number of internally disjoint u-v paths in G equals the minimum number of vertices that separate u and v. For a set $S = \{u_1, u_2, \ldots, u_k\}$ of k ($k \geq 2$) vertices in a graph G, an *S-path* is defined as a path between a pair of vertices of S that contains no other vertices of S. Two S-paths P_1 and P_2 are said to be *internally disjoint* if they are vertex-disjoint except for the vertices of S. If S is a set of independent vertices of a graph G, then a vertex set $U \subseteq V(G)$ with $U \cap S = \emptyset$ is said to *totally separate* S if every two vertices of S belong to different components of $G \setminus U$. Let S be a set of at least three independent vertices in a graph G. Let $\mu(G)$ denote the maximum number of internally disjoint S-paths and $\mu'(G)$ the minimum number of vertices that totally separate S. A natural extension of Menger' s theorem may well be suggested, namely: If S is a set of independent vertices of a graph G and $|S| \geq 3$, then $\mu(S) = \mu'(S)$. However, the statement is not true in general. Take for example, the graph G_0 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. For $S = \{v_1, v_2, v_3\}$, $\mu(S) = 1$ but $\mu'(S) = 2$. Mader proved that $\mu(S) \geq \frac{1}{2}\mu'(S)$. Moreover, the bound is sharp. Lovász conjectured an edge analogue of this result and Mader proved this conjecture and established its sharpness. For more details, we refer to [30–32].

1.3 Pendant-tree connectivity

The concept of pendant-tree connectivity [13] was introduced by Hager in 1985, which is specialization of generalized connectivity (or *k-tree-connectivity*) but a generalization of classical connectivity. For an S-Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pendant S-Steiner tree*. Two pendant S-Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local pendant-tree connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pendant S-Steiner trees in G. For an integer k with $2 \leq k \leq n$, *pendant-tree k-connectivity* is defined as $\tau_k(G) = \min\{\tau_G(S) \mid S \subseteq V(G), |S| = k\}$. Set $\kappa_k(G) = 0$ when G is disconnected. It is clear that

$$\begin{cases} \tau_k(G) = \kappa_k(G), & \text{for } k = 1, 2; \\ \tau_k(G) \leq \kappa_k(G), & \text{for } k \geq 3. \end{cases}$$

The relations between the pendant tree-connectivity and generalized connectivity are shown in the following Table 2.

	Pendant tree-connectivity	Generalized connectivity
Vertex subset	$S \subseteq V(G)$ $(\lvert S\rvert \geq 2)$	$S \subseteq V(G)$ $(\lvert S\rvert \geq 2)$
Set of Steiner trees	$\begin{cases} \mathscr{T}_S = \{T_1, T_2, \ldots, T_\ell\} \\ S \subseteq V(T_i), \\ d_{T_i}(v) = 1 \; for \; every \; v \in S \\ E(T_i) \cap E(T_j) = \emptyset, \end{cases}$	$\begin{cases} \mathscr{T}_S = \{T_1, T_2, \ldots, T_\ell\} \\ S \subseteq V(T_i), \\ E(T_i) \cap E(T_j) = \emptyset, \end{cases}$
Local parameter	$\tau(S) = \max \lvert\mathscr{T}_S\rvert$	$\kappa(S) = \max \lvert\mathscr{T}_S\rvert$
Global parameter	$\tau_k(G) = \min_{S \subseteq V(G), \lvert S\rvert = k} \tau(S)$	$\kappa_k(G) = \min_{S \subseteq V(G), \lvert S\rvert = k} \kappa(S)$

Table 2. Two tree-connectivities

It is clear that generalized k-connectivity (or *k-tree-connectivity*) and pendant-tree k-connectivity of a graph are indeed different. For example, let $H_2 = W_n$ be a wheel of order n. From Lemma 1.1, we have $\tau_3(H_2) \leq 1$. One can check that for any $S \subseteq V(H)$ with $|S| = 3$, $\tau_{H_2}(S) \geq 1$. Therefore, $\tau_3(H_2) = 1$. From Lemma 1.3, we have $\kappa_3(H_2) \leq \delta(H_2) - 1 = 3 - 1 = 2$. One can check that for any $S \subseteq V(G)$ with $|S| = 3$, $\kappa_{H_2}(S) \geq 2$. Therefore, $\kappa_3(H_2) = 2$.

In [13], Hager derived the following results.

Lemma 1.1 [13] *Let ℓ be an integer, and G be a graph. If $\tau_k(G) \geq \ell$, then $\delta(G) \geq k + \ell - 1$.*

Lemma 1.2 [13] *Let ℓ be an integer, and G be a graph. If $\tau_k(G) \geq \ell$, then $\kappa(G) \geq k + \ell - 2$.*

Li et al. [23] obtained the following result.

Lemma 1.3 [23] *Let G be a connected graph with minimum degree δ. If there are two adjacent vertices of degree δ, then $\kappa_k(G) \leq \delta(G) - 1$.*

1.4 Application background and our result

In addition to being a natural combinatorial measure, pendant tree k-connectivity and generalized k-connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a *Steiner tree*, and popularly used in the physical design of VLSI circuits (see [11, 12, 37]). In this application, a Steiner tree is needed to share an electric signal by a set of terminal nodes. Steiner tree is also used in computer communication networks (see [9]) and optical wireless communication networks (see [6]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G.

Product networks were proposed based upon the idea of using the cross product as a tool for “combining” two known graphs with established properties to obtain a new one that inherits properties from both [8]. There has been an increasing interest in a class of interconnection networks called Cartesian product networks; see [1, 8, 14, 19, 21].

The *Cartesian product* of two graphs G and H, written as $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

In this paper, we obtain the following lower bound of $\tau_3(G\Box H)$.

Theorem 1.4 *Let G and H be two connected graphs. Then*

$$\tau_3(G\Box H) \geq \min\left\{3\left\lfloor\frac{\tau_3(G)}{2}\right\rfloor, 3\left\lfloor\frac{\tau_3(H)}{2}\right\rfloor\right\}.$$

Moreover, the bound is sharp; see Remark 3.1.

2 Proof of main result

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G\Box H) = \{(u_i, v_j) \,|\, 1 \leq i \leq n,\ 1 \leq j \leq m\}$. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v) \,|\, 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u, v_j) \,|\, 1 \leq j \leq m\}$. In the sequel, let $K_{s,t}$, K_n and P_n denote the complete bipartite graph of order $s + t$, complete graph of order n, and path of order n, respectively. If G is a connected graph and $x, y \in V(G)$, then the *distance* $d_G(x, y)$ between x and y is the length of a shortest path connecting x and y in G.

We now introduce the general idea of the proof of Theorem 1.4, with a running example (corresponding to Figure 2.1). From the definition, Cartesian product graph $G\Box H$ is a graph obtained by replacing each vertex of G by a copy of H and replacing each edge of G by a perfect matching of a complete bipartite graph $K_{m,m}$. Recall that $V(G) = \{u_1, u_2, \ldots, u_n\}$. Clearly, $V(G\Box H) = \bigcup_{i=1}^{n} V(H(u_i))$. For example, let $G = K_8$ (see Figure 2.1 (a)). Set $V(K_8) = \{u_i \,|\, 1 \leq i \leq 8\}$ and $|V(H)| = m$. Then $K_8\Box H$ is a graph obtained by replacing each vertex of K_8 by a copy of H and replacing each edge of K_8 by a perfect matching of complete bipartite graph $K_{m,m}$ (see Figure 2.1 (e)). Clearly, $V(K_8\Box H) = \bigcup_{i=1}^{8} V(H(u_i))$.

