KIRKMAN YEARS IN PG(3,2)

by

Alphonse Baartmans, Michigan Technological University

Joseph Yucas,[1] Southern Illinois University

Abstract.

In this paper we give a bijection between the Kirkman years in $PG(3,2)$ and the packings of the 3-subsets of an 8-set with (7,3,1)-designs thus showing the existence of and providing contructions for Kirkman years in $PG(3,2)$.

It is well-known that the 35 lines in $PG(3,2)$ can be partitioned into 7 sets, each containing 5 parallel lines, providing a solution to Kirkman's schoolgirl problem, see e.g. [2]. A set of 5 parallel lines in $PG(3,2)$ is called a *Kirkman day* or a *parallel class* in $PG(3,2)$ and such a partition of the 35 lines into 7 days is called a *Kirkman week* or a *Kirkman triple system* in $PG(3,2)$. The main goal of this paper is to show that the collection of all Kirkman days in $PG(3,2)$ can be partitioned into disjoint Kirkman weeks. Such a partition we call a *Kirkman year* in $PG(3,2)$.

By a *packing* of the k-subsets of a set S with (v,k,λ)-designs we will mean a partition of all the k-subsets of S into (v,k,λ)-designs. Notice here we permit $v < |S|$. In [3], Sharry and Street used the term overlarge set for such a set of designs. We will construct a bijection between the packings of the 3-subsets of an 8-set with (7,3,1)-designs and the Kirkman years in $PG(3,2)$. Hence to construct Kirkman years one needs only to construct these packings. This has been accomplished by Sharry and Street in [3] where they show that there are exactly 11 different such packings.

Before proceeding with the construction of the bijection we set some notation. $\mathcal{P}$, $\mathcal{L}$, $\mathcal{D}$, $\mathcal{W}$, and $\mathcal{Y}$ will denote respectively the points, lines, Kirkman days, Kirkman weeks and Kirkman years in $PG(3,2)$. $\mathbf{S} = \{1, 2, \ldots, 8\}$ and $\mathbf{S}_j$ will denote the collection of j-subsets of $\mathbf{S}$. Finally, $\mathbf{D}$ will denote the collection of all (7,3,1)-designs whose point set is a 7-subset of $\mathbf{S}$ and $\mathbf{P}$ will denote the collection of all packings of the 3-subsets of $\mathbf{S}$ with (7,3,1)-designs.

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In [1] we gave a geometric construction of the Steiner System $S(4,7,23)$. The point set of this Steiner System was $\mathcal{P} \cup \mathbf{S}$. The blocks were of three types which we called planes, line sets and extended ovoid sets. The 70 line sets were of the form $K \cup l$ where $K \in \mathbf{S}_4$ and $l \in \mathcal{L}$. Each of the 70 elements of $\mathbf{S}_4$ appeared exactly once in the line sets and each of the 35 lines in $\mathcal{L}$ appeared exactly twice in the line sets. Define $\psi : \mathbf{S}_4 \to \mathcal{L}$ by specifying that $\psi(K)$ is the line appearing with K in the line sets. The following properties of ψ have been verified in [1]:

(1) For two 4-sets $K_1, K_2 \in \mathbf{S}_4$:

(i) $\psi(K_1) = \psi(K_2)$ iff $K_1 \cap K_2 = \emptyset$;

(ii) $|\psi(K_1) \cap \psi(K_2)| = 1$ iff $|K_1 \cap K_2| = 2$;

(iii) $\psi(K_1) \cap \psi(K_2) = \emptyset$ iff $|K_1 \cap K_2| = 1$ or 3.

(2) For any $a \in \mathbf{S}$, $\{\psi(K) \mid a \in K \in \mathbf{S}_4\} = \mathcal{L}$.

(3) Four points from $\mathcal{P} \cup \mathbf{S}$ appear at most once in the line sets $K \cup \psi(K)$, $K \in \mathbf{S}_4$.

For a line $l \in \mathcal{L}$ notice that by (2) there is a unique 4-set $K \in \mathbf{S}_4$ containing 1 with $\psi(K) = l$. We will denote this K by $\psi^{-1}(l)$. For a day D in $\mathcal{D}$ we let $S_D = \{\psi^{-1}(l) \mid l \in D\}$.

