Completing some spectra for 2-perfect cycle systems

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ABSTRACT: The determination of the spectrum for the decomposition of K_v into 2-perfect m-cycle systems is completed here for several small values of m. In particular, the cases $m = 9$, 12 and 16 are completed (except for three isolated cases). Other isolated 2-perfect m-cycle systems, some listed as unknown in a recent survey paper by Lindner and Rodger, have been found: namely, for K_v where $(m, v) = (7,21)$, $(11,33)$, $(11,45)$, $(13,39)$, $(17,35)$, $(17,51)$, $(17,69)$, $(19,39)$, $(19,77)$, $(19,115)$, $(23,93)$. The spectra for $m = 7, 11, 12, 13$ and 17 are now complete, with no isolated exceptions.

1 Introduction

An m-cycle decomposition of K_v is an edge-disjoint decomposition of K_v into cycles of length m. We write $(a, b, c, \ldots, x, y)$ to denote the cycle with edges $\{a, b\}, \{b, c\}, \ldots, \{x, y\}, \{y, a\}$. If K_v has vertex set V, and C denotes an edge-disjoint set of m-cycles which cover all the edges of K_v, then (V, C) is an m-cycle system of K_v.

If c is a cycle of length m, then let $c(i)$ denote the graph formed from c by joining all vertices in c at distance i. If (V, C) is an m-cycle system of K_v such that $(V, \{c(i) \mid c \in C\})$ is also a cycle system of K_v, then we call (V, C) an i-perfect m-cycle system. See the survey paper [10] by Lindner and Rodger and the references therein for more detail. We shall also use the concept of i-perfect m-cycle decompositions of graphs besides the complete graph; in particular, decompositions of complete tri- and quadri-partite graphs will be used in the constructions.

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Our basic construction (for cases not dealt with in [10], such as 2-perfect m-cycle systems with $m = 12, 15, 16$) is as follows.

For some admissible value of v with $v = ed + h$, we take the vertices of K_v to be

$$\{\infty_1, \ldots, \infty_h\} \cup \{(i, j) \mid 0 \leqslant i \leqslant e-1, \ 1 \leqslant j \leqslant d\}.$$

Now we take a decomposition into 2-perfect m-cycles of the complete k-partite graph $K_{d,d,\ldots,d}$ (k lots of d's here; usually $k = 3$ or $k = 4$). Next we require a group divisible design GD$(k, 1, M; e)$ where the group sizes belong to M. If $m \in M$ we usually require a decomposition of K_{md+h}, and also of $(K_{md+h} \setminus K_h)$, the complete graph on $md + h$ vertices with a "hole" of size h, if $h > 1$. For most cases, we have $M = \{m\}$, but if, say, $M = \{m, q^*\}$ (so one group in the GDD is of size q and the rest are all of size m) then we require a decomposition of K_{qd+h} and of $(K_{md+h} \setminus K_h)$. Sometimes if e is "too small", a suitable GDD does not exist, and then we may need a direct construction of K_{ed+h}.

If a GD$(k, 1, m; e)$ exists, then place such a design on the set $\{(i, j) \mid 0 \leqslant i \leqslant e-1\}$. Then we take m-cycles as follows:

(1) If $\{(x_1, j), (x_2, j), \ldots, (x_m, j)\}$ is *one* of the groups of the GDD, then on the vertices

$$\{\infty_1, \ldots, \infty_h\} \cup \{(x_1, j), (x_2, j), \ldots, (x_m, j) \mid j = 1, 2, \ldots, d\}$$

we place a 2-perfect m-cycle decomposition of K_{md+h}.

(2) For all other groups $\{(y_1, j), (y_2, j), \ldots, (y_m, j)\}$ of the GDD, we take a decomposition of $(K_{md+h} \setminus K_h)$ (or of K_{md+h} if $h = 0$ or 1) with the vertex set

$$\{\infty_1, \ldots, \infty_h\} \cup \{(y_1, j), (y_2, j), \ldots, (y_m, j) \mid j = 1, 2, \ldots, d\}.$$

(3) Finally, for each block $\{(z_1, j), (z_2, j), \ldots, (z_k, j)\}$ of the GDD, on the vertex set

$$\{(z_1, j) \mid 1 \leqslant j \leqslant d\} \cup \{(z_2, j) \mid 1 \leqslant j \leqslant d\} \cup \ldots \cup \{(z_k, j) \mid 1 \leqslant j \leqslant d\}$$

we place a decomposition of the complete k-partite graph $K_{d,d,\ldots,d}$.

The result is a suitable 2-perfect m-cycle system of K_{ed+h}.

We also use the following *GDD*s.

LEMMA 1.1 *There is a group divisible design on $2n \geqslant 6$ elements with block size 3 and group size 2 whenever $2n \equiv 0$ or 2 (mod 6); there is a group divisible design on $2n \geqslant 10$ elements with block size 3, one group of size 4 and the rest of size 2, when $2n \equiv 4$ (mod 6).*

Proof: The cases $2n \equiv 0$ or 2 (mod 6) first appeared in Hanani [7], Lemma 6.3; such group divisible designs also arise from any Steiner triple system by deleting one point. For the case $2n \equiv 4$ (mod 6), see for example page 276 of [13]. This gives a pairwise balanced design with number of elements congruent to 5 (mod 6), and with one block of size five and the rest of size three. Deletion of a point from the block of size five yields a suitable group divisible design, with one group of size four and the rest of size two.