In this section, we give the proof of Theorem 1.4. For two connected graphs G and H, we prove that $\tau_3(G\Box H) \geq \min\{3\lfloor\frac{\tau_3(G)}{2}\rfloor, 3\lfloor\frac{\tau_3(H)}{2}\rfloor\}$. By the symmetry of Cartesian product graphs, we assume $\tau_3(H) \geq \tau_3(G)$. We need to show that $\tau_3(G\Box H) \geq 3\lfloor\frac{\tau_3(G)}{2}\rfloor$. Set $\tau_3(G) = k$ and $\tau_3(H) = \ell$. From the definition of $\tau_3(G\Box H)$, it suffices to show that $\kappa_{G\Box H}(S) \geq 3\lfloor\frac{k}{2}\rfloor$ for any $S \subseteq V(G\Box H)$ and $|S| = 3$. Furthermore, from the definition of $\kappa_{G\Box H}(S)$, we need to find $3\lfloor\frac{k}{2}\rfloor$ internally disjoint pendant S-Steiner trees in $G\Box H$. Let $S = \{x, y, z\}$. Recall that $V(G) = \{u_1, u_2, \ldots, u_n\}$. From the above analysis, we know that $x, y, z \in V(G\Box H) = \bigcup_{i=1}^{n} V(H(u_i))$. Without loss of generality, let $x \in H(u_i)$, $y \in H(u_j)$ and $z \in H(u_k)$ (note that u_i, u_j, u_k are not necessarily different). For the above example, we have $x, y, z \in V(K_8\Box H) = \bigcup_{i=1}^{8} V(H(u_i))$. Without loss of generality, let $x \in H(u_1)$, $y \in H(u_2)$ and $z \in H(u_3)$ (see Figure 2.1 (e)).

Because $u_i, u_j, u_k \in V(G)$ and $\tau_3(G) = k$, there are k internally disjoint pendant Steiner trees connecting $\{u_i, u_j, u_k\}$, say $T_1, T_2, \ldots, T_k$. Note that $\bigcup_{i=1}^{k} T_i$ is a subgraph of G. Let y', z' be the vertices corresponding to y, z in $H(u_i)$. Since $\tau_3(H) = \ell$, there are ℓ internally disjoint pendant Steiner trees connecting $\{x, y', z'\}$ in $H(u_i)$, say $T_1', T_2', \ldots, T_\ell'$. Thus $(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T_j')$ is a subgraph of $G\Box H$. For the above example, we have $\tau_3(G) = \tau_3(K_8) = k = 5 \leq \ell$. It suffices to prove that $\tau_3(G\Box H) \geq 3\lfloor\frac{\tau_3(G)}{2}\rfloor = 3\lfloor\frac{k}{2}\rfloor$. Clearly, there are $k = 5$ internally disjoint pendant Steiner trees connecting $\{u_1, u_2, u_3\}$, say T_1, T_2, T_3, T_4, T_5 (see T_1, T_2, T_3, T_4 in Figure 2.1 $(b), (c)$). Note that $T_1 \cup T_2$ or $T_3 \cup T_4$ is a subgraph of G (see Figure 2.1 $(b), (c)$). Then $(\bigcup_{i=1}^{4} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)$ is a subgraph of $G\Box H$ (see Figure 2.1 $(d), (h)$).

If we can prove that $\tau_{(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)}(S) \geq 3\lfloor\frac{k}{2}\rfloor$ for $S = \{x, y, z\}$, then $\tau_{G\Box H}(S) \geq \tau_{(\bigcup_{i=0}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)}(S) \geq 3\lfloor\frac{k}{2}\rfloor$ since $(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)$ is a subgraph of $G\Box H$. Therefore, the problem is converted into finding out $3\lfloor\frac{k}{2}\rfloor$ internally disjoint pendant S-Steiner trees in $(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)$. Since

$$\bigcup_{i=1}^{\lfloor k/2\rfloor} (T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$$

is a subgraph of $(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)$, we only need to show that

$$\tau_{G\Box H}(S) \geq \tau_{\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1}\cup T_{2i})\Box(T'_{2i-1}\cup T'_{2i})}(S) \geq 3\lfloor k/2\rfloor.$$

The structure of $\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$ in $\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1} \cup T_{2i})\Box H$ is shown in Figure 2.2. In order to show this structure clearly, we take $2\lfloor k/2\rfloor$ copies of $H(u_j)$, and $2\lfloor k/2\rfloor$ copies of $H(u_k)$. Note that, these $2\lfloor k/2\rfloor$ copies of $H(u_j)$ (respectively, $H(u_k)$) represent the same graph. For the above example, if we can prove that $\tau_{(T_1\cup T_2\cup T_3\cup T_4)\Box\bigcup_{i=1}^{\lfloor k/2\rfloor}(T'_{2i-1}\cup T'_{2i})}(S) \geq 3\lfloor k/2\rfloor$ for $S = \{x, y, z\}$, then $\tau_{G\Box H}(S) \geq \tau_{(T_1\cup T_2\cup T_3\cup T_4)\Box\bigcup_{i=1}^{\lfloor k/2\rfloor}(T'_{2i-1}\cup T'_{2i})}(S) \geq 3\lfloor k/2\rfloor$, as desired. The problem is converted into finding out $3\lfloor k/2\rfloor$ internally disjoint pendant S-Steiner trees in $(T_1 \cup T_2 \cup T_3 \cup T_4)\Box\bigcup_{i=1}^{\lfloor k/2\rfloor}(T'_{2i-1} \cup T'_{2i})$ (see Figure 2.1 (h)).

For each $T_{2i-1} \cup T_{2i}$ and $T'_{2i-1} \cup T'_{2i}$ $(1 \leq i \leq \ell)$, if we can find 3 internally disjoint pendant S-Steiner trees in $(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$, say $T_{i,1}, T_{i,2}, T_{i,3}$, then the total number of internally disjoint pendant S-Steiner trees in $\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$ are $3\lfloor k/2\rfloor$, which implies that $\tau_{G\Box H}(S) \geq \tau_{\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1}\cup T_{2i})\Box(T'_{2i-1}\cup T'_{2i})}(S) \geq 3\lfloor k/2\rfloor$ (Note that we must guarantee that any two trees in $\{T_{i,j} \mid 1 \leq i \leq \lfloor k/2\rfloor,\ 1 \leq j \leq 3\}$ are internally disjoint).

Furthermore, from the arbitrariness of S, we can get $\tau_3(G\Box H) \geq 3\lfloor\frac{\tau_3(G)}{2}\rfloor$ and complete the proof of Theorem 1.4. For the above example, we need to find 3 internally disjoint pendant S-Steiner trees in $(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$ (see Figure 2.1 $(f), (g)$). Then the total number of internally disjoint pendant S-Steiner in $\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$ are $3\lfloor\frac{k}{2}\rfloor$, which implies

$$\tau_{G\Box H}(S) \geq \tau_{\bigcup_{i=1}^{\lfloor k/2\rfloor}(T_{2i-1}\cup T_{2i})\Box(T'_{2i-1}\cup T'_{2i})}(S) \geq 3\lfloor k/2\rfloor.$$

Thus the result follows by the arbitrariness of S.

From the above analysis, we need to consider the graph $(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$ and prove that for any $S = \{x, y, z\} \subseteq V((T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i}))$ there are three

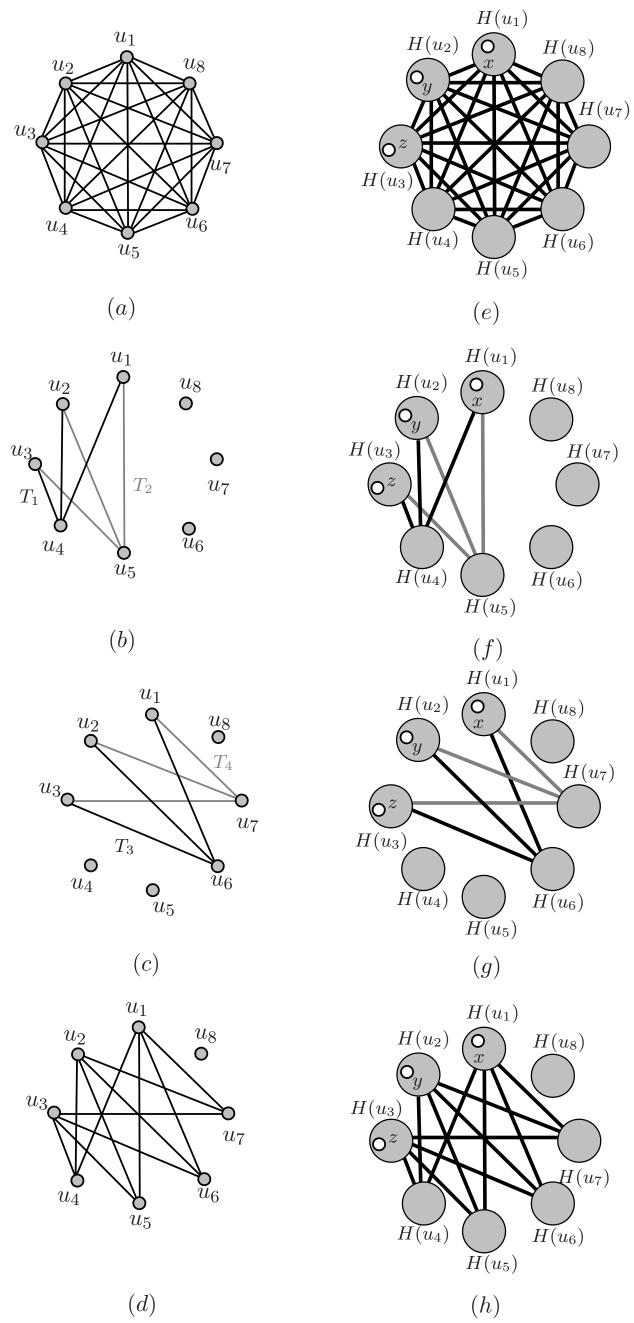

Figure 2.1: The structure of $\bigcup_{i=1}^{\lfloor k/2 \rfloor}(T_{2i-1} \cup T_{2i})\Box H$.