Proposition 1. Let $D \in \mathcal{D}$. Either $|\bigcap_{K \in S_D} K| = 3$ or there is a unique line $l_0 \in D$ with $\psi^{-1}(l_0) \cap \psi^{-1}(l) = \{1\}$ for every $l \in D \backslash \{l_0\}$.

Proof: Suppose $|\bigcap_{K \in S_D} K| \neq 3$. Since lines in D are parallel we have by (1)(iii) that $|\psi^{-1}(l) \cap \psi^{-1}(l')| = 1$ or 3 for $l \neq l' \in D$.

<u>Case 1</u>. If $|\psi^{-1}(l) \cap \psi^{-1}(l')| = 3$ for every $l \neq l' \in D$ then consider any two elements $\{1, a, b, c\}$ and $\{1, a, b, d\}$ of S_D. An element of S_D which does not contain 1, a and b must be a subset of $\{1, a, b, c, d\}$ since it must intersect both $\{1, a, b, c\}$ and $\{1, a, b, d\}$ in three places. Hence we may assume there is at least one more element of S_D containing 1, a and b for else we would have five 4-subsets of $\{1, a, b, c, d\}$ each contain 1. There are only $\binom{4}{3} = 4$ of these. Assume then that $\{1, a, b, e\}$ is in S_D. Now an element of S_D which does not contain all three of 1, a and b must contain 1, c, d, e and one of a or b. This clearly cannot be done with a 4-set.

<u>Case 2</u>. If there are two lines $l, l' \in D$ with $\psi^{-1}(l) \cap \psi^{-1}(l') = \{1\}$ then we may

assume $\{1,a,b,c\}$ and $\{1,d,e,f\}$ are in S_D with $\{a,b,c\}\cap\{d,e,f\}=\emptyset$. Another element of S_D cannot intersect both $\{1,a,b,c\}$ and $\{1,d,e,f\}$ in 3 places and cannot intersect both $\{1,a,b,c\}$ and $\{1,d,e,f\}$ in 1 place so we may assume S_D contains $\{1,a,b,g\}$ with $g\notin\{1,a,b,c,d,e,f\}$. Notice that any other element of S_D which intersects $\{1,d,e,f\}$ in three places must intersect one of $\{1,a,b,c\}$ or $\{1,a,b,g\}$ in two places since $\{1,a,b,c\}\cup\{1,d,e,f\}\cup\{1,a,b,g\}=\mathbf{S}$. Consequently every other element of S_D intersects $\{1,d,e,f\}$ in exactly one place and intersects $\{1,a,b,c\}$ in exactly three places. Take $l_0=\psi(\{1,d,e,f\})$.

□

Define $\Gamma:\mathbf{S}_3\to\mathcal{D}$ by

$$\Gamma(J)=\{\psi(J\cup\{x\})\mid x\in\mathbf{S}\backslash J\}.$$

Notice that the image of Γ is actually inside $\mathcal{D}$, for if $\alpha\in\psi(J\cup\{x\})\cap\psi(J\cup\{y\})$ then $J\cup\{\alpha\}$ is a 4-subset of $(J\cup\{x\})\cup\psi(J\cup\{x\})$ and of $(J\cup\{y\})\cup\psi(J\cup\{y\})$ contradicting (3).

Define $X:\mathcal{D}\to\mathbf{S}_3$ as follows:

$$\begin{aligned} X(D) &= \bigcap_{K\in S_D} K, \text{ if } \Big|\bigcap_{K\in S_D}\Big| = 3 \\ &= \psi^{-1}(l_0)\backslash\{1\} \text{ where } l_0 \text{ is the unique line in } D \\ &\qquad \text{of Proposition 1, otherwise.} \end{aligned}$$

Theorem 2. Γ and X are bijections between $\mathbf{S}_3$ and $\mathcal{D}$.