2 2-perfect odd-cycle systems

2.1 2-perfect 7-cycle systems

This was first dealt with in [12]. However, as stated in Lindner and Rodger's survey paper [10], the one outstanding case is K_{21}. We exhibit a suitable decomposition. Let the vertices of K_{21} be $\{\infty\} \cup \{(i,j) \mid 0 \leqslant i \leqslant 4,\ \ 1 \leqslant j \leqslant 4\}$. Then the following six cycles are suitable starter cycles, modulo $(5,-)$, with ∞ fixed of course.

$$\begin{array}{l}(\infty,(0,1),(3,1),(2,1),(0,2),(2,2),(1,2)),\\(\infty,(0,3),(2,3),(1,1),(1,2),(3,4),(2,4)),\\((0,1),(1,2),(4,1),(1,3),(2,2),(0,3),(4,3)),\\((0,1),(4,2),(1,3),(2,4),(2,2),(3,3),(1,4)),\\((0,1),(0,3),(4,4),(2,3),(4,1),(3,4),(0,4)),\\((0,1),(3,4),(4,2),(0,4),(2,2),(2,3),(2,4)).\end{array}$$

This completes the spectrum for 2-perfect 7-cycle systems.

2.2 2-perfect 9-cycle systems

Lindner and Rodger [9] showed that the necessary conditions for existence of a 2-perfect 9-cycle system of K_v, namely that $v \equiv 1$ or $9 \pmod{18}$, are sufficient for $v \equiv 1 \pmod{18}$, with the possible exception of $v = 55$. They also pointed out that existence of a 2-perfect 9-cycle decomposition of $K_{27} \setminus K_9$ (that is, K_{27} with a "hole" of size 9) would deal with the case $v \equiv 9 \pmod{18}$.

Here we complete the determination of the spectrum except for the case $v = 45$.

In all cases (1 or 9 (mod 18)) the construction follows that described in [2], so we omit details. For completeness both $v \equiv 9$ and $v \equiv 1 \pmod{18}$ can be dealt with this way. The case $v \equiv 1 \pmod{18}$ requires decompositions of K_{19}, K_{37} and $K_{9,9,9}$. Since 19 and 37 are primes, we know (see [9], Lemma 2.2) that there exists a 2-perfect 9-cycle decomposition of K_{19} and of K_{37}. Example 2.1 below gives a decomposition of $K_{9,9,9}$.

EXAMPLE 2.1 *A 2-perfect 9-cycle system of $K_{9,9,9}$:*

Elements are $\{(i,j) \mid 0 \leqslant i \leqslant 8,\ \ 1 \leqslant j \leqslant 3\}$.

Working modulo 9, we take the following three starter cycles:

$$\begin{array}{l}((0,1),(0,2),(0,3),(1,1),(2,2),(3,3),(2,1),(1,2),(6,3)),\\((0,1),(2,2),(1,3),(7,1),(1,2),(4,3),(8,1),(3,2),(7,3)),\\((0,1),(5,2),(7,3),(7,1),(4,2),(1,3),(8,1),(6,2),(4,3)).\end{array}$$

The case $v \equiv 9 \pmod{18}$ requires decompositions of $K_{9,9,9}$ (above), K_{27} (see [9], Lemma 2.3) and $K_{27} \setminus K_9$ (below). There is also an isolated case, K_{45}, for which no decomposition is yet known.

EXAMPLE 2.2 *A 2-perfect 9-cycle decomposition of $K_{27} \setminus K_9$, that is, a 2-perfect 9-cycle decomposition of K_{27} with a hole of size 9:*

The elements are $\{A, B, C, D, E, F, G, H, I\} \cup \{(i,j) \mid 0 \leqslant i \leqslant 2,\ \ 1 \leqslant j \leqslant 6\}$. The cycles, 35 of them, are as follows:

$((0,1),(2,4),(0,6),(1,1),(0,4),(1,6),(2,1),(1,4),(2,6))$ (uncycled),
$((0,3),(0,5),(1,2),(2,3),(2,5),(0,2),(1,3),(1,5),(2,2))$ (uncycled);
then the following starter cycles, mod $(3,-)$:
$((0,2),(1,6),(1,2),(0,6),(1,5),(2,3),(1,1),(2,6),(0,4)),$
$((0,1),(1,5),(2,1),(0,4),(1,3),(1,4),(0,3),(0,2),(0,5)),$
$(A,(0,1),(0,2),B,(1,1),(2,2),C,(0,3),(1,2)),$
$(A,(0,3),(0,1),D,(1,3),(2,1),E,(0,5),(0,4)),$
$(A,(0,5),(0,6),F,(1,5),(0,3),G,(1,4),(1,6)),$
$(B,(0,6),(1,6),D,(0,2),(1,2),F,(0,4),(1,4)),$
$(B,(0,3),(1,3),H,(1,2),(2,1),I,(1,5),(2,5)),$
$(C,(0,4),(1,5),D,(1,4),(0,5),H,(0,6),(2,5)),$
$(C,(0,6),(1,3),E,(0,4),(1,2),I,(1,4),(1,1)),$
$(E,(0,2),(1,5),G,(2,2),(0,4),H,(0,1),(0,6)),$
$(F,(0,1),(1,1),G,(2,6),(1,3),I,(0,6),(0,3)).$

This completes the spectrum for 2-perfect 9-cycle systems, except for the one case K_{45}.

2.3 2-perfect 11-cycle systems

Outstanding cases here are K_{33} and K_{45} (see [10]).