internally disjoint pendant S-Steiner trees in $(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i})$ for each i $(1 \leq i \leq \lfloor \frac{k}{2} \rfloor)$.

In the basis of such an idea, we study pendant tree 3-connectivity of Cartesian product of the union of two trees T_1, T_2 in G and the union of two trees T'_1, T'_2 in H first, and show that $\tau_3(T_{2i-1} \cup T_{2i})\Box(T'_{2i-1} \cup T'_{2i}) \geq 3$ in Subsection 2.2. After this preparation, we consider the graph $G\Box H$ where G, H are two general (connected) graphs and prove $\tau_3(G\Box H) \geq 3\lfloor \frac{\tau_3(G)}{2} \rfloor$ in Subsection 2.3. In Subsection 2.1, we investigate the pendant tree 3-connectivity of Cartesian product of a path P_n and a connected graph H. So the proof of Theorem 1.4 can be divided into the above mentioned three subsections. The first and second subsections are preparations of the last one.

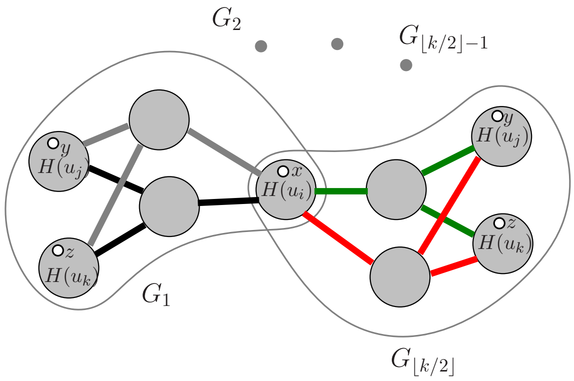

Figure 2.2: Structure of $\bigcup_{i=1}^{\lfloor k/2 \rfloor} G_i\Box H$, where $G_i = (T_{2i-1} \cup T_{2i})$.

2.1 Cartesian product of a path and a connected graph

A *subdivision* of G is a graph obtained from G by replacing edges with pairwise internally disjoint paths. Let G be a graph, and $S \subseteq V(G)$, $|S| = 3$. If T is an minimal pendant S-Steiner tree, then T is a subdivision of $K_{1,3}$, and hence T contains a vertex as its root. The following proposition is a preparation of Subsection 2.3.

Proposition 2.1 *Let H be a connected graph and P_n be a path with n vertices. Then $\tau_3(P_n\Box H) \geq \tau_3(H)$. Moreover, the bound is sharp.*

Suppose $\tau_3(H) = \ell$, $V(H) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Without loss of generality, let u_i and u_j be adjacent if and only if $|i - j| = 1$, where $1 \leq i \neq j \leq n$. It suffices to show that $\tau_{P_n\Box H}(S) \geq \ell$ for any $S = \{x, y, z\} \subseteq V(P_n\Box H)$, that is, there exist ℓ internally disjoint pendant S-Steiner trees in $P_n\Box H$. We proceed our proof by the following three lemmas.

Lemma 2.2 *If x, y, z belongs to the same $V(H(u_j))$ $(1 \leq j \leq n)$, then there exist ℓ internally disjoint pendant S-Steiner trees in $P_n\Box H$.*

Proof. Without loss of generality, we assume $x, y, z \in V(H(u_1))$. Since $\tau_3(H) = \ell$, it follows that there are ℓ internally disjoint pendant S-Steiner trees in $H(u_1)$, say $T_1, T_2, \ldots, T_\ell$. Clearly, they are ℓ internally disjoint pendant S-Steiner trees, as desired. ∎

Lemma 2.3 *If only two vertices of $\{x, y, z\}$ belong to some copy $H(u_j)$ $(1 \le j \le n)$, then there exist ℓ internally disjoint pendant S-Steiner trees in $P_n \Box H$.*

Proof. We may assume $x, y \in V(H(u_1))$ and $z \in V(H(u_j))$ $(2 \le j \le n)$. In the following argument, we can see that this assumption has no impact on the correctness of our proof. Let x', y' be the vertices corresponding to x, y in $H(u_j)$, z' be the vertex corresponding to z in $H(u_1)$.

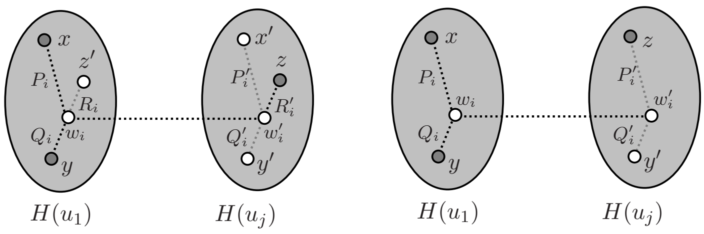

Figure 2.3: Graphs for Lemma 2.3.

Suppose $z' \notin \{x, y\}$. Since $\tau_3(H) = \ell$, it follows that $\tau_3(H(u_1)) = \tau_3(H(u_j)) = \ell$, and hence there exist ℓ internally disjoint pendant S-Steiner trees $T_1, T_2, \ldots, T_\ell$ in $H(u_1)$ and there exist ℓ internally disjoint pendant S-Steiner trees $T'_1, T'_2, \ldots, T'_\ell$ in $H(u_j)$ corresponding to $T_1, T_2, \ldots, T_\ell$ in $H(u_1)$, respectively. For each i $(1 \le i \le \ell)$, we let w_i, w'_i be the root of T_i, T'_i, respectively. Let P_i, Q_i, R_i denote the unique path connecting w_i and x, y, z', respectively. Let P'_i, Q'_i, R'_i denote the unique path connecting w'_i and x', y', z, respectively. Without loss of generality, let $w_i = (u_1, v_i)$ and $w'_i = (u_j, v_i)$. Then the trees T_i induced by the edges in $E(P_i) \cup E(Q_i) \cup E(R'_i) \cup \{(u_r, v_i)(u_{r+1}, v_i) \,|\, 1 \le r \le j-1\}$ $(1 \le i \le \ell)$ are ℓ internally disjoint pendant S-Steiner trees; see Figure 2.3 (a).