Proof: To show that $\Gamma\circ X$ is the identity on $\mathcal{D}$ let $D\in\mathcal{D}$. If $J=\bigcap_{K\in S_D}K$ has size 3 then clearly $\Gamma(J)=\{\psi(J\cup\{x\})\mid x\in\mathbf{S}\backslash J\}=D$. If not then suppose $\psi^{-1}(l_0)=\{1,a,b,c\}$. For $l_0\neq l\in D$, $\psi^{-1}(l)\cap\{a,b,c\}=\emptyset$ so $S_D=\{\{1,a,b,c\},\{1,d,e,f\},\{1,d,e,g\},\{1,d,f,g\},\{1,e,f,g\}\}$ where $\mathbf{S}=\{1,a,b,\ldots,g\}$. Consider all 4-subsets of $\mathbf{S}$ containing a, b and c. They are $\{a,b,c,1\}$, $\{a,b,c,d\}$, $\{a,b,c,e\}$, $\{a,b,c,f\}$ and $\{a,b,c,g\}$. Notice that the last four of these are just the complements of the last four in S_D and the first is the same as the first in S_D. It follows from (1)(i) that $\Gamma(\{a,b,c\})=D$. To show $X\circ\Gamma$ is the identity on $\mathbf{S}_3$ let $J\in\mathbf{S}_3$ and let $D=\Gamma(J)$. If $1\in J$ then $\bigcap_{K\in S_D}K=J$ and hence $X(D)=J$. If $1\notin J$ let $l_0=\psi(J\cup\{1\})$. Notice that for $x\in\mathbf{S}\backslash(J\cup\{1\})$, $\psi^{-1}(\psi(J\cup\{x\}))$ is

S$\backslash(J \cup \{x\})$. Consequently, $\psi^{-1}(l_0) \cap \psi^{-1}(l) = \{1\}$ for each $l \neq l_0$ in D. This shows that l_0 is the unique line of Proposition 1 and hence $X(D) = \psi^{-1}(l_0) = J$. □

Lemma 3. For $J_1, J_2 \in \mathrm{S}_3$, $\Gamma(J_1) \cap \Gamma(J_2) = \emptyset$ iff $|J_1 \cap J_2| = 1$.

Proof: Suppose first that $|J_1 \cap J_2| = 0$ and let a, b be the two points in $\mathrm{S}\backslash(J_1 \cup J_2)$. Notice that $K_1 = J_1 \cup \{a\}$ and $K_2 = J_2 \cup \{b\}$ are disjoint so by (1)(i), $\psi(K_1) = \psi(K_2)$ and $\Gamma(J_1) \cap \Gamma(J_2) \neq \emptyset$. Next suppose $|J_1 \cap J_2| = 1$ and let $x \in J_1 \cap J_2$. If $l \in \Gamma(J_1) \cap \Gamma(J_2)$ then $\{x\} \cup l$ is a 4-set of points from $\mathcal{P} \cup \mathrm{S}$ appearing in two line sets contradicting (3). Finally, f $|J_1 \cap J_2| = 2$ let $c \in J_1 \backslash J_2$ and $d \in J_2 \backslash J_1$. Then $\psi(J_1 \cup \{d\}) = \psi\{J_2 \cup \{c\}\}$ so $\Gamma(J_1) \cap \Gamma(J_2) \neq \emptyset$. □

Theorem 4. Γ induces a bijection $\hat{\Gamma} : \mathbf{D} \to \mathcal{W}$.

Proof: Since any two blocks of a (7,3,1)-design intersect in one place we see by Lemma 3 that $\hat{\Gamma}(\mathbf{D}) \subseteq \mathcal{W}$. Let $W \in \mathcal{W}$. $S_W = \{\Gamma^{-1}(D) \mid D \in W\}$ is a set of seven blocks of size 3 on say v points. Since any two days in W are disjoint we have by Lemma 3 that any two blocks of S_W intersect in exactly one place. Let $J \in S_W$. Since each of the remaining six blocks of S_W must intersect J in exactly one place and since $|\mathrm{S}| = 8$ we see that no point of J can occur more than 3 times in the blocks of S_W. On the other hand, a point of J occurring fewer than 3 times in the blocks of S_W forces another point of J to occur more than 3 times. Hence every point of J occurs exactly 3 times in the blocks of S_W. Since J was an arbitrary block of S_W we have that every point of S which occurs at all occurs exactly 3 times in the blocks of S_W. This shows that the dual of S_W is a (7,3,1)-design. Hence S_W is also. □

The following Corollary now follows from Theorem 2 and Theorem 4:

Corollary 5. $\hat{\Gamma}$ induces a bijection $\hat{\hat{\Gamma}} : \mathbf{P} \to \mathcal{Y}$.

References

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