Let the vertices of K_{33} be denoted by $\{(i,j) \mid 0 \leqslant i \leqslant 10,\ \ j = 1,2,3\}$. We need 48 cycles. The following cycles give a suitable decomposition of K_{33}:

Four fixed cycles:
$((0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1)),$
$((0,1),(2,1),(4,1),(6,1),(8,1),(10,1),(1,1),(3,1),(5,1),(7,1),(9,1)),$
$((0,2),(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(7,2),(8,2),(9,2),(10,2)),$
$((0,2),(2,2),(4,2),(6,2),(8,2),(10,2),(1,2),(3,2),(5,2),(7,2),(9,2));$
and four starter cycles, modulo $(11,-)$:
$((0,1),(4,1),(0,2),(0,3),(1,3),(6,2),(4,3),(10,3),(2,2),(1,1),(8,3)),$
$((0,1),(5,1),(8,1),(2,3),(3,1),(6,2),(5,3),(2,1),(2,2),(8,2),(0,3)),$
$((0,2),(3,2),(4,1),(2,2),(4,3),(10,2),(6,3),(8,1),(10,3),(6,2),(9,1)),$
$((0,2),(4,2),(10,1),(3,3),(2,2),(9,1),(10,3),(1,3),(4,3),(0,3),(5,1)).$

For a 2-perfect 11-cycle system of K_{45}, work with the integers modulo 45; then the following two starter cycles yield a suitable decomposition of K_{45}:

$$(0,1,36,9,37,7,12,26,30,39,23),(0,2,15,18,25,13,37,17,23,34,26).$$

This completes the spectrum for 2-perfect 11-cycle systems.

2.4 2-perfect 13-cycle systems

Here the only outstanding case is K_{39} (see [10]). For the element set we take $\{\infty\} \cup \{(i,j) \mid 0 \leqslant i \leqslant 18,\ \ j = 1,2\}$. Then the following three cycles are suitable starter cycles, working mod $(19, -)$.

$$(\infty,(0,1),(10,1),(8,1),(4,1),(5,1),(18,1),(15,1),(3,1),(17,1),(12,2),(1,2),(13,2)),$$
$$((0,1),(8,1),(0,2),(1,2),(1,1),(3,2),(5,2),(7,1),(11,2),(2,2),(9,1),(10,2),(6,2)),$$
$$((0,2),(14,2),(18,1),(7,2),(0,1),(3,2),(12,1),(17,2),(1,2),(11,1),(10,2),(16,2),(3,1)).$$

This completes the spectrum for 2-perfect 13-cycle systems.

2.5 2-perfect 15-cycle systems

The necessary conditions for existence of a 2-perfect 15-cycle system of K_v are that v is 1, 15, 21 or 25 (mod 30), and of course $v \geqslant 15$. We deal with these conditions in turn.

First let $v = 30n + 1$. We take $d = 15$, $e = 2n$ and $h = 1$, and use a decomposition of $K_{15,15,15}$ (given below). Then we merely need decompositions of K_{31} and K_{61}; these exist, by virtue of [9], Lemma 2.2.

Secondly, let $v = 30n + 15 = 5(6n + 3)$. We use $d = 5$ and $e = 6n + 3$, and also the existence of a resolvable Steiner triple system of order $6n + 3$. The we use decompositions of $K_{5,5,5}$ and K_{15} (see below).

EXAMPLE 2.3 *A 2-perfect 15-cycle system of $K_{5,5,5}$.*

The element set is $\bigcup_{j=1}^{3}\{(i,j) \mid 0 \leqslant i \leqslant 4\}$. The five cycles may be taken as

$$((0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)),$$
$$((0,1),(1,2),(4,3),(1,1),(2,2),(0,3),(2,1),(3,2),(1,3),(3,1),(4,2),(2,3),(4,1),(0,2),(3,3)),$$
$$((0,1),(2,2),(3,3),(1,1),(3,2),(4,3),(2,1),(4,2),(0,3),(3,1),(0,2),(1,3),(4,1),(1,2),(2,3)),$$
$$((0,1),(3,2),(2,3),(1,1),(4,2),(3,3),(2,1),(0,2),(4,3),(3,1),(1,2),(0,3),(4,1),(2,2),(1,3)),$$
$$((0,1),(4,2),(1,3),(1,1),(0,2),(2,3),(2,1),(1,2),(3,3),(3,1),(2,2),(4,3),(4,1),(3,2),(0,3)).$$

EXAMPLE 2.4 *A 2-perfect 15-cycle system of $K_{15,15,15}$.*

This is easily obtained from the previous example. From each of those five cycles we obtain nine new cycles, using a latin square of order 3. For example, from the first cycle above, we obtain the 9 cycles with the same second entries in each element, and with the first entries of the 15 elements being:

0	0	0	1	1	1	2	2	2	3	3	3	4	4	4,
0	5	5	1	6	6	2	7	7	3	8	8	4	9	9,
0	10	10	1	11	11	2	12	12	3	13	13	4	14	14,
5	0	10	6	1	11	7	2	12	8	3	13	9	4	14,
5	5	0	6	6	1	7	7	2	8	8	3	9	9	4,
5	10	5	6	11	6	7	12	7	8	13	8	9	14	9,
10	0	5	11	1	6	12	2	7	13	3	8	14	4	9,
10	5	10	11	6	11	12	7	12	13	8	13	14	9	14,
10	10	0	11	11	1	12	12	2	13	13	3	14	14	4.