Suppose $z' \in \{x, y\}$. Without loss of generality, let $z' = x$. Since $\tau_3(H) = \ell$, it follows from Lemma 1.2 that $\kappa(H) \ge \ell + 1$, and hence $\kappa(H(u_1)) \ge \ell + 1$ and $\kappa(H(u_j)) \ge \ell + 1$. Then there exist $\ell + 1$ internally disjoint paths connecting x and y in $H(u_1)$, say $R_1, R_2, \ldots, R_{\ell+1}$, and there exist $\ell + 1$ internally disjoint paths connecting z and y' in $H(u_j)$, say $R'_1, R'_2, \ldots, R'_{\ell+1}$. Note that there is at most one path in $\{R_1, R_2, \ldots, R_{\ell+1}\}$, say $R_{\ell+1}$, such that its length is 1, and there is at most one path in $\{R'_1, R'_2, \ldots, R'_{\ell+1}\}$, say $R'_{\ell+1}$, such that its length is 1. For each i $(1 \le i \le \ell)$, there is an internal vertex w_i in R_i, and there is an internal vertex w'_i in R'_i. Let P_i, Q_i denote the unique path connecting w_i and x, y, respectively. Let

P_i', Q_i' denote the unique path connecting w_i' and y', z, respectively. Without loss of generality, let $w_i = (u_1, v_i)$ and $w_i' = (u_j, v_i)$. Then the trees T_i induced by the edges in $E(P_i) \cup E(Q_i) \cup E(P_i') \cup \{(u_r, v_i)(u_{r+1}, v_i) \mid 1 \leq r \leq j-1\}$ $(1 \leq i \leq \ell)$ are ℓ internally disjoint pendant S-Steiner trees, as desired. ∎

Lemma 2.4 *If x, y, z are contained in distinct $H(u_j)$s, then there exist ℓ internally disjoint pendant S-Steiner trees in $P_n \Box H$.*

Proof. We may assume that $x \in V(H(u_a))$, $y \in V(H(u_b))$, $z \in V(H(u_c))$, where $1 \leq a < b < c \leq n$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let y', z' be the vertices corresponding to y, z in $H(u_a)$, x', z'' be the vertices corresponding to x, z in $H(u_b)$ and x'', y'' be the vertices corresponding to x, y in $H(u_c)$.

Suppose that x, y', z' are distinct vertices in $H(u_a)$. Since $\tau_3(H) = \ell$, it follows that $\tau_3(H(u_a)) = \tau_3(H(u_b)) = \tau_3(H(u_c)) = \ell$, and hence there exist ℓ internally disjoint pendant S-Steiner trees $T_1, T_2, \ldots, T_\ell$ in $H(u_a)$, and there exist ℓ internally disjoint pendant S-Steiner trees $T_1', T_2', \ldots, T_\ell'$ in $H(u_b)$, and there exist ℓ internally disjoint pendant S-Steiner trees $T_1'', T_2'', \ldots, T_\ell''$ in $H(u_c)$. For each i $(1 \leq i \leq \ell)$, we let w_i, w_i', w_i'' be the root of T_i, T_i', T_i'', respectively. Let P_i, Q_i, R_i denote the unique paths connecting w_i and x, y', z', respectively. Let P_i', Q_i', R_i' denote the unique paths connecting w_i' and x', y, z'', respectively. Let P_i'', Q_i'', R_i'' denote the unique paths connecting w_i'' and x'', y'', z, respectively. Without loss of generality, let $w_i = (u_a, v_i)$, $w_i' = (u_b, v_i)$ and $w_i'' = (u_c, v_i)$. Then the trees T_i induced by the edges in $E(P_i) \cup E(Q_i') \cup E(R_i'') \cup \{(u_r, v_i)(u_{r+1}, v_i) \mid a \leq r \leq b-1\} \cup \{(u_r, v_i)(u_{r+1}, v_i) \mid b \leq r \leq c-1\}$ $(1 \leq i \leq \ell)$ are ℓ internally disjoint pendant S-Steiner trees; see Figure 2.4 (a).

Suppose that two of x, y', z' are the same vertex in $H(u_a)$. Without loss of generality, let $x = y'$. Since $\tau_3(H) = \ell$, it follows from Lemma 1.2 that $\kappa(H) \geq \ell + 1$, and hence $\kappa(H(u_a)) \geq \ell + 1$, $\kappa(H(u_b)) \geq \ell + 1$ and $\kappa(H(u_c)) \geq \ell + 1$. Then there exist $\ell + 1$ internally disjoint paths connecting x and z' in $H(u_a)$, say $R_1, R_2, \ldots, R_{\ell+1}$, and there exist $\ell+1$ internally disjoint paths connecting y and z'' in $H(u_b)$, say $R_1', R_2', \ldots, R_{\ell+1}'$, and there exist $\ell+1$ internally disjoint paths connecting x'' and z in $H(u_c)$, say $R_1'', R_2'', \ldots, R_{\ell+1}''$. Note that there is at most one path in $\{R_1, R_2, \ldots, R_{\ell+1}\}$, say $R_{\ell+1}$, such that its length is 1, and there is at most one path in $\{R_1', R_2', \ldots, R_{\ell+1}'\}$, say $R_{\ell+1}'$, such that its length is 1, and there is at most one path in $\{R_1'', R_2'', \ldots, R_{\ell+1}''\}$, say $R_{\ell+1}''$, such that its length is 1. For each i $(1 \leq i \leq \ell)$, there is an internal vertex w_i in R_i, and there is an internal vertex w_i' in R_i', and there is an internal vertex w_i'' in R_i''. Let P_i, Q_i denote the unique path connecting w_i and x, z', respectively. Let P_i', Q_i' denote the unique path connecting w_i' and y, z'', respectively. Let P_i'', Q_i'' denote the unique path connecting w_i' and x'', z, respectively. Without loss of generality, let $w_i = (u_a, v_i)$, $w_i' = (u_b, v_i)$ and $w_i'' = (u_c, v_i)$. Then the trees T_i induced by the edges in $E(P_i) \cup E(P_i') \cup E(Q_i'') \cup \{(u_r, v_i)(u_{r+1}, v_i) \mid a \leq r \leq b-1\} \cup \{(u_r, v_i)(u_{r+1}, v_i) \mid b \leq r \leq c-1\}$ $(1 \leq i \leq \ell)$ are ℓ internally disjoint pendant S-Steiner trees; see Figure 2.4 (b).

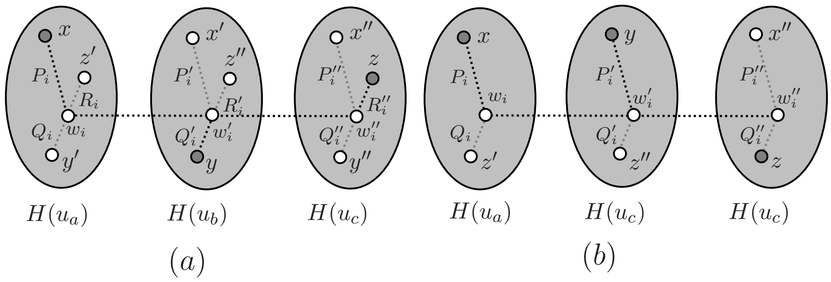

Figure 2.4: Graphs for Lemma 2.4.

Suppose that x, y', z' are the same vertex in $H(u_1)$. Since $\tau_3(H) = \ell$, it follows from Lemma 1.1 that $\delta(H) \geq \ell + 2$, and hence $\delta(H(u_a)) \geq \ell + 1$, $\delta(H(u_b)) \geq \ell + 1$ and $\delta(H(u_c)) \geq \ell + 1$. Then there are $\ell + 1$ neighbors of x in $H(u_a)$, say $(u_a, v_1), (u_a, v_2), \ldots, (u_a, v_{\ell+1})$. By the same reason, there are $\ell + 1$ neighbors of y in $H(u_b)$, say $(u_b, v_1), (u_b, v_2), \ldots, (u_b, v_{\ell+1})$, and there are $\ell + 1$ neighbors of z in $H(u_c)$, say $(u_c, v_1), (u_c, v_2), \ldots, (u_c, v_{\ell+1})$. Then the tree T_i induced by the edges in $\{x(u_a, v_i), y(u_b, v_i), z(u_c, v_i)\} \cup \{(u_s, v_i)(u_{s+1}, v_i) \,|\, a \leq s \leq b - 1\} \cup \{(u_s, v_i)(u_{s+1}, v_i) \,|\, b \leq s \leq c - 1\}$ is a pendant S-Steiner tree, where $1 \leq i \leq \ell + 1$. Therefore, the trees $T_1, T_2, \ldots, T_{\ell+1}$ are $\ell + 1$ internally disjoint pendant S-Steiner trees, as desired. ∎

From Lemmas 2.2, 2.3 and 2.4, we conclude that, for any $S \subseteq V(P_n \Box H)$, there exist ℓ internally disjoint pendant S-Steiner trees, and hence $\tau_{P_n \Box H}(S) \geq \ell$. From the arbitrariness of S, we have $\tau_3(P_n \Box H) \geq \ell$. The proof of Proposition 2.1 is complete.

2.2 Cartesian product of two trees in G and two trees in H

In this subsection, we consider the pendant tree 3-connectivity of Cartesian product of two trees in G and two trees in H, which is a preparation of the next subsection.

