AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(1)** (2018), Pages 12–14

A note on degree/diameter monotonicity of digraphs

Slobodan Filipovski

University of Primorska
Koper
Slovenia
slobodan.filipovski@famnit.upr.si

Abstract

Let $n_{d,k}$ be the largest order of a directed graph (digraph) with given maximum out-degree d and diameter k. In this note we show that $n_{d,k}$ is strictly monotonic increasing in each of d and k.

1 Introduction

By a *digraph* we mean a structure $G = (V, A)$, where $V(G)$ is a finite set of *vertices*, and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices $u, v \in V(G)$ called *arcs*. The *order* of the digraph G is the number of vertices in G. The *distance* from vertex u to vertex v in G is the length of the shortest path from u to v. The *diameter* k of a digraph G is the maximum distance between any two vertices in G.

An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in A(G)$. Similarly, an *out-neighbour* of a vertex v is a vertex w such that $(v, w) \in A(G)$. The *in-degree*, respectively *out-degree*, of a vertex $v \in V(G)$ is the number of its in-neighbours, respectively out-neighbours, in G. If both the in-degree and the out-degree equals d for every vertex, then the digraph G is called a *diregular digraph* of degree d.

Let (d, k)-digraph denote a directed graph of maximum out-degree d and diameter k, and let $n_{d,k}$ be the largest order of a (d, k)-digraph. Let n_i, for $0 \le i \le k$, be the number of vertices at distance i from a distinguished vertex. Then, $n_i \le d^i$, for $0 \le i \le k$. Hence,

$$n_{d,k} = \sum_{i=0}^{k} n_i \le 1 + d + \ldots + d^{k-1} + d^k = \begin{cases} \frac{d^{k+1}-1}{d-1}, & \text{if } d > 1, \\ k+1, & \text{if d=1.} \end{cases} \tag{1}$$

The number on the right-hand side of (1), denoted by $M_{d,k}$, is called the *Moore bound* for (d, k)-digraphs. A digraph whose order is equal to the Moore bound is called a *Moore digraph*. It is well known that $n_{d,k} = M_{d,k}$ only in the trivial cases when $d = 1$ (directed cycles of length $k + 1$) or $k = 1$ (complete digraphs of order $d + 1$).

In this note, we give a positive answer to the question concerning the degree/diameter problem of digraphs asked in [1]: is $n_{d,k}$ monotonic in d and k?

2 On degree and diameter monotonicity of digraphs

Employing *Kautz digraphs*, we show that $n_{d,k}$ is strictly monotonic increasing in k and in d. These graphs are diregular digraphs of large order. Moreover, they are iterated line digraphs of complete digraphs. The Kautz digraph of degree d and diameter k has order $d^k + d^{k-1}$ (see [1] and [2]).

Our proof is based on a relatively simple idea. The unknown optimal digraphs must have at least as many vertices as the Kautz digraph, but no more than the Moore digraph. Using the numbers $d^k + d^{k-1}$ and $d^k + d^{k-1} + \ldots + d + 1$ as a lower and an upper bound of $n_{d,k}$, respectively, we give an elementary proof of our claim.

Diameter monotonicity

Theorem 2.1 *For each $k, d \geq 1$ holds $n_{d,k+1} > n_{d,k}$.*

Proof. If $d = 1$, then $n_{d,k} = n_{1,k} = M_{1,k} = k + 1$. This implies

$$n_{d,k+1} = n_{1,k+1} = k + 2 > k + 1 = n_{1,k} = n_{d,k}.$$

If $d > 1$, then we have

$$n_{d,k+1} \geq d^{k+1} + d^k > d^{k+1} > \frac{d^{k+1} - 1}{d - 1} \geq n_{d,k}.$$

∎

Degree monotonicity

Theorem 2.2 *For each $k, d \geq 1$ holds $n_{d+1,k} > n_{d,k}$.*

Proof. If $d = 1$, then we get

$$n_{d+1,k} = n_{2,k} \geq 2^k + 2^{k-1} > k + 1 = n_{1,k} = n_{d,k}.$$

If $d > 1$, then we have

$$\begin{aligned} n_{d+1,k} &\geq (d+1)^k + (d+1)^{k-1} \\ &= (d+2)(d+1)^{k-1} \\ &\geq \frac{d^2}{d-1}(d+1)^{k-1} \geq \frac{d^2}{d-1}d^{k-1} \\ &> \frac{d^{k+1}-1}{d-1} \geq n_{d,k}. \end{aligned}$$

∎

Acknowledgements

The author would like to thank his adviser Robert Jajcay for introducing him to degree/diameter problems. Thanks also belong to the referees whose suggestions helped in improving the note.

The author acknowledges financial support from the Slovenian Research Agency (research core funding No. P1-0285 and Young Researchers Grant).

References

[1] M. Miller and J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, *Electron. J. Combin.* DS 14, 2013.

[2] W. H. Kautz, Bounds on directed (d, k) graphs, *Theory of cellular logic networks and machines*, Air Force Cambridge Research Laboratories-68-0668 Final Report (1968) 20–28.

(Received 3 May 2016; revised 12 Apr 2017, 8 Sep 2017)