EXAMPLE 2.5 *A 2-perfect 15-cycle system of K_{15}.*

The element set is $\{(i,j) \mid 0 \leqslant i \leqslant 6,\ \ j=1,2\} \cup \{\infty\}$. We have one starter mod 7:

$$(\infty,(0,1),(1,1),(0,2),(3,2),(1,2),(5,1),(6,2),(2,1),(2,2),(4,1),(6,1),(3,1),(5,2),(4,2)).$$

Thirdly, let $v = 30n+21 = 5(6n+4)+1$. We take 5 layers with $6n+4$ elements per layer, and an infinity element. We use the existence of a group divisible design $\mathrm{GD}(3,1,\{4^*,6\};6n+4)$ which has blocks of size three, one group of size 4 and the rest all of size 6. (This exists; see Main Theorem in [6].)

So we need, besides a decomposition of $K_{5,5,5}$, decompositions of K_{21} and of K_{31}; the latter exists by virtue of [9], Lemma 2.2, and a decomposition of the former we give here:

EXAMPLE 2.6 *For a decomposition of K_{21}, we take two starter cycles, on the element set $\{(i,j) \mid 0 \leqslant i \leqslant 6,\ \ j=1,2,3\}$; they are cycled mod $(7,-)$:*

$$(0,1),(1,1),(0,3),(5,1),(6,3),(6,1),(0,2),(4,1),(2,2),(1,3),(6,2),(4,2),(5,2),(5,3),(3,3)),$$
$$(0,1),(5,1),(5,2),(2,2),(3,1),(6,1),(3,2),(0,3),(2,1),(4,2),(5,3),(6,3),(2,3),(6,2),(4,3)).$$

The final case, $v \equiv 25 \pmod{30}$, is at present incomplete, as no suitable decomposition of K_{25} has yet been found. We could complete the spectrum if we found this, and the isolated case K_{55}.

2.6 2-perfect 17-cycle systems

Since 17 is prime, by Theorem 3.9 of [10] we need only find suitable 2-perfect 17-cycle decompositions of K_{35}, K_{51}, K_{69} and K_{103}. Since 103 is prime, Lemma 3.10 of [10] covers the last of these. The other three cases are dealt with below.

Here is a 2-perfect 17-cycle decomposition of K_{35}; it has one starter cycle mod 35:

$$(0,1,3,6,2,8,13,4,28,10,18,33,11,25,32,9,19).$$

Here is a 2-perfect 17-cycle decomposition of K_{51}; it is based on the element set $\{\infty\} \cup \{(i,j) \mid 0 \leqslant i \leqslant 24,\ \ j=1,2\}$. The following three starter cycles are cycled mod $(25,-)$.

$$((0,1),(1,1),(15,2),(16,1),(3,2),(13,2),(4,2),(2,1),(10,1),(7,2),(4,1),$$
$$(15,1),(24,1),(3,1),(12,2),(1,2),(5,1)),$$
$$((0,1),(2,1),(8,1),(1,1),(1,2),(3,1),(4,2),(10,1),(0,2),(7,1),(17,2),$$
$$(9,2),(3,2),(2,2),(23,2),(21,2),(8,2)),$$
$$(\infty,(0,1),(11,2),(8,2),(2,1),(17,1),(5,2),(12,2),(20,1),(23,1),(11,1),$$
$$(2,2),(22,1),(4,2),(9,2),(5,1),(0,2)).$$

And here is a 2-perfect 17-cycle decomposition of K_{69}; it has two starters cycles mod 69:

$$(0,1,33,2,30,32,15,25,19,27,61,18,9,24,4,46,22),$$
$$(0,3,8,15,33,52,4,17,42,31,1,47,59,23,9,13,29).$$

This completes the spectrum for 2-perfect 17-cycle systems.

2.7 2-perfect 19-cycle systems

Since 19 is prime, we need only find 2-perfect 19-cycle decompositions of K_{39}, K_{57}, K_{77} and K_{115}. Moreover, only the first of these is essential for the construction, and the other three are isolated cases.

For K_{39} we have one starter cycle mod 39:

$$(0, 1, 3, 6, 2, 8, 13, 4, 26, 10, 35, 28, 20, 9, 36, 21, 11, 31, 18).$$

For K_{77} we have two starter cycles mod 77:

$$(0, 1, 76, 38, 71, 58, 32, 15, 44, 33, 29, 43, 7, 56, 40, 45, 13, 28, 9),$$
$$(0, 3, 9, 1, 23, 30, 7, 53, 13, 56, 32, 20, 67, 46, 21, 71, 14, 24, 59).$$

For K_{115} we have three starter cycles mod 115:

$$(0, 1, 77, 88, 24, 90, 28, 102, 33, 80, 87, 30, 60, 70, 46, 61, 104, 45, 8),$$
$$(0, 2, 6, 1, 14, 11, 17, 26, 4, 37, 21, 3, 39, 27, 7, 47, 28, 78, 61),$$
$$(0, 14, 41, 86, 109, 61, 29, 1, 95, 64, 9, 34, 78, 113, 36, 62, 10, 39, 81).$$

A suitable decomposition of K_{57} into 2-perfect 19-cycles has not yet been found; otherwise, the spectrum is complete.

2.8 2-perfect 23-cycle systems

As stated in [10] (in Section 3), the spectrum of 2-perfect 23-cycle systems is the set of all $v \equiv 1$ or 23 (modulo 46) except possibly 69 and 93.