Proposition 2.5 *Let G, H be two graphs. For $S = \{x, y, z\} \subseteq V(G \Box H)$, we assume that u_1, u_2, u_3 are three vertices in $V(G)$ such that $x \in V(H(u_1))$, $y \in V(H(u_2))$, and $z \in V(H(u_3))$. Let T_1, T_2 be two minimal pendant Steiner trees connecting $\{u_1, u_2, u_3\}$ in G. Let y', z' be the vertices corresponding to y, z in $H(u_1)$. Let T_1', T_2' be two pendant Steiner trees connecting $\{x, y', z'\}$ in $H(u_1)$. Then*

$$\tau_{(T_1 \cup T_2) \Box (T_1' \cup T_2')}(S) \geq 3.$$

Proof. Since T_1, T_2 are two minimal pendant Steiner trees connecting $\{u_1, u_2, u_3\}$, it follows that T_1, T_2 are subdivisions of $K_{1,3}$ and hence have roots, say u_r, u_s, respectively. Note that $x \in V(H(u_1))$, $y \in V(H(u_2))$ and $z \in V(H(u_3))$. Let y', z' be the vertices corresponding to y, z in $H(u_1)$, x', z'' be the vertices corresponding to x, z in $H(u_2)$ and x'', y'' be the vertices corresponding to x, y in $H(u_3)$. Let x_1, y_1, z_1 be the vertices in $H(u_r)$ corresponding to x, y', z' in $H(u_1)$, respectively, and let x_2, y_2, z_2 be the vertices in $H(u_s)$ corresponding to x, y', z' in $H(u_1)$, respectively.

- Let R_1, R_2, R_3 be the three paths connecting u_r and u_1, u_2, u_3, respectively.
  - Set $R_1 = u_1 p_1 p_2 \dots p_a u_r$, where $p_i \in V(G), 1 \leq i \leq a$.
  - Set $R_2 = u_2 p'_1 p'_2 \dots p'_b u_r$, where $p'_i \in V(G), 1 \leq i \leq b$.
  - Set $R_3 = u_3 p''_1 p''_2 \dots p''_c u_r$, where $p''_i \in V(G), 1 \leq i \leq c$.
- Let R'_1, R'_2, R'_3 be the three paths connecting u_s and u_1, u_2, u_3, respectively.
  - Set $R'_1 = u_1 q_1 q_2 \dots q_d u_s$, where $q_i \in V(G), 1 \leq i \leq d$.
  - Set $R'_2 = u_2 q'_1 q'_2 \dots q'_e u_s$, where $q'_i \in V(G), 1 \leq i \leq e$.
  - Set $R'_3 = u_3 q''_1 q''_2 \dots q''_f u_s$, where $q''_i \in V(G), 1 \leq i \leq f$.

We distinguish the following three cases to show this proposition.

Case 1. The vertices x, y', z' are distinct vertices in $H(u_1)$.

In order to show the structure of pendant S-Steiner trees clearly, we assume all of the following.

- Let w, t be the roots of T'_1, T'_2, respectively.
- Let w', w'', w_1, w_2 be the vertices corresponding to w in $H(u_2), H(u_3), H(u_r), H(u_s)$, respectively.
- Let t', t'', t_1, t_2 be the vertices corresponding to t in $H(u_2), H(u_3), H(u_r), H(u_s)$, respectively.
- Let $P_{1,1}, P_{1,2}, P_{1,3}$ be the three paths connecting w and x, y', z' in T'_1, respectively.
- Let $Q_{1,1}, Q_{1,2}, Q_{1,3}$ be the three paths connecting t and x, y', z' in T'_2, respectively.
- Let $P_{2,j}, P_{3,j}, P_{r,j}, P_{s,j}$ $(1 \leq j \leq 3)$ be the paths corresponding to $P_{1,j}$ in $H(u_2), H(u_3)$, $H(u_r), H(u_s)$, respectively.
- Let $Q_{2,j}, Q_{3,j}, Q_{r,j}, Q_{s,j}$ $(1 \leq j \leq 3)$ be the paths corresponding to $Q_{1,j}$ in $H(u_2), H(u_3)$, $H(u_r), H(u_s)$, respectively.
- Without loss of generality, let $x = (u_1, v_1)$, $y' = (u_1, v_2)$, $z' = (u_1, v_3)$, $w = (u_1, v_4)$ and $t = (u_1, v_5)$.

Let T be the S-Steiner tree induced by the edges in

$$
\begin{aligned}
& E(P_{1,1}) \cup E(P_{r,2}) \cup E(P_{r,3}) \\
& \cup \{w(p_1, v_4)\} \cup \{(p_i, v_4)(p_{i+1}, v_4) \mid 1 \leq i \leq a-1\} \cup \{(p_a, v_4) w_1\} \\
& \cup \{y(p'_1, v_2)\} \cup \{(p'_i, v_2)(p'_{i+1}, v_2) \mid 1 \leq i \leq b-1\} \cup \{(p'_b, v_2) y_1\} \\
& \cup \{z(p''_1, v_3)\} \cup \{(p''_i, v_3)(p''_{i+1}, v_3) \mid 1 \leq i \leq c-1\} \cup \{(p''_c, v_3) z_1\},
\end{aligned}
$$

and T' be the S-Steiner tree induced by the edges in

$$\begin{aligned}&E(Q_{s,1}) \cup E(Q_{2,2}) \cup E(Q_{3,3})\\ &\cup\{x(q_1, v_1)\} \cup \{(q_i, v_1)(q_{i+1}, v_1) \,|\, 1 \leq i \leq d-1\} \cup \{(q_d, v_1)x_2\}\\ &\cup\{t'(q'_1, v_5)\} \cup \{(q'_i, v_5)(q'_{i+1}, v_5) \,|\, 1 \leq i \leq e-1\} \cup \{(q'_e, v_5)t_2\}\\ &\cup\{t''(q''_1, v_5)\} \cup \{(q''_i, v_5)(q''_{i+1}, v_5) \,|\, 1 \leq i \leq f-1\} \cup \{(q''_c, v_5)t_2\},\end{aligned}$$

and T'' be the S-Steiner tree induced by the edges in

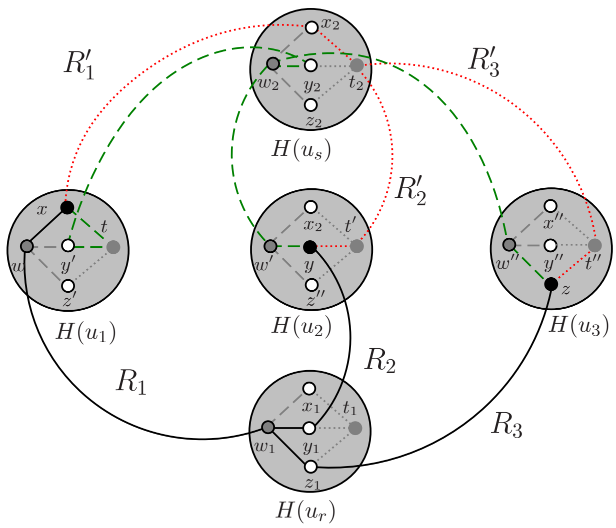

Figure 2.5: Graphs for Case 1 of Proposition 2.5.

$$\begin{aligned}&E(P_{s,2}) \cup E(Q_{1,2}) \cup E(Q_{1,1}) \cup E(P_{2,2}) \cup E(P_{3,3})\\ &\cup\{y'(q_1, v_2)\} \cup \{(q_i, v_2)(q_{i+1}, v_2) \,|\, 1 \leq i \leq d-1\} \cup \{(q_d, v_2)y_2\}\\ &\cup\{w'(q'_1, v_4)\} \cup \{(q'_i, v_4)(q'_{i+1}, v_4) \,|\, 1 \leq i \leq e-1\} \cup \{(q'_e, v_4)w_2\}\\ &\cup\{w''(q''_1, v_4)\} \cup \{(q''_i, v_4)(q''_{i+1}, v_4) \,|\, 1 \leq i \leq f-1\} \cup \{(q''_f, v_4)w_2\}.\end{aligned}$$

Since T, T', T'' are internally disjoint, it follows that

$$\tau_{(T_1 \cup T_2)\Box(T'_1 \cup T'_2)}(S) \geq 3,$$

as desired.

Case 2. Two of x, y', z' are the same vertex in $H(u_1)$.

Without loss of generality, let $x = y'$. Note that there are two paths P_1, Q_1 connecting x and z' in T_1, T_2, respectively. Observe that the length of P_1 is 1 but the length of Q_1 is at least 2, or the length of Q_1 is 1 but the length of P_1 is at least 2, or the lengths of Q_1 and P_1 are at least 2. We now assume that the length of P_1 is at least 2. Then there exists an internal vertex in P_1, say t, and hence t divides

P_1 into two paths, say $P_{1,1}, P_{1,2}$. In order to show the structure of pendant S-Steiner trees clearly, we assume the following.

- Let x_1, x_2 be the vertices corresponding to x in $H(u_r), H(u_s)$, respectively.
- Let z_1, z_2 be the vertices corresponding to z' in $H(u_r), H(u_s)$, respectively.
- Let t', t'', t_1, t_2 be the vertices corresponding to t in $H(u_2), H(u_3), H(u_r), H(u_s)$, respectively.
- Without loss of generality, let $x = (u_1, v_1)$, $z' = (u_1, v_2)$ and $t = (u_1, v_3)$.
- Let $P_{2,j}, P_{3,j}, P_{r,j}, P_{s,j}$ $(j = 1, 2)$ be the paths corresponding to $P_{1,j}$ in $H(u_2)$, $H(u_3), H(u_r), H(u_s)$, respectively.
- Let Q_2, Q_3, Q_r, Q_s be the paths corresponding to Q_1 in $H(u_2), H(u_3), H(u_r)$, $H(u_s)$, respectively.

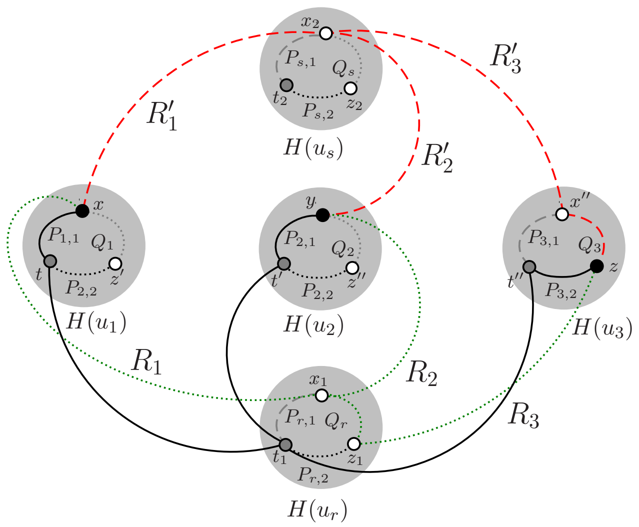

Figure 2.6: Graphs for Case 2 of Proposition 2.5.

Let T be the S-Steiner tree induced by the edges in

$$\begin{aligned}
&\{x(p_1, v_1)\} \cup \{(p_i, v_1)(p_{i+1}, v_1) \,|\, 1 \le i \le a-1\} \cup \{(p_a, v_1)x_1\} \\
&\cup\{y(p_1', v_1)\} \cup \{(p_i', v_1)(p_{i+1}', v_1) \,|\, 1 \le i \le b-1\} \cup \{(p_b', v_1)x_1\} \\
&\cup E(Q_r) \cup \{z(p_1'', v_2)\} \cup \{(p_i'', v_2)(p_{i+1}'', v_2) \,|\, 1 \le i \le c-1\} \cup \{(p_c'', v_2)z_1\},
\end{aligned}$$

and T' be the S-Steiner tree induced by the edges in

$$\begin{aligned}
&\{x(q_1, v_1)\} \cup \{(q_i, v_1)(q_{i+1}, v_1) \,|\, 1 \le i \le d-1\} \cup \{(q_d, v_1)x_2\} \\
&\cup\{y(q_1', v_1)\} \cup \{(q_i', v_1)(q_{i+1}', v_1) \,|\, 1 \le i \le e-1\} \cup \{(q_e', v_1)x_2\} \\
&\cup\{x''(q_1'', v_1)\} \cup \{(q_i'', v_1)(q_{i+1}'', v_1) \,|\, 1 \le i \le f-1\} \cup \{(q_f'', v_1)x_2\} \cup E(Q_3),
\end{aligned}$$

and T'' be the S-Steiner tree induced by the edges in

$$\begin{aligned}&E(P_{1,1}) \cup E(P_{2,1}) \cup E(P_{3,2})\\&\cup\{t(p_1, v_3)\} \cup \{(p_i, v_3)(p_{i+1}, v_3) \mid 1 \le i \le a-1\} \cup \{(p_a, v_3)t_1\}\\&\cup\{t'(p'_1, v_3)\} \cup \{(p'_i, v_3)(p'_{i+1}, v_3) \mid 1 \le i \le b-1\} \cup \{(p'_b, v_3)t_1\}\\&\cup\{t''(p''_1, v_3)\} \cup \{(p''_i, v_3)(p''_{i+1}, v_3) \mid 1 \le i \le c-1\} \cup \{(p''_c, v_3)t_1\}.\end{aligned}$$

Since T, T', T'' are internally disjoint, it follows that

$$\tau_{(T_1 \cup T_2)\Box(T'_1 \cup T'_2)}(S) \ge 3,$$

as desired.

Case 3. x, y', z' are the same vertex in $H(u_1)$.

Let w, t be the neighbors of x in T'_1, T'_2, respectively. Let w', w'', w_1, w_2 be the vertices corresponding to w in $H(u_2), H(u_3), H(u_r), H(u_s)$, respectively. Let t', t'', t_1, t_2 be the vertices corresponding to t in $H(u_2), H(u_3), H(u_r), H(u_s)$, respectively. Without loss of generality, let $x = (u_1, v_1)$, $w = (u_1, v_2)$ and $t = (u_1, v_3)$.

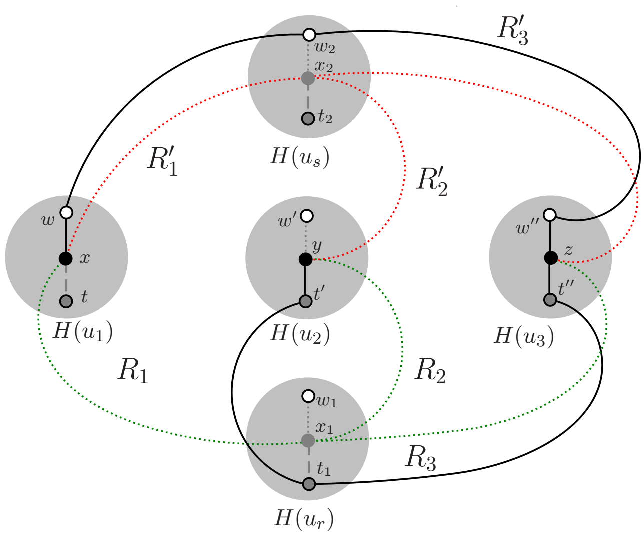

Figure 2.7: Graphs for Case 3 of Proposition 2.5.