A 2-perfect 23-cycle system of K_{93} is given by the following two starter cycles modulo 93:

$$(0, 1, 23, 88, 83, 52, 73, 60, 21, 25, 77, 42, 12, 62, 59, 75, 56, 45, 78, 70, 79, 81, 15),$$
$$(0, 6, 16, 2, 9, 21, 41, 64, 90, 22, 56, 88, 35, 83, 59, 30, 85, 68, 31, 77, 33, 15, 51).$$

At present a suitable 2-perfect 23-cycle system of K_{69} has not been found.

3 2-perfect even cycle systems

The spectrum for 2-perfect m-cycle systems, with m even, is far less determined. The case $m = 4$ is impossible; the case $m = 6$ is dealt with in [8] (see also [3]); the case $m = 8$ is dealt with in [1]. Treatment of $m = 10$ is omitted here; about half the spectrum has so far been determined. In this section we completely solve the spectrum for $m = 12$, and solve the spectrum for $m = 16$ apart from two isolated cases, $(m, v) = (16, 289)$ and $(16, 353)$.

3.1 2-perfect 12-cycle systems

The necessary condition for a 2-perfect 12-cycle decomposition of K_v to exist is that $v \equiv 1$ or $9 \pmod{24}$.

When $v \equiv 1 \pmod{24}$, let $v = 24n + 1$; in our construction we use $K_{12,12,12}$, K_{25} and K_{49}.

When $v \equiv 9 \pmod{24}$, let $v = 24n + 9$. This case requires $K_{12,12,12}$, K_{33}, K_{57} and $K_{33} \setminus K_9$.

EXAMPLE 3.1 *A 2-perfect 12-cycle decomposition of K_{25}:*

Element set is $\mathbb{Z}_{25}$; one starter cycle mod 25:

$$(0, 1, 4, 12, 14, 5, 16, 11, 17, 24, 9, 21).$$

EXAMPLE 3.2 *A 2-perfect 12-cycle decomposition of K_{49}:*

Element set is $\mathbb{Z}_{49}$; two starter cycles mod 49:

$$(0, 1, 12, 19, 32, 4, 23, 39, 16, 6, 3, 5), (0, 4, 10, 45, 21, 39, 48, 11, 31, 9, 26, 34).$$

EXAMPLE 3.3 *A 2-perfect 12-cycle decomposition of K_{33}:*

Element set is $\{(i, j) \mid 0 \leqslant i \leqslant 10, \quad j = 1, 2, 3\}$. We take the following four starter cycles mod $(11, -)$:

$$\begin{aligned}
&((0,1),(3,1),(1,3),(2,2),(8,3),(6,3),(9,2),(7,1),(0,2),(1,2),(2,1),(5,2)),\\
&((0,1),(2,1),(3,2),(5,1),(1,3),(9,1),(3,3),(5,2),(8,3),(4,1),(0,2),(8,2)),\\
&((0,1),(7,1),(2,3),(10,3),(6,2),(4,2),(9,1),(9,2),(9,3),(3,3),(1,1),(0,3)),\\
&((0,1),(1,1),(6,1),(7,3),(0,3),(4,2),(9,2),(5,2),(10,3),(8,2),(9,3),(8,3)).
\end{aligned}$$

EXAMPLE 3.4 *A 2-perfect 12-cycle decomposition of K_{57}:*

Element set is $\{(i, j) \mid 0 \leqslant i \leqslant 18, \quad j = 1, 2, 3\}$. The following seven starter cycles mod $(19, -)$ give a suitable decomposition:

$$\begin{aligned}
&((0,1),(1,1),(4,3),(12,2),(10,3),(0,3),(18,2),(17,1),(11,2),(4,2),(4,1),(9,2)),\\
&((0,1),(15,1),(4,2),(17,1),(13,3),(4,1),(6,3),(7,2),(4,3),(9,1),(13,2),(11,2)),\\
&((0,1),(5,1),(3,3),(18,3),(16,2),(13,2),(6,1),(16,1),(11,2),(11,3),(10,1),(12,1)),\\
&((0,1),(3,1),(0,3),(12,2),(11,2),(15,2),(1,2),(14,3),(9,2),(18,3),(14,1),(6,1)),\\
&((0,2),(11,2),(17,3),(12,1),(14,2),(5,2),(18,2),(9,3),(1,3),(16,2),(0,3),(12,3)),\\
&((0,3),(6,3),(13,1),(12,2),(2,1),(18,2),(1,1),(16,2),(12,3),(17,2),(14,1),(14,3)),\\
&((0,3),(16,3),(9,1),(2,2),(10,3),(2,1),(1,3),(18,3),(5,1),(15,3),(14,3),(8,1)).
\end{aligned}$$

EXAMPLE 3.5 *A 2-perfect 12-cycle decomposition of $K_{33} \setminus K_9$:*

The nine hole elements are $\{A, B, C, D, E, F, G, H, I\}$, and the other twenty-four elements are $\{(i,j) \mid 0 \leqslant i \leqslant 2,\ \ 1 \leqslant j \leqslant 8\}$. The graph $K_{33} \setminus K_9$ has 12×41 edges, and so we want 41 12-cycles; we have 13 starters mod $(3, -)$, and two cycles that are fixed (not cycled).