Let T be the S-Steiner tree induced by the edges in

$$\begin{aligned}&\{x(p_1, v_1)\} \cup \{(p_i, v_1)(p_{i+1}, v_1) \mid 1 \le i \le a-1\} \cup \{(p_a, v_1)x_1\}\\&\cup\{y(p'_1, v_1)\} \cup \{(p'_i, v_1)(p'_{i+1}, v_1) \mid 1 \le i \le b-1\} \cup \{(p'_b, v_1)x_1\}\\&\cup\{z(p''_1, v_1)\} \cup \{(p''_i, v_1)(p''_{i+1}, v_1) \mid 1 \le i \le c-1\} \cup \{(p''_c, v_1)x_1\},\end{aligned}$$

and T' be the S-Steiner tree induced by the edges in

$$\begin{aligned}&\{x(q_1,v_1)\}\cup\{(q_i,v_1)(q_{i+1},v_1)\,|\,1\leq i\leq d-1\}\cup\{(q_d,v_1)x_2\}\\ &\cup\{y(q'_1,v_1)\}\cup\{(q'_i,v_1)(q'_{i+1},v_1)\,|\,1\leq i\leq e-1\}\cup\{(q'_e,v_1)x_2\}\\ &\cup\{z(q''_1,v_1)\}\cup\{(q''_i,v_1)(q''_{i+1},v_1)\,|\,1\leq i\leq f-1\}\cup\{(q''_f,v_1)x_2\},\end{aligned}$$

and T'' be the S-Steiner tree induced by the edges in

$$\begin{aligned}&\{xw,w(q_1,v_2)\}\cup\{(q_i,v_2)(q_{i+1},v_2)\,|\,1\leq i\leq d-1\}\cup\{(q_d,v_2)w_2\}\\ &\cup\{w''(q''_1,v_2)\}\cup\{(q''_i,v_2)(q''_{i+1},v_2)\,|\,1\leq i\leq f-1\}\cup\{(q''_f,v_2)w_2\}\\ &\cup\{w''z,zt'',t''(p''_1,v_3)\}\cup\{(p''_i,v_3)(p''_{i+1},v_3)\,|\,1\leq i\leq c-1\}\cup\{(p''_c,v_3)t_1\}\\ &\cup\{yt',t'(p'_1,v_3)\}\cup\{(p'_i,v_3)(p'_{i+1},v_3)\,|\,1\leq i\leq b-1\}\cup\{(p''_b,v_3)t_1\}.\end{aligned}$$

Since T,T',T'' are internally disjoint, we have

$$\tau_{(T_1\cup T_2)\Box(T'_1\cup T'_2)}(S)\geq 3,$$

as desired.

From the above argument, there exist three internally disjoint pendant S-Steiner trees, which implies $\tau_{T\Box H}(S)\geq 3$. The proof is now complete. ∎

2.3 Cartesian product of two general graphs

After the above preparations, we are ready to prove Theorem 1.4 in this subsection.

Proof of Theorem 1.4: Suppose $\tau_3(G)=k$ and $\tau_3(H)=\ell$. Assume without loss of generality $k\leq\ell$. If $\ell=0$ or $k=0$ then the result follows. So we assume that $k\geq 1$ and $\ell\geq 1$. Recall that $V(G)=\{u_1,u_2,\ldots,u_n\}$, $V(H)=\{v_1,v_2,\ldots,v_m\}$. From the definition of $\tau_3(G\Box H)$ and the symmetry of Cartesian product graphs, we need to prove that $\tau_{G\Box H}(S)\geq 3\lfloor k/2\rfloor$ for any $S=\{x,y,z\}\subseteq V(G\Box H)$. Furthermore, it suffices to show that there exist $3\lfloor k/2\rfloor$ internally disjoint pendant S-Steiner trees in $G\Box H$. Clearly, $V(G\Box H)=\bigcup_{i=1}^{n}V(H(u_i))$. Without loss of generality, let $x\in V(H(u_i))$, $y\in V(H(u_j))$ and $z\in V(H(u_k))$.

Case 1. The vertices x,y,z belong to the same $V(H(u_i))$ $(1\leq i\leq n)$.

Without loss of generality, let $x,y,z\in V(H(u_1))$. From Lemma 1.1, $\delta(G)\geq\tau_3(G)+2=k+2$ and hence the vertex u_1 has at least $k+2$ neighbors in G. Select $k+2$ neighbors from them, say $u_2,u_3,\ldots,u_{k+3}$. Without loss of generality, let $x=(u_1,v_1)$, $y=(u_1,v_2)$ and $z=(u_1,v_3)$. Note that there is a pendant Steiner tree T'_i connecting $\{(u_i,v_1),(u_i,v_2),(u_i,v_3)\}$. Then the tree induced by the edges in $E(T')\cup\{x(u_i,v_1),y(u_i,v_2),z(u_i,v_3)\}$ are $k+2$ internally disjoint pendant S-Steiner trees in $G\Box H$, which contain no edge of $H(u_1)$. Since $\tau_3(H)=\ell$, it follows that there are ℓ internally disjoint pendant S-Steiner trees in $H(u_1)$. Observe that these ℓ pendant S-Steiner trees and the trees T_i $(2\leq i\leq k+3)$ are internally disjoint. So the total number of internally disjoint pendant S-Steiner trees is $k+\ell+2>3\lfloor k/2\rfloor$, as desired.

Case 2. Only two vertices of $\{x, y, z\}$ belong to some copy $H(u_j)$ $(1 \leq j \leq n)$.

Without loss of generality, let $x, y \in H(u_1)$ and $z \in H(u_2)$. From Lemma 1.2, $\kappa(G) \geq \tau_3(G) + 1 = k + 1$ and hence there exist $k + 1$ internally disjoint paths connecting u_1 and u_2 in G, say $P_1, P_2, \ldots, P_{k+1}$. Clearly, there exists at most one of $P_1, P_2, \ldots, P_{k+1}$, say P_{k+1}, such that $P_{k+1} = u_1u_2$. We may assume that the length of P_i $(1 \leq i \leq k)$ is at least 2. From Proposition 2.1, there exist ℓ internally disjoint pendant S-Steiner trees in $P_{k+1}\Box H$, say $T_1, T_2, \ldots, T_\ell$. For each P_i $(1 \leq i \leq k)$, since P_i is a path of length at least 2, it follows that there exists an internal vertex in P_i, say u_i. Let Q_i, R_i be the two paths connecting u_i and u_1, u_2 in P_i, respectively. Set $Q_i = u_1, u_1', u_2', \ldots, u_s', u_i$ and $R_i = u_2, u_1'', u_2'', \ldots, u_t'', u_i$. In the following argument, we can see that this assumption has no impact on the correctness of our proof. Let x', y' be the vertices corresponding to x, y in $H(u_2)$, z' be the vertex corresponding to z in $H(u_1)$, and x'', y'', z'' be the vertices corresponding to x, y, z in $H(u_i)$.

Suppose $z' \notin \{x, y\}$. Without loss of generality, let $x = (u_1, v_1)$, $y = (u_1, v_2)$ and $z = (u_2, v_3)$. Since $\tau_3(H) = \ell \geq 1$, it follows that there is a pendant Steiner tree connecting $\{x'', y'', z''\}$ in $H(u_i)$, say T^i. Furthermore, the tree T_i' $(1 \leq i \leq k)$ induced by the edges in

$$\begin{aligned}
&E(T^i) \cup \{x(u_1', v_1)\} \cup \{(u_j', v_1)(u_{j+1}', v_1) \mid 1 \leq j \leq s\} \cup \{x''(u_s', v_1)\} \cup \{y(u_1', v_2)\} \\
&\cup \{(u_j', v_2)(u_{j+1}', v_2) \mid 1 \leq j \leq s\} \cup \{x''(u_s', v_2)\} \cup \{z(u_1'', v_3)\} \\
&\cup \{(u_j'', v_2)(u_{j+1}'', v_2) \mid 1 \leq j \leq t\} \cup \{z''(u_s'', v_3)\}
\end{aligned}$$

is a pendant S-Steiner tree. Obviously, the trees $T_1, T_2, \ldots, T_\ell, T_1', T_2', \ldots, T_k'$ are $k + \ell \geq 3\lfloor k/2 \rfloor$ internally disjoint pendant S-Steiner trees.

Suppose $z' \in \{x, y\}$. Without loss of generality, let $z' = x$, $x = (u_1, v_1)$, $y = (u_1, v_2)$. Then $z = (u_2, v_1)$. Since $\tau_3(H) \geq 1$, it follows that there is a path connecting x'' and y'', say P'. Furthermore, the tree T_i' $(1 \leq i \leq k)$ induced by the edges in

$$\begin{aligned}
&E(P') \cup \{x(u_1', v_1)\} \cup \{(u_j', v_1)(u_{j+1}', v_1) \mid 1 \leq j \leq s\} \cup \{x''(u_s', v_1)\} \cup \{y(u_1', v_2)\} \\
&\cup \{(u_j', v_2)(u_{j+1}', v_2) \mid 1 \leq j \leq s\} \cup \{x''(u_s', v_2)\} \cup \{z(u_1'', v_3)\} \\
&\cup \{(u_j'', v_2)(u_{j+1}'', v_2) \mid 1 \leq j \leq t\} \cup \{z''(u_s'', v_3)\}
\end{aligned}$$

is a pendant S-Steiner tree. Obviously, the trees $T_1, T_2, \ldots, T_\ell, T_1', T_2', \ldots, T_k'$ are $k + \ell \geq 3\lfloor k/2 \rfloor$ internally disjoint pendant S-Steiner trees.

Case 3. The vertices x, y, z are contained in distinct $H(u_i)$s.

Without loss of generality, let $x \in V(H(u_1))$, $y \in V(H(u_2))$ and $z \in V(H(u_3))$. Since $\tau_3(G) = k$, it follows that there exist k internally disjoint pendant Steiner trees connecting $\{u_1, u_2, u_3\}$ in G, say $T_1, T_2, \ldots, T_k$. Let y', z' be the vertices corresponding to y, z in $H(u_1)$, x', z'' be the vertices corresponding to x, z in $H(u_i)$ and x'', y'' be the vertices corresponding to x, y in $H(u_j)$. Since $\tau_3(H) = \ell$, it follows that there exist ℓ internally disjoint pendant Steiner trees connecting $\{x, y', z'\}$ in $H(u_1)$, say $T_1', T_2', \ldots, T_\ell'$. Note that $\bigcup_{i=1}^{k} T_i$ is a subgraph of G, $\bigcup_{j=1}^{\ell} T_j'$ is a subgraph of H, and $(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T_j')$ is a subgraph of $G\Box H$. From Proposition 2.5, for any T_i, T_j $(1 \leq i \neq j \leq k)$ and any T_r', T_s' $(1 \leq r \neq s \leq \ell)$, $(T_i \cup T_j)\Box(T_r \cup T_s)$ contains internally disjoint pendant S-Steiner trees. Since $k \leq \ell$, there exist $3\lfloor k/2 \rfloor$ internally disjoint pendant S-Steiner trees in $(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)$, and hence there are $3\lfloor k/2 \rfloor$ internally disjoint pendant S-Steiner trees in $G\Box H$.

From the above argument, we conclude, for any $S \subseteq V(G\Box H)$, that

$$\tau_{G\Box H}(S) \geq \tau_{(\bigcup_{i=1}^{k} T_i)\Box(\bigcup_{j=1}^{\ell} T'_j)}(S) \geq 3\lfloor k/2 \rfloor,$$

which implies that $\tau_3(G\Box H) \geq 3\lfloor k/2 \rfloor = 3\lfloor \tau_3(G)/2 \rfloor$. The proof is complete. ∎

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian product networks.

Given a vertex x and a set U of vertices, an (x, U)-*fan* is a set of paths from x to U such that any two of them share only the vertex x. The *size* of an (x, U)-fan is the number of internally disjoint paths from x to U.

Lemma 3.1 (Fan Lemma, [39], p. 170) *A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an (x, U)-fan of size k.*

In [38], Špacapan obtained the following result.

Lemma 3.2 [38] *Let G and H be two nontrivial graphs. Then*

$$\kappa(G\Box H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}.$$

3.1 Grid graph, mesh, and torus

A *two-dimensional grid graph* $G_{n,m}$ that is the Cartesian product $P_n\Box P_m$ of path graphs on m and n vertices. For more details on grid graph, we refer to [3, 16].

Proposition 3.3 *Let n and m be two integers with $n \geq 3, m \geq 3$. The network $P_n\Box P_m$ has no pendant Steiner tree connecting any three nodes.*

Proof. From Theorem 1.4, we have $\tau_3(P_n\Box P_m) \geq 3\lfloor \frac{\tau_3(P_n)}{2} \rfloor + 3\lfloor \frac{\tau_3(P_m)}{2} \rfloor = 0$. Choose a vertex of degree 2 in $P_n\Box P_m$, say x. Let y, z be two neighbors of x. Then there is no pendant Steiner tree connecting $\{x, y, z\}$. Therefore, $\tau_3(P_n\Box P_m) = 0$. ∎

Remark 3.1. For $P_n\Box P_m$ $(n \geq 3, m \geq 3)$, $\tau_3(P_n\Box P_m) = 0 = 3\lfloor \frac{\tau_3(P_n)}{2} \rfloor + 3\lfloor \frac{\tau_3(P_m)}{2} \rfloor$. So the graph $P_n\Box P_m$ is a sharp example of Theorem 1.4.

An *n-dimensional mesh* is the Cartesian product of n paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An n-dimensional hypercube is a special case of an n-dimensional mesh, in which the n linear arrays are all of size 2; see [18].

Corollary 3.4 *Let k be a positive integer with $k \geq 3$. For n-dimensional mesh $P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_n}$,*

$$0 \leq \tau_k(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_n}) \leq n - k + 2.$$

Proof. From Lemma 3.2, $\kappa(P_{m_1} \Box P_{m_2} \Box \ldots \Box P_{m_n}) \leq \delta(P_{m_1} \Box P_{m_2} \Box \ldots \Box P_{m_{n-1}}) + \delta(P_{m_n}) = n$, and hence

$$0 \leq \tau_k(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_n}) \leq n - k + 2$$

by Lemma 1.2. ∎

An *n-dimensional torus* is the Cartesian product of n cycles $C_{m_1}, C_{m_2}, \cdots, C_{m_n}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. [19] showed that there are n edge-disjoint spanning trees in an n-dimensional torus.

Proposition 3.5 *Let k be a positive integer with $k \geq 3$. For network $C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}$,*

$$1 \leq \tau_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}) \leq 2n - k + 2,$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

Proof. Set $G = C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}$. From Lemma 3.2, we have $\kappa(G) = 2n$, and hence

$$\tau_k(G) \leq 2n - k + 2$$

by Lemma 1.2. Since $\kappa(G) = 2n > k$, it follows from Lemma 3.1 that there is at least one pendant S-Steiner tree for any $S \subseteq V(G)$ and $|S| = k$, and hence

$$1 \leq \tau_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}) \leq 2n - k + 2,$$

as desired. ∎

3.2 Generalized hypercube and hyper Petersen network

Let K_m be a clique of m vertices, $m \geq 2$. An *n-dimensional generalized hypercube* [8, 10] is the product of m cliques. We have the following:

Proposition 3.6 *Let k be a positive integer with $k \geq 3$. For network $K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_n}$ where $m_i \geq k$ $(1 \leq i \leq n)$,*

$$\tau_k(K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_n}) \leq \sum_{i=1}^{n} m_i - n - k + 2.$$

Proof. From Lemma 3.2, we have $\kappa(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) = \sum_{i=1}^{n} m_i - n$, and hence

$$\tau_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) \leq \sum_{i=1}^{n} m_i - n - k + 2$$

by Lemma 1.2. ∎

An *n-dimensional hyper Petersen network HP_n* is the product of the well-known Petersen graph and Q_{n-3} [7], where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$-dimensional hypercube. Note that HP_3 is just the Petersen graph.

Proposition 3.7 *(a) The network HP_3 has one pendant Steiner tree connecting any three nodes.*

(b) The network HP_4 has two internally disjoint pendant Steiner trees connecting any three nodes. The number of internally disjoint pendant Steiner trees is the maximum.

Proof. (a) Note that HP_3 is just the Petersen graph. Set $G = HP_3$. Since $\delta(G) = 3$, it follows that $\tau_3(G) \leq 1$ by Lemma 1.1. From Lemma 3.1, there exists an (x, S)-*fan* for any $S \subseteq V(G)$ and $|S| = 3$, where $x \in V(G) \setminus S$. Thus $\tau(S) \geq 1$, and hence $\tau_3(G) = 1$, that is, HP_3 has one pendant Steiner tree connecting any three nodes.

(b) Since $\delta(G) = 4$, it follows from Lemma 1.1 that $\tau_3(HP_4) \leq 2$. One can check that for any $S \subseteq V(G)$ and $|S| = 3$, $\tau(S) \geq 2$. So $\tau_3(G) = 2$. ∎

4 Concluding Remarks

In this paper, we have proved that $\tau_3(G\Box H) \geq \min\{3\lfloor\frac{\tau_3(G)}{2}\rfloor, 3\lfloor\frac{\tau_3(H)}{2}\rfloor\}$ for any two connected graphs G and H. For general k, we can propose the following problem: Give exact value or sharp upper and lower bounds of $\tau_k(G * H)$, where $*$ is a kind of graph products.

Acknowledgements

The authors are very grateful to the editor and referees for their valuable comments and suggestions, which improved the presentation of this paper.

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(Received 23 Aug 2016; revised 2 July 2017, 21 Sep 2017)