The two fixed cycles are

$$((0,1),(2,3),(0,4),(2,2),(2,1),(1,3),(2,4),(1,2),(1,1),(0,3),(1,4),(0,2)),$$
$$((0,5),(1,6),(0,8),(2,7),(2,5),(0,6),(2,8),(1,7),(1,5),(2,6),(1,8),(0,7)).$$

Then the following 13 cycles are cycled mod $(3, -)$:

$$(A,(0,1),(2,1),E,(1,2),(2,2),B,(1,1),(1,3),F,(0,2),(2,3)),$$
$$(C,(0,1),(1,2),G,(1,1),(0,2),D,(2,4),(2,2),H,(0,4),(1,3)),$$
$$(A,(0,4),(0,3),B,(2,7),(2,6),C,(2,8),(1,1),D,(0,7),(1,7)),$$
$$(A,(0,6),(0,4),C,(2,7),(0,5),F,(0,7),(1,2),I,(1,6),(1,5)),$$
$$(A,(0,8),(1,5),G,(2,4),(1,6),F,(0,1),(0,6),H,(2,8),(2,2)),$$
$$(B,(0,5),(2,5),H,(2,3),(0,8),G,(0,3),(1,5),I,(2,1),(2,8)),$$
$$(D,(0,5),(0,4),F,(1,8),(0,8),E,(1,4),(2,4),I,(2,3),(0,6)),$$
$$(B,(0,6),(1,8),D,(0,3),(2,5),E,(0,7),(1,6),G,(1,7),(2,4)),$$
$$(C,(0,5),(0,3),E,(1,6),(2,7),H,(1,1),(1,7),I,(0,8),(2,2)),$$
$$((0,1),(0,5),(1,1),(2,5),(0,4),(2,1),(0,7),(0,8),(0,3),(1,3),(0,6),(2,4)),$$
$$((0,2),(0,6),(1,2),(2,6),(0,5),(2,4),(2,1),(1,6),(1,8),(2,2),(0,7),(2,5)),$$
$$((0,3),(1,7),(1,3),(0,7),(1,1),(2,3),(1,8),(0,5),(2,2),(2,7),(2,4),(0,2)),$$
$$((0,4),(1,8),(2,4),(2,8),(2,5),(2,2),(0,3),(0,6),(2,6),(1,1),(0,8),(1,7)).$$

EXAMPLE 3.6 *Last, but certainly not least, we give a decomposition of $K_{12,12,12}$ into 2-perfect 12-cycles.*

We have 36 12-cycles, based on the vertex set $\{(i,j) \mid 0 \leqslant i \leqslant 11,\ \ 1 \leqslant j \leqslant 3\}$.

$$((0,1),(1,2),(10,3),(3,1),(0,2),(5,3),(2,1),(3,2),(0,3),(1,1),(2,2),(3,3)),$$
$$((0,1),(2,2),(9,3),(3,1),(1,2),(7,3),(2,1),(0,2),(3,3),(1,1),(3,2),(1,3)),$$
$$((0,1),(3,2),(11,3),(3,1),(2,2),(4,3),(2,1),(1,2),(1,3),(1,1),(0,2),(2,3)),$$
$$((0,1),(0,2),(8,3),(3,1),(3,2),(6,3),(2,1),(2,2),(2,3),(1,1),(1,2),(0,3)),$$
$$((0,1),(5,2),(6,3),(3,1),(4,2),(1,3),(2,1),(7,2),(8,3),(1,1),(6,2),(11,3)),$$
$$((0,1),(6,2),(5,3),(3,1),(5,2),(3,3),(2,1),(4,2),(11,3),(1,1),(7,2),(9,3)),$$
$$((0,1),(7,2),(7,3),(3,1),(6,2),(0,3),(2,1),(5,2),(9,3),(1,1),(4,2),(10,3)),$$
$$((0,1),(4,2),(4,3),(3,1),(7,2),(2,3),(2,1),(6,2),(10,3),(1,1),(5,2),(8,3)),$$
$$((0,1),(9,2),(2,3),(3,1),(8,2),(9,3),(2,1),(11,2),(4,3),(1,1),(10,2),(7,3)),$$
$$((0,1),(10,2),(1,3),(3,1),(9,2),(11,3),(2,1),(8,2),(7,3),(1,1),(11,2),(5,3)),$$
$$((0,1),(11,2),(3,3),(3,1),(10,2),(8,3),(2,1),(9,2),(5,3),(1,1),(8,2),(6,3)),$$
$$((0,1),(8,2),(0,3),(3,1),(11,2),(10,3),(2,1),(10,2),(6,3),(1,1),(9,2),(4,3)),$$

$((4,1),(1,2),(6,3),(7,1),(0,2),(1,3),(6,1),(3,2),(8,3),(5,1),(2,2),(11,3)),$
$((4,1),(2,2),(5,3),(7,1),(1,2),(3,3),(6,1),(0,2),(11,3),(5,1),(3,2),(9,3)),$
$((4,1),(3,2),(7,3),(7,1),(2,2),(0,3),(6,1),(1,2),(9,3),(5,1),(0,2),(10,3)),$
$((4,1),(0,2),(4,3),(7,1),(3,2),(2,3),(6,1),(2,2),(10,3),(5,1),(1,2),(8,3)),$
$((4,1),(5,2),(2,3),(7,1),(4,2),(9,3),(6,1),(7,2),(4,3),(5,1),(6,2),(7,3)),$
$((4,1),(6,2),(1,3),(7,1),(5,2),(11,3),(6,1),(4,2),(7,3),(5,1),(7,2),(5,3)),$
$((4,1),(7,2),(3,3),(7,1),(6,2),(8,3),(6,1),(5,2),(5,3),(5,1),(4,2),(6,3)),$
$((4,1),(4,2),(0,3),(7,1),(7,2),(10,3),(6,1),(6,2),(6,3),(5,1),(5,2),(4,3)),$
$((4,1),(9,2),(10,3),(7,1),(8,2),(5,3),(6,1),(11,2),(0,3),(5,1),(10,2),(3,3)),$
$((4,1),(10,2),(9,3),(7,1),(9,2),(7,3),(6,1),(8,2),(3,3),(5,1),(11,2),(1,3)),$
$((4,1),(11,2),(11,3),(7,1),(10,2),(4,3),(6,1),(9,2),(1,3),(5,1),(8,2),(2,3)),$
$((4,1),(8,2),(8,3),(7,1),(11,2),(6,3),(6,1),(10,2),(2,3),(5,1),(9,2),(0,3)),$

$((8,1),(1,2),(2,3),(11,1),(0,2),(9,3),(10,1),(3,2),(4,3),(9,1),(2,2),(7,3)),$
$((8,1),(2,2),(1,3),(11,1),(1,2),(11,3),(10,1),(0,2),(7,3),(9,1),(3,2),(5,3)),$
$((8,1),(3,2),(3,3),(11,1),(2,2),(8,3),(10,1),(1,2),(5,3),(9,1),(0,2),(6,3)),$
$((8,1),(0,2),(0,3),(11,1),(3,2),(10,3),(10,1),(2,2),(6,3),(9,1),(1,2),(4,3)),$
$((8,1),(5,2),(10,3),(11,1),(4,2),(5,3),(10,1),(7,2),(0,3),(9,1),(6,2),(3,3)),$
$((8,1),(6,2),(9,3),(11,1),(5,2),(7,3),(10,1),(4,2),(3,3),(9,1),(7,2),(1,3)),$
$((8,1),(7,2),(11,3),(11,1),(6,2),(4,3),(10,1),(5,2),(1,3),(9,1),(4,2),(2,3)),$
$((8,1),(4,2),(8,3),(11,1),(7,2),(6,3),(10,1),(6,2),(2,3),(9,1),(5,2),(0,3)),$
$((8,1),(9,2),(6,3),(11,1),(8,2),(1,3),(10,1),(11,2),(8,3),(9,1),(10,2),(11,3)),$
$((8,1),(10,2),(5,3),(11,1),(9,2),(3,3),(10,1),(8,2),(11,3),(9,1),(11,2),(9,3)),$
$((8,1),(11,2),(7,3),(11,1),(10,2),(0,3),(10,1),(9,2),(9,3),(9,1),(8,2),(10,3)),$
$((8,1),(8,2),(4,3),(11,1),(11,2),(2,3),(10,1),(10,2),(10,3),(9,1),(9,2),(8,3)).$

This completes the spectrum for 2-perfect 12-cycle systems.

3.2 2-perfect 16-cycle systems

The necessary condition for a 2-perfect 16-cycle decomposition of K_v to exist is that $v \equiv 1 \pmod{32}$. So let $v = 32n + 1$. We have four cases, according as n is 0 or 1 (mod 3), or 2 or 5 (mod 6).

First, let $n = 3m$, so that $v = 4(24m) + 1$. We take $d = 4$ in our construction, and use a GD$(4, 1, 24; 24m)$; this exists for $m \geqslant 4$ [5]. Then we need decompositions of $K_{4,4,4,4}$ and K_{97}; see below. There are also the isolated cases K_{193} and K_{289}. A decomposition of the former of these is given below.

Secondly, let $n = 3m + 1$, so that $v = 4(24m + 8) + 1$. We again take $d = 4$ and use a GD$(4, 1, 8; 24m + 8)$; this exists for all $m \geqslant 1$ [5]. Then we use decompositions of $K_{4,4,4,4}$ and K_{33}.

Thirdly, let $n = 6m + 2$, so that $v = 16(12m + 4) + 1$. This time we use $d = 16$, together with a resolvable BIBD$(12m+4, 4, 1)$, and decompositions of $K_{16,16,16,16}$ and K_{65}. These are given below.

Fourthly and finally, let $n = 6m + 5$, so that $v = 4(48m + 40) + 1$. This time, with $d = 4$, we use a GD$(4, 1, \{16, 40^*\}; 48m + 40)$, which exists for $m \geqslant 3$. (The existence follows from Theorem 4 of [4] and Lemma 2.27 of [5], using a GD$(4, 1, 8; 32)$.) Then we use decompositions of $K_{4,4,4,4}$, K_{65} and K_{161}; these are given below. We also have the isolated cases (corresponding to $m = 1$ and 2) K_{353} and K_{545}. A construction for the latter is also given below; the former remains open.

EXAMPLE 3.7 *A decomposition of $K_{4,4,4,4}$; element set is $\{(i,j) \mid 1 \leqslant i,j \leqslant 4\}$. The second component, j, of each element, determines to which part of the partition of $K_{4,4,4,4}$ the element belongs.*

(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4),
(1,1),(2,2),(3,3),(4,4),(2,1),(3,2),(4,3),(1,4),(3,1),(4,2),(1,3),(2,4),(4,1),(1,2),(2,3),(3,4),
(1,1),(3,2),(1,4),(2,3),(2,1),(4,2),(2,4),(3,3),(3,1),(1,2),(3,4),(4,3),(4,1),(2,2),(4,4),(1,3),
(1,1),(4,2),(3,4),(1,3),(2,1),(1,2),(4,4),(2,3),(3,1),(2,2),(1,4),(3,3),(4,1),(3,2),(2,4),(4,3),
(1,1),(2,3),(3,2),(3,4),(2,1),(3,3),(4,2),(4,4),(3,1),(4,3),(1,2),(1,4),(4,1),(1,3),(2,2),(2,4),
(1,1),(3,3),(1,2),(2,4),(2,1),(4,3),(2,2),(3,4),(3,1),(1,3),(3,2),(4,4),(4,1),(2,3),(4,2),(1,4).

EXAMPLE 3.8 *A decomposition of K_{33}, given by one starter cycle mod 33:*

$$(0,1,3,6,2,8,13,26,7,14,23,5,15,31,19,11).$$

EXAMPLE 3.9 *A decomposition of K_{65}, given by two starter cycles, mod 65:*

$$(0,1,56,62,57,20,9,54,58,2,26,8,33,48,60,3),$$
$$(0,2,15,31,50,7,42,35,3,17,46,20,58,10,44,21).$$

EXAMPLE 3.10 *A decomposition of K_{97}, given by three starter cycles, mod 97:*

$$(0,1,19,4,42,15,79,51,94,3,35,84,59,89,8,5),$$
$$(0,2,42,87,83,25,4,39,10,71,30,41,75,66,22,12),$$
$$(0,7,21,34,12,35,81,61,1,32,40,95,71,45,64,47).$$

EXAMPLE 3.11 *A decomposition of K_{161}, given by five starter cycles, mod 161:*

$$(0,1,155,20,94,103,6,22,123,4,127,97,18,114,131,2),$$
$$(0,3,107,32,140,146,77,153,87,112,143,26,61,95,39,11),$$
$$(0,4,95,17,115,44,87,41,64,9,33,133,100,108,121,18),$$
$$(0,5,15,3,24,38,19,4,31,78,42,62,84,16,110,81),$$
$$(0,37,89,143,27,86,135,22,111,72,156,115,53,13,123,73).$$

EXAMPLE 3.12 *A decomposition of K_{193}, given by six starter cycles, mod 193:*

$$(0,1,135,18,60,187,40,134,186,46,168,14,94,101,183,2),$$
$$(0,3,65,133,76,43,38,73,98,175,137,50,143,36,185,8),$$
$$(0,4,124,138,164,192,160,8,179,44,50,60,107,131,27,45),$$
$$(0,9,84,105,2,130,4,82,125,175,63,187,85,1,98,79),$$
$$(0,11,26,46,23,77,60,4,17,146,109,192,54,115,164,34),$$
$$(0,27,67,168,11,83,32,117,47,166,3,34,82,177,148,60).$$

EXAMPLE 3.13 *A 2-perfect 16-cycle decomposition of $K_{16,16,16,16}$:*

We use six cycles in the decomposition of $K_{4,4,4,4}$ given in Example 3.7. From each one of these cycles, 16 new cycles are formed, on the set $\{(i,j)|1 \leqslant i \leqslant 16, 1 \leqslant j \leqslant 4\}$, as follows. From the cycle
$((a_1,x_1),(a_2,x_2),(a_3,x_3),(a_4,x_4),(b_1,x_1),(b_2,x_2),(b_3,x_3),(b_4,x_4),$
$(c_1,x_1),(c_2,x_2),(c_3,x_3),(c_4,x_4),(d_1,x_1),(d_2,x_2),(d_3,x_3),(d_4,x_4)),$
we successively replace (g_1,g_2,g_3,g_4), for $g = a, b, c$ and d, by $(g_1+4\alpha, g_2+4\beta, g_3+4\gamma, g_4+4\delta)$, where $\alpha,\beta,\gamma,\delta$ is a row from a 16×4 resolvable orthogonal array. It is straightforward to check that the result is a 2-perfect 16-cycle system of $K_{16,16,16,16}$.

EXAMPLE 3.14 *A 2-perfect 16-cycle system of K_{545}.*

Note that $545 = 1 + (16 \times 34)$, so let $d = 16$ in the construction. There exists a $GD(4,1,\{4,10^*\};34)$; this, together with decompositions of K_{65}, K_{161} and $K_{16,16,16,16}$, complete the existence proof. (The GDD used here may be constructed from a resolvable $GDD(3,1,4;24)$ by adjoining ten new elements, one to each parallel class.)

4 Concluding remarks

We tabulate below the expected and actual spectra for 2-perfect m-cycle systems for values of m we have considered here. References are given in the "Comments" column. The column headed "Spectrum (*)" lists the expected spectrum, if there are any undecided values in the last column. This table updates and extends the 2-perfect part of the table given in [10]. It now remains for someone to settle the remaining undecided values, especially 25 for $m = 15$!

m	Spectrum (*)	Comments	Undecided values
3	1 or 3 (mod 6)	Steiner triple system	
4	∅	Not possible	
5	1 or 5 (mod 10), not 15	[11]	
6	1 or 9 (mod 12)	[8], [3]	
7	1 or 7 (mod 14)	[12], [10] and Section 2.1 above	
8	1 (mod 16)	[1]	
9	1 or 9 (mod 18)	[9], [10] and Section 2.2 above	45
11	1 or 11 (mod 22)	[9], [10] and Section 2.3 above	
12	1 or 9 (mod 24)	Section 3.1 above	
13	1 or 13 (mod 26)	[9], [10] and Section 2.4 above	
15	1, 15, 21 or 25 (mod 30)	Section 2.5 above	25 (mod 30)
16	1 mod (32)	Section 3.2 above	289, 353
17	1 or 17 (mod 34)	[10] and Section 2.6 above	
19	1 or 19 (mod 38)	[10] and Section 2.7 above	57
23	1 or 23 (mod 46)	[10] and Section 2.8 above	69

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